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RAMANUJAN COMPLEXES OF TYPE $\tilde{A}_d$*

BY

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With love and admiration to Hillel Furstenberg, a teacher and friend

ABSTRACT

We define and construct Ramanujan complexes. These are simplicial complexes which are higher dimensional analogues of Ramanujan graphs (constructed in [LPS]). They are obtained as quotients of the buildings of type $\tilde{A}_{d-1}$ associated with $\mathrm{PGL}_d(F)$ where F is a local field of positive characteristic.

1. Introduction

A finite k-regular graph X is called a Ramanujan graph if for every eigenvalue λ of the adjacency matrix $A = A_X$ of X either $\lambda = \pm k$ or $|\lambda| \le 2\sqrt{k-1}$. This term was defined in [LPS] where some explicit constructions of such graphs were presented; see also [Ma1], [Lu1], [Mo]. These graphs were obtained as quotients of the k-regular tree $T = T_k$, for $k = q+1$, q a prime power, divided by the action

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of congruence subgroups of $G = \mathrm{PGL}_2(F)$. Here F is a non-archimedean local field with residue field of order q, and the tree T is the Bruhat–Tits building associated with G, which is a building of type $\tilde{A}_1$. The (proved) Ramanujan conjecture for GL_2 was an essential ingredient in the proof that the graphs are indeed Ramanujan; see [Lu1]. The number $2\sqrt{k-1}$ plays a special role in the definition of Ramanujan graphs because of the Alon–Boppana theorem (see [LPS]), which proves that this is the best possible bound for an infinite family of k-regular graphs. A conceptual explanation was given by Greenberg [Gr], [Lu1, Thm. 4.2.7] (see also [GZ]): for a connected graph X, let $\rho(X)$ denote the norm of the adjacency operator A on $\mathrm{L}^2(X)$ (so $\rho(T_k) = 2\sqrt{k-1}$); then, Greenberg showed that no upper bound on the non-trivial eigenvalues of finite quotients of X is better than $\rho(X)$. These considerations motivated Cartwright, Solé and Żuk [CSZ] to suggest a generalization of the notion of Ramanujan graphs from finite quotients of T_k — which is an $\tilde{A}_1$ building — to the simplicial complexes obtained as finite quotients of $\mathcal{B} = \mathcal{B}_d(F)$, the Bruhat–Tits building of type $\tilde{A}_{d-1}$ associated with the group $G = \mathrm{PGL}_d(F)$. The vertices $\mathcal{B}^0$ of the building are labelled by a 'color' function $\varrho\colon \mathcal{B}^0 \to \mathbb{Z}/d\mathbb{Z}$, and we may look at the $d-1$ colored adjacency operators A_k, $k = 1, \dots, d-1$ on $\mathrm{L}^2(\mathcal{B}^0)$, called the Hecke operators. They are defined by

$$(A_k f)(x) = \sum f(y) \tag{1.1}$$

where the summation is over the neighbors y of x such that $\varrho(y) - \varrho(x) = k$ in $\mathbb{Z}/d\mathbb{Z}$. These operators A_k are bounded, normal, and commute with each other. Thus, they have a simultaneous spectral decomposition, and the spectrum $\mathfrak{S}_d$ of $(A_1, \dots, A_{d-1})$ on $\mathrm{L}^2(\mathcal{B}^0)$ was computed explicitly as a subset of $\mathbb{C}^{d-1}$ (see Subsection 2.3 below). This set is, of course, contained in the Cartesian product $\mathfrak{S}_{d,1} \times \cdots \times \mathfrak{S}_{d,d-1}$, where $\mathfrak{S}_{d,k}$ is the spectrum of A_k, but it is not equal to the product.

Definition 1.1 (following [CSZ]): A finite quotient X of $\mathcal{B}$ is called a Ramanujan complex if the eigenvalues of every non-trivial simultanenous eigenvector $v \in \mathrm{L}^2(X)$, $A_k v = \lambda_k v$, satisfy $(\lambda_1, \dots, \lambda_{d-1}) \in \mathfrak{S}_d$.

(See Subsection 2.3 for more detailed explanations, and in particular for a description of the trivial eigenvalues. See also [JL] for a definition and construction of Ramanujan complexes which are not simplicial.) Cartwright et al. [CSZ] also suggested a way of obtaining such Ramanujan complexes: assume F is a local field of positive characteristic; let Γ be a cocompact arithmetic lattice of $G = \mathrm{PGL}_d(F)$ of inner type, and $\Gamma(I)$ a congruence subgroup of Γ. They conjectured that the quotients $\Gamma(I)\backslash\mathcal{B}$ are Ramanujan complexes. The work of Lafforgue in the last few years, which proved the Ramanujan conjecture for GL_d in characteristic p (an extension of Drienfeld's work for GL_2 in characteristic p and of Deligne's for GL_2 in characteristic zero), provided hope that these combinatorial applications could be deduced. The current work, which started from the challenge to prove the conjecture in [CSZ], shows that for general d, the story is more subtle. It turns out that most of these quotients are indeed Ramanujan, but not all. To describe our results, let us first introduce some notation. Let k be a global field of characteristic $p > 0$, and D a division algebra of degree d over k. Denote by G' the k-algebraic group $D^\times/k^\times$, and fix a suitable embedding of G' as a linear group (see Section 5). Let T be the finite set of valuations of k for which D does not split. We assume that for every $\nu \in T$, $D_\nu = D\otimes_k k_\nu$ is a division algebra. Let ν_0 be a valuation of k which is not in T, and $F = k_{\nu_0}$. Let

$$R_0 = \{x \in k \colon \nu(x) \geq 0 \quad \text{for every } \nu \neq \nu_0\}. \tag{1.2}$$

Then $\Gamma = G'(R_0)$ is a discrete subgroup of $G'(F)$, and the latter is isomorphic to $G(F) = \mathrm{PGL}_d(F)$, as F splits D. By general results, Γ is in fact a cocompact lattice in $G(F)$ — an "arithmetic lattice of inner type". Let $\mathcal{B} = \mathcal{B}_d(F)$ be the Bruhat–Tits building of $G(F)$; then $\mathcal{B}^0 \cong G(F)/K$, where $K = G(\mathcal{O})$ is a maximal compact subgroup ($\mathcal{O}$ is the ring of integers in F). $G(F)$ acts on $\mathcal{B}$ by left translation. For $0 \neq I \lhd R_0$ an ideal (note that R_0 is a principal ideal domain), we have the principal congruence subgroup

$$\Gamma(I) = G'(R_0, I) = \mathrm{Ker}(G'(R_0) \to G'(R_0/I)). \tag{1.3}$$

In the following two theorems we assume the global Jacquet–Langlands correspondence for function fields; see Remark 1.6 below regarding this assumption.

THEOREM 1.2: *If d is prime, then for every $0 \neq I \lhd R_0$, $\Gamma(I)\backslash\mathcal{B}$ is a Ramanujan complex.*

So for d prime, the Cartwright–Solé–Żuk conjecture is indeed true. On the other hand, for general d:

THEOREM 1.3: (a) *For every d, if I is prime to some valuation $\theta \in T$, i.e. $\theta(a) = 0$ for some $\theta \in T$ and some $a \in I$, then $\Gamma(I)\backslash\mathcal{B}$ is a Ramanujan complex.*
(b) *If d is not a prime, then there exist (infinitely many) ideals I such that $\Gamma(I)\backslash\mathcal{B}$ are not Ramanujan.*

Theorem 1.2 may suggest that in positive characteristic, if d is a prime, then every finite quotient of $\mathcal{B}$ is Ramanujan. We do not know if this is indeed the case (which would be truly remarkable), but at least in the zero characteristic analog there are counterexamples. Indeed, in Section 6 we show that if E is a non-archimedean local field of characteristic zero, then congruence quotients of $\mathcal{B} = \mathcal{B}_d(E)$ can be non-Ramanujan for every $d \geq 4$. This happens if Γ is taken to be an arithmetic group of outer type.

THEOREM 1.4: *Let E be a non-archimedean local field of characteristic zero, and assume $d \geq 4$. Then $\mathcal{B}_d(E)$ has infinitely many non-Ramanujan quotients.*

For a discussion of the case $d = 3$, see [B1]. The proof of Theorems 1.2 and 1.3(a) follows in principle the line of proof for Ramanujan graphs, as in [Lu1]. The problem is transferred to representation theory.

PROPOSITION 1.5: *Let Γ be a cocompact lattice in $G(F) = \mathrm{PGL}_d(F)$. Then $\Gamma\backslash\mathcal{B}$ is a Ramanujan complex iff every irreducible spherical infinite-dimensional sub-representation of $\mathrm{L}^2(\Gamma\backslash G(F))$ is tempered.*

The strategy now is to start with an irreducible sub-representation ρ of $\mathrm{L}^2(\Gamma(I)\backslash G'(F))$. By Strong Approximation, one can show (see Subsection 3.2 below) that ρ is a local factor of an adèlic automorphic representation $\pi' = \otimes\pi'_\nu$ in $\mathrm{L}^2(G'(k)\backslash G'(\mathbb{A}))$ such that $\pi'_{\nu_0} = \rho$, where $\mathbb{A}$ is the ring of adèles of k. We can view π' as an automorphic representation of $D^\times(\mathbb{A})$. Then, the Jacquet–Langlands correspondence associates with π' an automorphic representation $\pi = \otimes\pi_\nu$ in $\mathrm{L}^2(\mathrm{GL}_d(k)\backslash\mathrm{GL}_d(\mathbb{A}))$, such that $\pi_{\nu_0} = \pi'_{\nu_0}$. We then appeal to the work of Lafforgue, who proved that if π is cuspidal, then π_ν is tempered for every unramified ν, and in particular $\pi_{\nu_0} = \rho$ is tempered. Now, the cuspidality issue is exactly what distinguishes between the cases where d is a prime and where d is a composite number. If d is prime, then all infinite-dimensional irreducible sub-representations of $\mathrm{L}^2(\mathrm{GL}_d(k)\backslash\mathrm{GL}_d(\mathbb{A}))$ are cuspidal (and the others are one-dimensional, and are responsible for the "trivial" eigenvalues, see Subsection 2.3). Thus Theorem 1.2 can be proved. On the other hand, when d is not a prime, there is a "residual spectrum" and π may be there, in which case π_{ν_0} is not tempered. Theorem 1.3 (both parts (a) and (b)) is proved by a careful analysis of the image of the Jacquet–Langlands map, as described in [HT]. The proof of Theorem 1.4 is different. We apply the method of Burger–Li–Sarnak [BLS1], [BLS2] who showed how the existence of large "extended arithmetic subgroups" in $\Gamma(I)$ can affect the spectrum. For arithmetic lattices associated to Hermitian forms (unlike the case of inner type), such "large" subgroups do exist, but anisotropic Hermitian forms (with enough variables) exist only if $\mathrm{char}(F) = 0$.

Remark 1.6: The global Jacquet–Langlands correspondence is proved in the literature for fields of characteristic zero (see Theorem 4.4 below and [HT, Thm. VI.1.1]). It is likely that the theorem is valid in exactly the same formulation in positive characteristic, and it seems (to some experts we consulted) that a proof can be worked out using existing knowledge. So far, this task has not been carried out. We hope that our work will give some additional motivation to complete this gap in the literature. W. Li [Li] managed to prove the existence of Ramanujan complexes of type $\tilde{A}_d$ in positive characteristic, avoiding the use of the Jacquet–Langlands correspondence, and in fact also not using Lafforgue's theorem, appealing to [LRS] instead. In order to apply this method, one needs the division algebra to be ramified in at least four places, and therefore it does not cover the case of algebras ramified in two places. This case is crucial for our next work, [LSV], in which we give an explicit construction of Ramanujan complexes. On the other hand, we have to assume that in the ramification points the algebra is completely ramified, while Li requires this assumption in only two prime places. We recently learned that Alireza Sarveniazi [Sa] has also given a construction of Ramanujan complexes.

The paper is organized as follows: in Section 2 we describe briefly the building $\mathcal{B}$, the operators A_k, the local representation theory, and, in particular, we prove Proposition 1.5 above. In Section 3 we show how strong approximation enables one to pass from the local theory to the global one. In Section 4 we survey the global theory: Lafforgue's theorem, the residual spectrum, and the Jacquet–Langlands correspondence. After the preparations we prove Theorems 1.2 and 1.3 in Section 5 and Theorem 1.4 in Section 6. Much of the material of Sections 2–4 is well known to experts, but since we expect (and hope) the paper will have readers outside representation theory and automorphic forms, we tried to present the material in a suitable way for non-experts.

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2. Affine buildings and representations of the local group

In this section, F is a non-Archimedean local field of arbitrary characteristic, $\mathcal{O}$ its ring of integers, and $\varpi \in \mathcal{O}$ a uniformizer. Recall that a complex is a structure composed of i-cells, where the Let $\nu_0: F \to \mathbb{Z}$ denote the valuation of F.

2.1. AFFINE BUILDINGS OF TYPE $\tilde{A}_{d-1}$. Recall that a complex is a structure composed of i-cells, where the 0-cells are called vertices, and every i-cell is a set of $i+1$ vertices. A complex is simplicial if every subset of a cell is also a cell. We will now describe the affine building $\mathcal{B} = \mathcal{B}_d(F)$ associated to $\mathrm{PGL}_d(F)$, which is an (infinite) simplicial complex. Consider the $\mathcal{O}$-lattices of full rank in F^d. We define an equivalence relation on lattices by setting $L \sim sL$ for every $s \in F^\times$. Since $F^\times/\mathcal{O}^\times$ is the infinite cyclic group generated by ϖ, an equivalent definition is that $L \sim \varpi^i L$ for every $i \in \mathbb{Z}$. By $\mathcal{B}^i$ we denote the set of i-cells of $\mathcal{B}$. The vertices $\mathcal{B}^0$ are the equivalence classes of lattices. There is an edge (1-cell)(x,x'), from $x = [L]$ to $x' = [L'] \in \mathcal{B}^0$, if $\varpi L \subsetneq L' \subsetneq L$. Notice that this is a symmetric relation, since then $\varpi L' \subsetneq \varpi L \subsetneq L'$. The quotient $L/\varpi L$ is a vector space of dimension d over the field $\mathcal{O}/\varpi\mathcal{O} \cong \mathbb{F}_q$. As i-cells of $\mathcal{B}$ we take the complete subgraphs of size $i+1$ of $\mathcal{B}^0$. It immediately follows that $\mathcal{B}$ has $(d-1)$-cells (corresponding to maximal flags in quotients $L/\varpi L$). It also follows that there are no higher dimensional cells. We call $L_0 = \mathcal{O}^d \subseteq F^d$ the **standard lattice**. For every lattice L, there is some i such that $\varpi^i L \subseteq L_0$ (it then follows that every two lattices of maximal rank are commensurable). We define a color function $\varrho: \mathcal{B}^0 \to \mathbb{Z}/d$, by

$$\varrho(L) = \log_q [L_0 : \varpi^i L] \tag{2.1}$$

for i large enough; the color is well defined since $\log_q [\varpi^i L : \varpi^{i+1} L] = d$. In a similar way, the color of an ordered edge $(x,y) \in \mathcal{B}^1$ is defined to be $\varrho(x) - \varrho(y) \pmod{d}$. The group $\mathrm{GL}_d(F)$ acts on lattices by its action on bases; the scalar matrices carry a lattice to an equivalent lattice, so $G = \mathrm{PGL}_d(F)$ acts (transitively) on the vertices of $\mathcal{B}$. Since the action of $\mathrm{GL}_d(F)$ preserves inclusion of lattices, G respects the structure of $\mathcal{B}$, and in particular the color of edges. Note that $\mathrm{GL}_d(F)$ does not preserve the color of vertices, but $\mathrm{SL}_d(F)$ does. The stabilizer of $[L_0]$ is the maximal compact subgroup $K = \mathrm{PGL}_d(\mathcal{O})$. We can thus identify $\mathcal{B}^0$ with G/K, where G acts by multiplication from the left. A coset $gK \in G/K$ corresponds to the lattice generated by the columns of g (so $[L_0]$ corresponds to the identity matrix). The color of gK can then be computed from the determinant of g:

$$\det(g) \equiv \varpi^{\varrho(gK)} \pmod{{F^{\times}}^{d}},$$

where ${F^{\times}}^{d}$ is the subgroup of d-powers in $F^{\times}$. Let $\omega_k = \operatorname{diag}(\varpi, \dots, \varpi, 1, \dots, 1)$, where $\det(\omega_k) = \varpi^k$. The lattice corresponding to $\omega_k K$ is obviously a neighbor of color k of $[L_0]$. Let Ω_k be the set of neighbors of color k of $[L_0]$. Then K acts (as a subgroup of G) transitively on Ω_k, so that $K\omega_k K = \bigcup yK$, where the union is over $yK \in \Omega_k$. Multiplying from the left by an arbitrary $g \in G$, we see that the neighbors forming an edge of color k with gK are $\{gyK\}_{yK\in\Omega_k}$. It follows that the operators A_k (defined in Equation (1.1)) act on functions $f\colon G/K \to \mathbb{C}$ by

$$(A_k f)(gK) = \sum_{yK\in\Omega_k} f(gyK) = \sum_{yK\in\Omega_k} \int_{yK} f(gx)dx = \int_{K\omega_k K} f(gx)dx;$$

the integrals are normalized so that $\int_K dx = 1$. See [M] and [B2] for details.

2.2. Spherical representations. In this section let $K = \mathrm{GL}_d(\mathcal{O})$, which is a maximal compact subgroup of $G = \mathrm{GL}_d(F)$. As in [Lu1], we study the spectrum of the operators A_k via representations of $\mathrm{GL}_d(F)$. An irreducible admissible representation of G is called **H-spherical** if the representation space has an H-fixed vector, where $H \leq G$ is a subgroup. The K-spherical representations are simply called **spherical**. (A representation is **smooth** if every $v \in V$ is fixed under some open compact subgroup, and **admissible** if, moreover, the spaces fixed by each open compact subgroup are finite dimensional.)

The Hecke operators A_k of the preceding subsection (defined in the same way, as functions of G/K for $G = \mathrm{GL}_d(F)$ rather than $G = \mathrm{PGL}_d(F)$) generate the Hecke algebra $\mathrm{H}(G,K)$ of all bi-K-invariant compactly supported functions on G, with multiplication defined by

$$(A * A')(g) = \int_G A(x)A'(x^{-1}g)dx.$$

The A_k commute with each other, and freely generate $\mathrm{H}(G,K)$ (cf. [M, Sec. V]). Let $\rho\colon G \to \operatorname{End}(V)$ be an admissible representation; the Hecke algebra acts on the representation space (see [C, Eq. (9)]) by

$$A \cdot v = \int_G A(x)(\rho(x))(v)dx, \tag{2.2}$$

which is an integration over a compact set since A is compactly supported. It projects V to the K-fixed subspace V^K (which is finite dimensional as the representation is admissible). Moreover, if V is an irreducible G-module, then V^K

is an irreducible $\mathrm{H}(G,K)$-module. Since $\mathrm{H}(G,K)$ is commutative and finitely generated, V^K is one-dimensional in this case, and consequently, every $v \in V^K$ is an eigenvector of all the A_k. We describe how spherical representations are parameterized by d-tuples of complex numbers, called the Satake parameters. For details, the reader is referred to [C]. Let B denote a Borel subgroup of G (e.g. the upper triangular matrices), U its unipotent radical, and $T \cong B/U \cong (F^\times)^d$ a maximal torus of B. We then have $B = UT$ and $G = BK = UTK$.

Recall that $F^\times/\mathcal{O}^\times = \langle \varpi \rangle$. A character $\chi\colon F^\times \to \mathbb{C}^\times$ is spherical if it is trivial on the maximal compact subgroup of $F^\times$, namely $\mathcal{O}^\times$. Such a character is, thus, determined by $z = \chi(\varpi)$, which is an arbitrary complex number. The character is called **unitary** iff $z \in S^1 = \{w \in \mathbb{C}\colon |{=}|_w 1\}$. Every character $\chi\colon T \to \mathbb{C}^\times$ can be written as $\chi(\mathrm{diag}(t_1,\dots,t_d)) = \chi_1(t_1)\cdots\chi_d(t_d)$, for characters $\chi_i\colon F^\times \to \mathbb{C}^\times$. χ is said to be **unramified** if the χ_i are spherical. Since $T \cong B/U$, χ extends to a character of B. The symmetric group S_d acts on the characters by permuting the χ_i. The unitary induction of representations from B to G is defined using the **modular function**

$$\Delta(b) = |a_1|_F^{d-1}|a_2|_F^{d-3}\cdots|a_d|_F^{1-d}, \quad b \in B \tag{2.3}$$

where $a_1, a_2, \dots, a_d$ are the entries on the diagonal of b (in that order), and $|\cdot|_F$ is the absolute value function of F, normalized so that $|{=}|_x q^{-\nu_0(x)}$ where $q = |\mathcal{O}/\varpi\mathcal{O}|$. The induced representation $I_\chi = \mathrm{Ind}_B^G(\chi)$ is the space of locally constant functions $f\colon G \to \mathbb{C}$ such that

$$f(bg) = \Delta^{1/2}(b)\chi(b)f(g), \quad b \in B, g \in G$$

with the action of G from the right (by $g \cdot f(x) = f(xg)$). The inclusion of the modular function Δ guarantees that if χ is unitary, then there is an inner product $\langle f, f' \rangle = \int_K f(x)\overline{f'(x)}dx$ on I_χ, for which the action of G is unitary (these are called the spherical principal series representations). However, the space I_χ still can be unitary even if χ is not unitary (these are called spherical complementary series representations); see Subsection 2.4. We remark that I_χ need not be irreducible. Two spaces I_χ and $I_{\chi'}$ are isomorphic iff $\chi' = w\chi$ for some $w \in S_d$ ([C, Subsec. 3.3], [Bu, Sec. 2.6]). Notice that $|\cdot|$ is spherical, so the modular function Δ is an unramified character. If $g \in B \cap K$ then g is upper triangular, with its diagonal entries invertible in $\mathcal{O}$. Since $G = BK$ and χ is unramified, it follows that

$$f_\chi(bk) := \Delta^{1/2}(bk)\chi(bk) = \Delta^{1/2}(b)\chi(b), \quad b \in B, k \in K, \tag{2.4}$$

is a well defined K-fixed function (unique in I_χ), which makes the induced representation $\rho\colon G \to \mathrm{End}(I_\chi)$ spherical. By definition, ρ is determined by the numbers $z_i = \chi_i(\varpi) = \chi(\mathrm{diag}(1,\dots,1,\varpi,1,\dots,1))$, called the Satake parameters of χ, where χ_i are the diagonal components of χ, which is a sub-representation of $\Delta^{-1/2}\rho|_B$. The representations which are well defined on $\mathrm{PGL}_d(F)$ are those with $z_1 \cdots z_d = 1$ (since they need to be trivial on the center of $\mathrm{GL}_d(F)$). Let $\sigma_k(z_1,\dots,z_d)$ be the kth elementary symmetric function, i.e. $\sigma_k(z_1,\dots,z_d) = \sum_{i_1<\dots<i_k} z_{i_1}\cdots z_{i_k}$.

PROPOSITION 2.1: *The function f_χ is an eigenfunction of the A_k, $A_k f_\chi = \lambda_k f_\chi$, where $\lambda_k = q^{k(d-k)/2}\sigma_k(z_1,\dots,z_d)$.*

Proof: Since $\mathrm{H}(G,K)$ acts on I_χ and preserves the K-fixed subspace $\langle f_\chi\rangle$, f_χ is an eigenvector of the A_k. It is enough to compute $A_k f_\chi$ at the point $g = 1$ (noting that $f_\chi(1) = 1$). For every subset $C \subseteq \{1,\dots,d\}$ of size k, let $\Omega_{k,C}$ be the set of upper triangular matrices m such that $m_{ii} = \varpi$ if $i \in C$, $m_{ii} = 1$ if $i \notin C$, m_{ij} is in some fixed lifting of $\mathcal{O}/\varpi\mathcal{O}$ to $\mathcal{O}$ if $i \in C$ and $j \notin C$, and $m_{ij} = 0$ otherwise. For example, for $d = 4$ and $k = 2$ the sets are

$$\begin{pmatrix} \varpi & 0 & * & * \\ 0 & \varpi & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \varpi & * & 0 & * \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \varpi & * \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \varpi & * & * & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \varpi \end{pmatrix},$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \varpi & 0 & * \\ 0 & 0 & \varpi & * \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \varpi & * & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \varpi \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \varpi & 0 \\ 0 & 0 & 0 & \varpi \end{pmatrix}.$$

There is a one-to-one correspondence between neighbors $yK \in \Omega_k$ of $[L_0]$ and subspaces of co-dimension k of $L_0/\varpi L_0$, so using the correspondence between matrices and lattices mentioned above, $\Omega_k = \bigcup_C \Omega_{k,C}$. Fix a subset C, and let $s = \sum_{i\in C} i$. The number of matrices in $\Omega_{k,C}$ is $q^{(d-k+1)+\dots+d-s}$, while $\Delta^{1/2}(y) = \Pi_{i\in C}|\varpi|_F^{(d+1)/2-i} = q^{-k(d+1)/2+s}$ for every $yK \in \Omega_{k,C}$. It follows that the sum of $\Delta^{1/2}(y)\chi(y)$ over $yK \in \Omega_{k,C}$ is $q^{k(d-k)/2}\chi(y) = q^{k(d-k)/2}\Pi_{i\in C} z_i$, so by summing over all C we obtain

$$(A_k f_\chi)(1) = \sum_{yK\in\Omega_k} \Delta^{1/2}(y)\chi(y) = q^{k(d-k)/2}\sigma_k(z_1,\dots,z_d). \qquad \blacksquare$$

Now let $\rho\colon G \to \mathrm{End}(V)$ be an irreducible spherical representation of $G = \mathrm{GL}_d(F)$ with a unique (up to scalar multiples) K-invariant vector $v_0 \in V$. Let

$\hat{V}$ be the representation contragredient to V. The space $\hat{V}^K$ is dual to V^K and thus one-dimensional. Choose $\hat{v_0} \in \hat{V}^K$ such that $\langle v_0, \hat{v_0}\rangle = 1$ (where $\langle , \rangle$ is the action of $\hat{V}$ on V). Define the bi-K-invariant function $\psi(g) = \langle \rho(g)v_0, \hat{v_0}\rangle$ (so that $\psi(1) = 1$). $\psi(g)$ is called a **spherical function**. As explained earlier, v_0 is an eigenvector of $\mathrm{H}(G, K)$. The action of $\mathrm{H}(G, K)$ on $V^K = \langle v_0\rangle$ defines a homomorphism ω: $\mathrm{H}(G, K) \to \mathbb{C}$ by

$$A \cdot v_0 = \omega(A)v_0. \tag{2.5}$$

The action of G (from the right) on the space of functions $\{f: G \to \mathbb{C}\}$ induces an action of the Hecke algebra on this space, and by Equation (2.2) and the definition of ψ, we find that $A \cdot \psi = \omega(A)\psi$. Using Equation (2.2) one can check that

$$\psi(g) = \omega(1_{KgK})/\mu(KgK), \tag{2.6}$$

where 1_{KgK} is the characteristic function of $KgK \subseteq G$, and μ is the normalized measure.

PROPOSITION 2.2: *Using the notation as above, if ρ_1 and ρ_2 are irreducible spherical representations, then $\rho_1 \cong \rho_2$ iff $\psi_1 = \psi_2$ iff $\omega_1 = \omega_2$.*

Proof: Let v_{1_0} and v_{2_0} be the (unique) K-fixed vectors of ρ_1 and ρ_2. The equivalence of $\psi_1 = \psi_2$ and $\omega_1 = \omega_2$ follows at once from Equation (2.6). If $\rho_1 \cong \rho_2$ then it is obvious that $\psi_1 = \langle \rho_1(\cdot)v_{1_0}, \hat{v_{1_0}}\rangle = \langle \rho_2(\cdot)v_{2_0}, \hat{v_{2_0}}\rangle = \psi_2$. In the other direction, assume $\psi_1 = \psi_2$, and let V_i be the representation space of ρ_i. Define a map from V_i to $\langle \psi_i G\rangle$, the representation spanned by ψ_i, by sending $v \in V_i$ to the function $g \mapsto \langle \rho(g)v, \hat{v_{i_0}}\rangle$, for $i = 1, 2$. This is easily seen to be a non-zero homomorphism (since $v_{i_0} \mapsto \psi_i \neq 0$), which is an isomorphism since V_i is irreducible. Then $\rho_1 \cong \langle \psi_1 G\rangle = \langle \psi_2 G\rangle \cong \rho_2$. ∎

As a G-module, I_χ has finite composition length, so it has only finitely many irreducible subquotients.

PROPOSITION 2.3 ([C]): *Every irreducible spherical representation of* $\mathrm{GL}_d(F)$ *is isomorphic to a subquotient of I_χ for some unramified character χ, which is unique up to permutation.*

Proof: Let ρ: $G \to \mathrm{End}(V)$ be an irreducible spherical representation of G with v_0 as its K-invariant vector. Let ω: $\mathrm{H}(G, K) \to \mathbb{C}$ be its corresponding homomorphism, defined by Equation (2.5). It can be shown [C, Cor. 4.2] that every such homomorphism is of the form

$$\omega_\chi(A) = \int_G A(x) f_\chi(x) dx$$

for some unramified character $\chi: T \to \mathbb{C}$ (unique up to permutation) where f_χ is defined in Equation (2.4). Then

$$(A \cdot f_\chi)(1) = \int_G A(x) f_\chi(x) dx = \omega_\chi(A) = \omega_\chi(A) \cdot f_\chi(1),$$

and since f_χ is an eigenvector, $A \cdot f_\chi = \omega_\chi(A) f_\chi$ for every $A \in \mathrm{H}(G, K)$. Let W be an irreducible subquotient of I_χ in which f_χ has a non-zero image. By the previous proposition ρ is isomorphic to W, since $\omega_\chi = \omega$. ∎

Thus, every spherical representation is determined by the Satake parameters $z_i = \chi_i(\varpi) = \chi(\mathrm{diag}(1, \dots, 1, \varpi, 1, \dots, 1))$, for some unramified χ, uniquely determined up to permutation.

Proposition 2.4: *Let $f: G/K \to \mathbb{C}$ be a simultaneous eigenvector of $A_1, \dots, A_{d-1}$. Then there is an unramified character χ such that f_χ has the same eigenvalues.*

Proof: Consider $f: G \to \mathbb{C}$, which is invariant with respect to K. Let $\langle fG \rangle$ denote the linear span of the G-orbits of f, where G acts from the right. Taking this space modulo a maximal sub-module not containing f, we obtain an irreducible spherical representation, where f is a (unique) K-fixed vector. By the previous proposition, it is isomorphic to a subquotient of I_χ for an unramified character χ, where f_χ is the unique K-fixed vector. By Proposition 2.2, since the two representation spaces are isomorphic, they induce the same homomorphism $\omega: \mathrm{H}(G, K) \to \mathbb{C}$, namely $A_k \cdot f = \omega(A_k) f$ and $A_k \cdot f_\chi = \omega(A_k) f_\chi$. ∎

Let $\rho: G \to \mathrm{End}(V)$ be a unitary representation, and $\langle , \rangle$ the inner product defined on V. The functions of the form

$$\rho_{v,w}: g \mapsto \langle \rho(g) v, w \rangle$$

where $v, w \in V$ are called the **matrix coefficients** of ρ. Notice that if V has a K-fixed vector v_0 and $\langle v_0, v_0 \rangle = 1$, then ρ_{v_0,v_0} is a spherical function. In the special case of I_χ, $\rho_{f_\chi, f_\chi}(g) = \int_K f_\chi(xg) dx$. If V is irreducible, then fixing $w \neq 0$, the map $v \mapsto \rho_{v,w}$ is an isomorphism of representations (where G acts on the space of functions from the right). A representation is called **tempered**, if for some $0 \neq v, w \in V$, $\rho_{v,w} \in \mathrm{L}^{2+\epsilon}(G)$ for every $\epsilon > 0$. The following equivalence is well known:

PROPOSITION 2.5: *An irreducible spherical unitary representation is tempered iff its Satake parameters have absolute value* 1.

2.3. RAMANUJAN COMPLEXES AND THE SPECTRUM OF A_k. Let Γ be a cocompact lattice of $G = \mathrm{PGL}_d(F)$. Then Γ acts on $\mathcal{B} = G/K$ by left translation, and $\Gamma\backslash\mathcal{B}$ is a finite complex. The color function defined on $\mathcal{B}^0$ (Equation (2.1)) may not be preserved by the map $\mathcal{B} \to \Gamma\backslash\mathcal{B}$. However, the colors defined on $\mathcal{B}^1$ by $\varrho(x,y) = \varrho(x) - \varrho(y) \pmod d$ are preserved, since they are determined by the index of (a representative of) y as a sublattice in (a representative of) x. Since the Hecke algebra $\mathrm{H}(G,K)$ acts on G from the right, and Γ is acting from the left, the operators A_k on $\mathrm{L}^2(\mathcal{B}) = \mathrm{L}^2(G/K)$ induce colored adjacency operators on $\Gamma\backslash\mathcal{B}$.

It should be noted that if Γ is torsion free, then $\gamma x \neq x$ for any $\gamma \neq 1$ and any $x \in \mathcal{B}^0$, so the underlying graph of $\Gamma\backslash\mathcal{B}$ is simple. Every cocompact lattice has a finite index torsion free subgroup. The trivial eigenvectors appear in $\mathrm{L}^2(\Gamma\backslash\mathcal{B})$ but not in $\mathrm{L}^2(\mathcal{B})$, since the former complex is finite. The trivial eigenvectors can be constructed as follows. The trivial representation of G is obviously spherical. Taking $\chi = \Delta^{-1/2}$, we see that $f_\chi(g) = 1$ for every g (see Equation (2.4)), and the action of G on the subspace $\mathbb{C}f_\chi \subseteq I_\chi$ is trivial. The Satake parameters of the trivial representation are thus $z_i = \Delta^{-1/2}(\mathrm{diag}(1,\dots,1,\varpi,1,\dots,1)) = q^{-(d-2i+1)/2}$. More generally, since $G(F)/\mathrm{PSL}_d(F)K \cong F^\times/{F^\times}^d$, G has d one-dimensional spherical representations. Fixing ζ such that $\zeta^d = 1$,

$$\chi(g) = \Delta^{-1/2}(g)\zeta^{\nu_0(\det(g))} \tag{2.7}$$

corresponds to a one-dimensional representation, with the K-fixed vector $f_\chi(g) = \zeta^{\nu_0(\det(g))}$ (since $\det(k) \in \mathcal{O}^\times$ for every $k \in K$). The Satake parameters in this case are $\zeta q^{-(d-1)/2}, \dots, \zeta q^{(d-1)/2}$, and the eigenvalues are

$$\zeta^k \cdot q^{k(d-k)/2}\sigma_k(q^{-(d-1)/2}, q^{-(d-3)/2}, \dots, q^{(d-1)/2}).$$

Let $t = [\Gamma : \Gamma \cap \mathrm{PSL}_d(F)]$. By Equation (2.7), the trivial eigenvector f_χ is well defined on $\Gamma\backslash\mathcal{B}$ iff $\zeta^{d/t} = 1$, and the respective d/t eigenvectors give rise to the *trivial* eigenvalues. Since G is infinite, these f_χ do not belong to $\mathrm{L}^2(\mathcal{B})$. Let $\mathfrak{S}_{d,k} \subseteq \mathbb{C}$ denote the spectrum of the operator A_k acting on $\mathrm{L}^2(\mathcal{B})$.

Definition 2.6: The complex $\Gamma\backslash\mathcal{B}$ is pseudo-Ramanujan if for each $k = 1,\dots,d-1$, the non-trivial eigenvalues of A_k acting on $\mathrm{L}^2(\Gamma\backslash\mathcal{B})$ belong to $\mathfrak{S}_{d,k}$.

Let $\mathfrak{S}_d \subseteq \mathbb{C}^{d-1}$ denote the simultaneous spectrum of $(A_1, \ldots, A_{d-1})$ acting on the space $\mathrm{L}^2(\mathcal{B})$, namely, the set of $(\lambda_1, \ldots, \lambda_{d-1}) \in \mathbb{C}^{d-1}$ for which there exist a sequence of unit vectors $v_n \in \mathrm{L}^2(\mathcal{B})$ such that $\lim_{n\to\infty}(A_k v_n - \lambda_k v_n) = 0$ for every $k = 1, \ldots, d-1$.

Definition 2.7: The complex $\Gamma\backslash\mathcal{B}$ is Ramanujan if for every non-trivial simultaneous eigenvector $f \in \mathrm{L}^2(\Gamma\backslash\mathcal{B})$ of the A_k, the eigenvalues $(\lambda_1, \ldots, \lambda_{d-1})$ belong to $\mathfrak{S}_d$.

Since the A_k commute, every eigenvalue of A_k can be obtained by a simultaneous eigenvector. Hence, a Ramanujan complex is pseudo-Ramanujan. On the other hand, $\mathfrak{S}_d$ is not the Cartesian product of the $\mathfrak{S}_{d,k}$. For example, inverting the direction of edges in $\mathcal{B}$ carries A_k to A_{d-k}, so the operators A_k and A_{d-k} are adjoint to each other. In particular, for every $(\lambda_1, \ldots, \lambda_d) \in \mathfrak{S}_d$ we have that $\lambda_{d-k} = \bar{\lambda}_k$.

Remark 2.8: The spectrum $\mathfrak{S}_{d,k}$ of A_k on $\mathrm{L}^2(\mathcal{B})$ is equal to the projection of $\mathfrak{S}_d$ on the kth component.

Remark 2.9: If $d = 2$ or $d = 3$, then $\Gamma\backslash\mathcal{B}$ is Ramanujan iff it is pseudo-Ramanujan (indeed, for $d = 2$ the definitions coincide, and for $d = 3$, A_2 is the adjoint operator of A_1).

Let $S = \{(z_1, \ldots, z_d)\colon |{=}|_{z_i} 1, z_1 \cdots z_d = 1\}$ and $\sigma\colon S \to \mathbb{C}^{d-1}$ be the map defined by $(z_1, \ldots, z_d) \mapsto (\lambda_1, \ldots, \lambda_{d-1})$, where

$$\lambda_k = q^{k(d-k)/2}\sigma_k(z_1, \ldots, z_d).$$

The theorem below is proved in [Cw]. For completeness, we sketch the proof, following ideas from [CM] (where the result was proved for $d = 3$). First, we will need an easy calculus lemma:

LEMMA 2.10: *Let $(a_n), (b_n)$ be positive series. If* $\limsup(a_n b_n^{2+\epsilon}) \le 1$ *for every* $\epsilon > 0$ *and $\{a_n\}$ is bounded, then* $\limsup(a_n b_n^2) \le 1$.

Proof: Otherwise let $C > 1$ be an upper bound of $\{a_n\}$ and $p = \limsup(a_n b_n^2) > 1$, and take $\epsilon < 2\log(p)/\log(C)$. Then

$$p = \limsup(a_n^{\epsilon/(2+\epsilon)} a_n^{2/(2+\epsilon)} b_n^2) \le C^{\epsilon/(2+\epsilon)} \limsup(a_n b_n^{2+\epsilon})^{2/(2+\epsilon)} < C^{\epsilon/2},$$

a contradiction. ∎

THEOREM 2.11: *The simultaneous spectrum $\mathfrak{S}_d$ is equal to $\sigma(S)$.*

Proof: Let $\underline{z} = (z_1, \dots, z_d) \in S$. Then the corresponding character χ is unitary, and the irreducible subquotient generated by f_χ of the induced representation I_χ is tempered (Proposition 2.5). Thus, the corresponding spherical function ψ_χ is in $\mathrm{L}^{2+\epsilon}(G)$ for every $\epsilon > 0$. We already saw that ψ_χ is an eigenvector; however, it does not belong to $\mathrm{L}^2(G)$. In order to show that $\sigma(\underline{z})$ is in the spectrum, we twist ψ_χ to elements of $\mathrm{L}^2(G)$ which are "almost" eigenvectors, and their almost-eigenvalues converge to $\sigma(\underline{z})$. By Proposition 2.1 (and since ψ_χ is the spherical function associated to f_χ), $A_k\psi_\chi = \lambda_k\psi_\chi$ where $\lambda_k = q^{k(d-k)/2}\sigma_k(z_1, \dots, z_d)$. For every vertex $x \in \mathcal{B}^0 = G/K$, let $w(x)$ denote the distance (in $\mathcal{B}^1$) of x from the origin $[L_0]$. Recall [M, V.(2.2)] that every double coset in $K\backslash G/K$ has a unique representative of the form $\mathrm{diag}(\varpi^{\ell_1}, \dots, \varpi^{\ell_d})$ where $\ell_1 \geq \cdots \geq \ell_{d-1} \geq \ell_d = 0$; we call this representative of KgK the **type** of gK, and note that the number of vertices of this type is $\mu(KgK)$. Its distance from $[L_0]$ is equal to ℓ_1, so there are $\binom{n+d-2}{d-2} < (n+d)^d$ types of distance n. For $\delta > 0$, define a function ψ_χ^δ on $\mathcal{B}^0$ by $\psi_\chi^\delta(x) = (1-\delta)^{w(x)}\psi_\chi(x)$. For each n, let g_n denote the type of the vertex of distance n for which $\mu(Kg_nK)|\psi_\chi(g_nK)|^2$ is maximal. To see that $\psi_\chi^\delta \in \mathrm{L}^2(\mathcal{B}^0)$, compute that

$$\begin{aligned}\sum_{x\in\mathcal{B}^0}(1-\delta)^{2w(x)}|\psi_\chi(x)|^2 &= \sum_{n=0}^{\infty}(1-\delta)^{2n}\sum_{w(x)=n}|\psi_\chi(x)|^2\\ &\leq \sum_{n=0}^{\infty}(1-\delta)^{2n}(n+d)^d\mu(Kg_nK)|\psi_\chi(g_nK)|^2,\end{aligned}$$

and the convergence follows from the root test once we show that $\limsup\,(\mu(Kg_nK)|\psi_\chi(g_nK)|^2)^{1/n} \leq 1$. But since $\psi_\chi \in \mathrm{L}^{2+\epsilon}(\mathcal{B}^0)$ for every $\epsilon > 0$, we have

$$\limsup\,(\mu(Kg_nK)|\psi_\chi(g_nK)|^{2+\epsilon})^{1/n} \leq 1,$$

and the result follows from $\mu(Kg_nK)^{1/n} \leq q^d$ by the lemma. By the definition of A_k, $A_k\psi_\chi^\delta(x)$ is a sum of $\psi_\chi^\delta(y)$ for neighbors y of x, and the distance of neighbors satisfies $|w(y) - w(x)| \leq 1$. Since $A_k\psi_\chi - \lambda_k\psi_\chi = 0$, it follows that for some constant c, $||A_k\psi_\chi^\delta - \lambda_k\psi_\chi^\delta|| \leq c\delta||\psi_\chi^\delta||$ for every $\frac{1}{2} > \delta > 0$, showing that $(\lambda_1, \dots, \lambda_{d-1}) \in \mathfrak{S}_d$. Now let $(\lambda_1, \dots, \lambda_{d-1}) \in \mathfrak{S}_d$, and let $z_1, \dots, z_d \in \mathbb{C}$ be numbers satisfying $q^{k(d-k)/2}\sigma_k(z_1, \dots, z_d) = \lambda_k$, with the added property that $z_1 \cdots z_d = 1$ (the z_i are unique up to order). We need to show that $(z_1, \dots, z_d) \in S$, implying $(\lambda_1, \dots, \lambda_{d-1}) \in \sigma(S)$. Let $v_n \in \mathrm{L}^2(\mathcal{B}^0)$

be unit vectors such that $A_k v_n - \lambda_k v_n \to 0$, for all k, and define a homomorphism $\omega\colon \mathrm{H}(G,K) \to \mathbb{C}$ by $||Av_n - \omega(A)v_n|| \to 0$ (here we use the fact that the A_k are bounded and generate $\mathrm{H}(G,K)$). Then ω is continuous in the norm of the operators on $\mathrm{L}^2(\mathcal{B}^0)$, and in particular $|\omega(A)| \leq ||A||$ for every $A \in \mathrm{H}(G,K)$ (otherwise take ϵ such that $|\omega(A)| > ||A|| + \epsilon$, then $(\frac{A}{||A||+\epsilon})^n$ converges to zero but $\omega((\frac{A}{||A||+\epsilon})^n)$ does not). For $\ell \geq 1$, let $H_\ell \in \mathrm{H}(G,K)$ be the characteristic function of $K\operatorname{diag}(\varpi^{d\ell}, 1, \dots, 1)K$. We show that while $\omega(H_\ell)$ is a certain combination of $z_r^{-d\ell}$, the bound $||H_\ell||$ is polynomial in ℓ, thus implying that $|z_r| \geq 1$ for every r. The vector ψ_1 associated to the trivial character $\chi = 1$ is strictly positive (since $f_1(x) > 0$ for every $x \in G/K$ and $\psi_1(x) = \int_K f_1(kx)dk$), so if $H_\ell \psi_1 = b\psi_1$ and $H_\ell^* \psi_1 = b'\psi_1$, we have $||H_\ell|| \leq \sqrt{bb'}$ by Schur's criterion [P, p. 102]. Let $p = ((d-1)\ell, -\ell, \dots, -\ell)$. From [M, (3.5)] and [M, (3.3)], and using the limit

$$\lim_{(x_1,\dots,x_d)\to(1,\dots,1)} \sum_{k=1}^{d} \frac{x_k^m}{\prod_{i\neq k}(x_k - x_i)} = \binom{m}{d-1}, \tag{2.8}$$

we obtain $bb' = (1-q^{-1})^{2(d-1)}\binom{d\ell}{d-1}\binom{d(\ell+1)-2}{d-1}q^{d(d-1)\ell} < (d\ell)^{2d}q^{d(d-1)\ell}$, so $||H_\ell|| < (d\ell)^d q^{d(d-1)\ell/2}$. (Note that the action of $\mathrm{H}(G,K)$ on the spherical functions in [M] is via the multiplication of the Hecke algebra, unlike ours; see Equation (2.2).) In a similar manner, $\omega(H_\ell)$ is equal to $\widehat{c_{-p}}(\omega_s)$ of [M, (3.3)], and has the form $q^{d(d-1)\ell/2}\sum_{r=1}^{d} \alpha_r z_r^{-d\ell}$ where

$$\alpha_r = \prod_{i\neq r} \frac{z_i - q^{-1}z_r}{z_i - z_r}$$

if all the z_i are different (see [M, III.(2.2)]). From the continuity of ω we proved

$$\sum_{r=1}^{d} \alpha_r z_r^{-d\ell} \leq C\ell^d$$

for some constant C and every ℓ. Order the z_i by absolute value, so that $|z_1| \leq \cdots \leq |z_d|$. Then $\alpha_1 \neq 0$, and from the last bound it follows that $|z_1| \geq 1$; but $z_1 \cdots z_d = 1$, so $(z_1, \dots, z_d) \in S$. If the z_i are not assumed to be different, one computes the coefficients of the $z_i^{-\ell d}$ by Equation (2.8), and the same arguments apply. ∎

The sets $\mathfrak{S}_{d,k}$ are explicitly described in [CS]: $\mathfrak{S}_{d,k}$ is the simply connected domain with boundary the complex curve

$$\{q^{k(d-k)/2}\sigma_k(e^{i\theta}, \dots, e^{i\theta}, e^{-(d-1)i\theta})\colon \theta \in [0, 2\pi]\}$$

where $i = \sqrt{-1}$. Notice that the equations $\lambda_k = q^{k(d-k)/2}\sigma_k(z_1, \dots, z_d)$ always have a solution, but unless $(\lambda_1, \dots, \lambda_d) \in \mathfrak{S}_d$, the z_i do not have to be unitary—even if each $\lambda_k \in \mathfrak{S}_{d,k}$. In terms of characters, Theorem 2.11 implies that the eigenvalues corresponding to f_χ (see Proposition 2.1) are in the simultaneous spectrum of $(A_1, \dots, A_{d-1})$ acting on $\mathrm{L}^2(\mathcal{B})$ iff χ is unitary. This can be used to give a representation theoretic definition of being Ramanujan, as in Proposition 1.5.

Proof of Proposition 1.5: Assume every irreducible spherical infinite-dimensional sub-representation of $\mathrm{L}^2(\Gamma\backslash G(F))$ is tempered. As Γ is cocompact, $\mathrm{L}^2(\Gamma\backslash G(F))$ is a direct sum of irreducible representations. Let $f \in \mathrm{L}^2(\Gamma\backslash G(F)/K)$ be a non-trivial simultaneous eigenvector of the A_k, with $A_k f = \lambda_k f$. By Proposition 2.4, the λ_k are determined by some unramified character χ. Consider f as a K-fixed vector in $\mathrm{L}^2(\Gamma\backslash G(F))$. Since the only finite-dimensional representations of $G(F)$ are the trivial ones, the representation $\langle fG(F)\rangle$ is infinite-dimensional. Let V be an irreducible quotient of $\langle fG(F)\rangle$ in which $f \neq 0$; then V is an irreducible infinite-dimensional spherical sub-representation of $\mathrm{L}^2(\Gamma\backslash G(F))$, so by assumption V is tempered. It then follows from Proposition 2.5 that χ is unitary, and so $(\lambda_1, \dots, \lambda_{d-1}) \in \mathfrak{S}_d$. In the other direction, let V be an irreducible spherical infinite-dimensional sub-representation of $\mathrm{L}^2(\Gamma\backslash G(F))$; then its unique K-fixed vector f is a simultaneous eigenvector of the A_k, where $A_k f = \lambda_k f$. By assumption $(\lambda_1, \dots, \lambda_d) \in \mathfrak{S}_d$. The eigenvalues induce a homomorphism $\omega = \omega_\chi$ for some unitary character χ, and by Proposition 2.3, V is isomorphic to a subquotient of I_χ. Consequently, V is tempered. ∎

2.4. BOUNDS ON THE SPECTRUM OF A_k ON $\mathrm{L}^2(\Gamma\backslash\mathcal{B})$.

In the previous subsection we computed the spectrum of $(A_1, \dots, A_{d-1})$ in their action on $\mathrm{L}^2(\mathcal{B})$. For a discrete subgroup $\Gamma \leq G$, let $\mathrm{spec}_{\Gamma\backslash\mathcal{B}}(A_1, \dots, A_{d-1})$ denote the spectrum of these operators in their action on $\mathrm{L}^2(\Gamma\backslash\mathcal{B})$ (which is a finite set). In this subsection we apply the classification of unitary representations of $\mathrm{GL}_d(F)$ to give an upper bound on $\mathrm{spec}_{\Gamma\backslash\mathcal{B}}(A_1, \dots, A_{d-1})$ (which is independent of Γ). In addition we state an Alon–Boppana type theorem, due to W. Li, that for suitable families of quotients $\{\Gamma_i\backslash\mathcal{B}\}$ of $\mathcal{B}$, $\overline{\bigcup \mathrm{spec}_{\Gamma_i\backslash\mathcal{B}}(A_1, \dots, A_{d-1})} \supseteq \mathfrak{S}_d$. Let $f \in \mathrm{L}^2(\Gamma\backslash G/K)$ be a simultaneous eigenvector of the A_k. Lift f to $\mathrm{L}^2(\Gamma\backslash G)$, and recall that the representation $\langle fG\rangle$ is unitary (since the action of G on $\mathrm{L}^2(\Gamma\backslash G)$ is unitary) and spherical (since f is K-fixed). The unitary spherical representations were described by Tadić [T], as part of the classification of all the unitary representations of $\mathrm{GL}_d(F)$. Such a spherical representation is induced by a character $\chi = \chi_1 \oplus \cdots \oplus \chi_d$, where the χ_i are combined into blocks. For the Satake parameters $(z_{i_1}, \dots, z_{i_s})$ of each block $\chi_{i_1}, \dots, \chi_{i_s}$, one of the following three options holds: either $s = 1$ and $z_{i_1} \in S^1 = \{z \in \mathbb{C}: |z| = 1\}$; $(z_{i_1}, \dots, z_{i_s})$ is of the form

$$(q^{(s-1)/2}z, \dots, q^{(1-s)/2}z)$$

for $z \in S^1$; or (if $s = 2s'$ is even) it is of the form

$$(q^{(s'-1)/2+\alpha}z, \dots, q^{(1-s')/2+\alpha}z, q^{(s'-1)/2-\alpha}z, \dots, q^{(1-s')/2-\alpha}z)$$

for $z \in S^1$ and $0 < \alpha < 1/2$. This set of possible parameters $(z_1, \dots, z_d)$ determines the eigenvalues $(\lambda_1, \dots, \lambda_{d-1})$ via Proposition 2.1. In particular, if $d \geq 3$, we obtain for the non-trivial eigenvalues

$$|\lambda_k| \leq q^{k(d-k)/2} \cdot \sigma_k(q^{(d-2)/2}, \dots, q^{(2-d)/2}, 1) \approx q^{k(d-k-\frac{1}{2})}$$

for every $k \leq d/2$ (and $\lambda_{d-k} = \bar{\lambda}_k$). One can see that if $d \geq 3$ then $|\lambda_k| < \left[{d \atop k}\right]_q$ for every non-trivial unitary representation (where $\left[{d \atop k}\right]_q$ denotes the number of subspaces of dimension k in $\mathbb{F}_q^d$, which is the number of neighbors of color k of each vertex). In particular, the non-trivial eigenvalues of $A = A_1 + \cdots + A_{d-1}$ are bounded away from the trivial one. This demonstrates the fact that $\mathrm{PGL}_d(F)$ has Kazhdan property (T) and the quotient graphs $\Gamma\backslash\mathcal{B}^1$ are expanders for every Γ [Lu2]. On the other hand, for $d = 2$ the eigenvalues $q^{1/2}\sigma_1(q^\alpha, q^{-\alpha})$ approach the degree $q + 1$ when $\alpha \to 1/2$, in accordance with the fact that $\mathrm{PGL}_2(F)$ does not have property (T). For the lower bound, we quote

THEOREM 2.12 ([Li, Thm. H]): *Let X_i be a family of finite quotients of $\mathcal{B}$ with unbounded injective radius. Then* $\overline{\bigcup \mathrm{spec}_{X_i}(A_1, \dots, A_{d-1})} \supseteq \mathfrak{S}_d$.

This also follows from a multi-dimensional version of [GZ].

2.5. SUPER-CUSPIDAL AND SQUARE-INTEGRABLE REPRESENTATIONS. Let G denote the group $\mathrm{GL}_d(F)$ or $\mathrm{PGL}_d(F)$, and $Z = \mathrm{Z}(G)$ its center. Let $\rho\colon G \to \mathrm{End}(V)$ be a unitary representation. Recall that the **matrix coefficients** of ρ are the functions $\rho_{v,w}\colon g \mapsto \langle \rho(g)v, w\rangle$ where $v, w \in V$. A unitary representation of G is called **super-cuspidal**, if its matrix coefficients are compactly supported modulo the center. Notice that the irreducible representations of $\mathrm{GL}_1(F)$ are all super-cuspidal (as the group equals its center). We say that a unitary representation ρ is **square-integrable**, if $\rho_{v,w} \in \mathrm{L}^2(G/Z)$ for every $v, w \in V$.

A representation is square-integrable iff it is isomorphic to a sub-representation of $\mathrm{L}^2(G)$ [Kn, Prop. 9.6]. Note that super-cuspidal representations are square-integrable, and square-integrable representations are tempered. Let $s|d$ be any divisor, and let $P_s(F)$ denote the parabolic subgroup corresponding to the partition of d into s equal parts. For a representation ψ of $\mathrm{GL}_{[}d/s](F)$, we denote

$$(2.9)\quad M_s(\psi) = \mathrm{Ind}_{P_s(F)}^{\mathrm{GL}_d(F)}(|\det|_F^{(1-s)/2}\psi \oplus |\det|_F^{(3-s)/2}\psi \oplus \cdots \oplus |\det|_F^{(s-1)/2}\psi).$$

The unique irreducible sub-representation of $M_s(|\det|_F^{(s-1)/2}\psi)$ will be denoted by $C_s(\psi)$. It is known that if ψ is irreducible and super-cuspidal, then the induced representation $M_s(|\det|_F^{(s-1)/2}\psi)$ has precisely 2^{s-1} irreducible subquotients, two of which (if $s > 1$) are unitary [HT, p. 32] (notice that $M_1(\psi) = \psi$). These subquotients are $\mathrm{C}_s(\psi)$, and a certain irreducible quotient, called the **generalized Steinberg representation** (or sometimes "special representation") and denoted by $\mathrm{Sp}_s(\psi)$.

PROPOSITION 2.13 ([HT, p. 32], [Z]): *For $s > 1$ and ψ an irreducible super-cuspidal representation of $\mathrm{GL}_{d/s}(F)$, $\mathrm{Sp}_s(\psi)$ is square-integrable, and $\mathrm{C}_s(\psi)$ is not tempered. Every square-integrable representation of $\mathrm{GL}_d(F)$ is either super-cuspidal, or of the form $\mathrm{Sp}_s(\psi)$ for a unique divisor s of d and a unique super-cuspidal representation ψ of $\mathrm{GL}_{d/s}(F)$.*

Remark 2.14: If $s > 1$, $\mathrm{C}_s(\psi)$ is not tempered for any unitary representation ψ.

Example 2.15: Let $\phi\colon F^\times \to \mathbb{C}^\times$ be a character, and $\psi = |\cdot|_F^{(1-d)/2}\phi$. Then $\mathrm{C}_d(\psi) = \phi \circ \det$, which is one-dimensional.

Proof: Let $B(F)$ denote the standard Borel subgroup of $\mathrm{GL}_d(F)$. By definition, $\mathrm{C}_d(\psi)$ is the unique irreducible sub-representation of

$$M_d(\phi) = \mathrm{Ind}_{B(F)}^{\mathrm{GL}_d(F)}(|\cdot|_F^{(1-d)/2}\phi \oplus \cdots \oplus |\cdot|_F^{(d-1)/2}\phi),$$

which is the unitary induction of $\Delta^{-1/2}\cdot(\phi\circ\det)$ to $\mathrm{GL}_d(F)$. In particular, this representation, when restricted to $B(F)$, contains the representation $\phi \circ \det$, which is thus a sub-representation of $M_d(\phi)$, so by definition $\mathrm{C}_d(\psi) = \phi \circ \det$. ∎

3. From local to global

3.1. THE GLOBAL FIELD. Let k be a global field, $\mathcal{V} = \{\nu\}$ its nonarchimedean discrete valuations, and $\mathcal{V}_\infty$ the Archimedean valuations. For $\nu \in \mathcal{V}$, k_ν is the completion, $\mathcal{O}_\nu = \{x\colon \nu(x) \geq 0\}$ the valuation ring of k_ν (which is the closed unit ball of k_ν and thus compact), and $P_\nu = \{x\colon \nu(x) > 0\}$ the valuation ideal. Note that the ring of ν-adic integers $k \cap \mathcal{O}_\nu$ of k is a local ring, with maximal ideal $k \cap P_\nu$. Fix a valuation $\nu_0 \in \mathcal{V}$, and set $F = k_{\nu_0}$. Consider the intersection

$$R_0 = \{x \in k\colon \forall(\nu \in \mathcal{V} - \{\nu_0\})\nu(x) \geq 0\} = \bigcap_{\nu \in \mathcal{V}-\{\nu_0\}} (k \cap \mathcal{O}_\nu).$$

Recall that the valuations of $k = \mathbb{F}_q(y)$ are all nonarchimedean. They are indexed by the prime polynomials of $\mathbb{F}_q[y]$ and $1/y$. For a prime p the valuation is $\nu_p(p^i f/g) = i$ when f and g are prime to p, and the valuation corresponding to $1/y$ is the minus degree valuation, defined by $\nu_{1/y}(f/g) = \deg(g) - \deg(f)$. If $\nu_0 = \nu_{1/y}$ then $R_0 = \mathbb{F}_q[y]$. For every $x \in k^\times$ we have that

$$\nu_{1/y}(x) + \sum_p \deg(p)\nu_p(x) = 0, \tag{3.1}$$

so $\nu_0(x) \leq 0$ for every $x \in R_0$. As a result R_0 is discrete in F. It also follows that $R_0 \cap \mathcal{O}_{\nu_0} = \mathbb{F}_q$. For every ν, choose a uniformizer $\varpi_\nu \in R_0$, so that $\nu(\varpi_\nu) = 1$. Then, the completion of k at ν is $\mathbb{F}_q((\varpi_\nu))$, and the local ring of integers is $\mathbb{F}_q[[\varpi_\nu]]$. Note that if $\nu_0 = \nu_{1/y}$, we can choose the uniformizers, ϖ_ν, to be the prime polynomials of $\mathbb{F}_q[y]$, and $\varpi_{\nu_0} = 1/y$. Let $\mathcal{I}$ denote the set of functions $\vec{\imath}\colon \mathcal{V} \to \mathbb{N} \cup \{0\}$, such that $i_\nu = 0$ for almost all ν and $i_{\nu_0} = 0$. The ideals of R_0 are indexed by functions $\vec{\imath} \in \mathcal{I}$, in the following way: For $\vec{\imath} \in \mathcal{I}$, we define

$$I_{\vec{\imath}} = \{x \in R_0\colon \nu(x) \geq i_\nu\} = \bigcap_{\nu \in \mathcal{V}-\{\nu_0\}} (k \cap P_\nu^{i_\nu}), \tag{3.2}$$

where we make the notational convention that ${P_\nu}^0 = \mathcal{O}_\nu$. From our choice of the uniformizers, it follows that if $\nu_0 = \nu_{1/y}$, then $I_{\vec{\imath}}$ is the (principal) ideal generated by $\prod_{\nu \neq \nu_0} \varpi_\nu^{i_\nu}$. Notice that for the zero vector $\vec{\imath} = 0$ ($i_\nu = 0$ for all ν), we obtain the trivial ideal $I_0 = R_0$. Let $\times k_\nu$ be the direct product of the fields k_ν over all the valuations $\nu \in \mathcal{V} \cup \mathcal{V}_\infty$ of k, and recall that the ring $\mathbb{A}$ of adèles over k is defined to be the restricted product

$$\mathbb{A} = \{x = (x_\nu) \in \times k_\nu\colon \nu(x_\nu) \geq 0 \quad \text{for almost all } \nu\}. \tag{3.3}$$

The field k embeds in $\mathbb{A}$ diagonally. In a similar manner to the construction of R_0, we define

$$\tilde{R}_0 = \{(x_\nu) \in \mathbb{A}\colon \forall(\nu \in \mathcal{V} - \{\nu_0\})\nu(x_\nu) \geq 0\} \\ = F \times \prod_{\nu \in \mathcal{V}-\{\nu_0\}} \mathcal{O}_\nu \times \prod_{\nu \in \mathcal{V}_\infty} k_\nu. \tag{3.4}$$

The ideals of finite index of $\tilde{R}_0$ are again indexed by $\mathcal{I}$, and are of the form

$$\tilde{I}_{\vec{\imath}} = \{(x_\nu) \in \mathbb{A}\colon \forall(\nu \in \mathcal{V} - \{\nu_0\})\nu(x_\nu) \geq i_\nu\}, \tag{3.5}$$

and with respect to the diagonal embedding, we have $R_0 = k \cap \tilde{R}_0$ and $I_{\vec{\imath}} = k \cap \tilde{I}_{\vec{\imath}}$ for every $\vec{\imath} \in \mathcal{I}$. In fact, $\tilde{R}_0$ and $\tilde{I}_{\vec{\imath}}$ are the topological closures of R_0 and $I_{\vec{\imath}}$, respectively.

3.2. STRONG APPROXIMATION. Let $\mathcal{G}$ be a connected, simply connected, almost simple linear algebraic group, defined over k (e.g. SL_d), with a fixed embedding into GL_r for some r. For a subring R of a k-algebra A, we denote $\mathcal{G}(R) = \mathcal{G}(A) \cap \mathrm{GL}_r(R)$. For simplicity of notation (and as our applications are mainly for positive characteristic), we assume $\mathcal{G}(k_\nu)$ is compact for all Archimedean places ν. The diagonal embedding $k \hookrightarrow \mathbb{A}$, which is obviously discrete, induces a discrete embedding $\mathcal{G}(k) \hookrightarrow \mathcal{G}(\mathbb{A})$. Let T be the set of valuations θ such that $\mathcal{G}(k_\theta)$ is compact; this is a finite set [PR]. Fix a valuation $\nu_0 \in \mathcal{V} - T$, and let $F = k_{\nu_0}$ denote the completion with respect to this special valuation. $\mathcal{G}(k)$ is a lattice of finite co-volume in $\mathcal{G}(\mathbb{A})$, and moreover, if $T \neq \emptyset$, $\mathcal{G}(k)$ is a cocompact lattice [PR, Thm. 5.5].

THEOREM 3.1 (Strong Approximation [Pr], [PR]): *The product $\mathcal{G}(k)\mathcal{G}(F)$ is dense in $\mathcal{G}(\mathbb{A})$.*

So for every open subgroup U of $\mathcal{G}(\mathbb{A})$,

$$\mathcal{G}(k)\mathcal{G}(F)U = \mathcal{G}(\mathbb{A}). \tag{3.6}$$

COROLLARY 3.2: *Let $U \subseteq \mathcal{G}(\mathbb{A})$ be a compact subgroup such that $\mathcal{G}(F)U$ is open, and $\mathcal{G}(F) \cap U = 1$. Set $\Gamma_U = \mathcal{G}(k) \cap \mathcal{G}(F)U$. Then its projection to $\mathcal{G}(F)$ (which we will also denote by Γ_U) is discrete, and*

$$\mathcal{G}(k)\backslash\mathcal{G}(\mathbb{A})/U \cong \Gamma_U\backslash\mathcal{G}(F). \tag{3.7}$$

For example, if $U = \prod_{\nu \in \mathcal{V}-\{\nu_0\}} \mathcal{G}(\mathcal{O}_\nu) \times \prod_{\nu \in \mathcal{V}_\infty} \mathcal{G}(k_\nu)$, then $\Gamma = \Gamma_U$ is the arithmetic subgroup $\mathcal{G}(R_0)$. More generally, let $\vec{\imath} = (i_\nu) \in \mathcal{I}$ be a function corresponding to an ideal $\tilde{I}_{\bar{\imath}}$, and let

$$U_{\bar{\imath}} = \prod_{\nu \in \mathcal{V}-\{\nu_0\}} \mathcal{G}(\mathcal{O}_\nu, P_\nu^{i_\nu}) \times \prod_{\nu\in\mathcal{V}_\infty} \mathcal{G}(k_\nu)$$

where $\mathcal{G}(\mathcal{O}_\nu, P_\nu^{i_\nu}) = \operatorname{Ker}(\mathcal{G}(\mathcal{O}_\nu) \to \mathcal{G}(\mathcal{O}_\nu/P_\nu^{i_\nu}))$ is a congruence subgroup. Then $\mathcal{G}(F)U_{\bar{\imath}} = \mathcal{G}(\tilde{R}_0, \tilde{I}_{\bar{\imath}}) = \operatorname{Ker}(\mathcal{G}(\tilde{R}_0)) \to \mathcal{G}(\tilde{R}_0/\tilde{I}_{\bar{\imath}})$ is an open subgroup of $\mathcal{G}(\mathbb{A})$, and we set

$$(3.8) \qquad \Gamma_{\bar{\imath}} = \mathcal{G}(R_0, I_{\bar{\imath}}) = \mathcal{G}(k) \cap \mathcal{G}(F)U_{\bar{\imath}},$$

called the **principal congruence subgroup mod** $I_{\bar{\imath}}$ of $\mathcal{G}(R_0)$. Again, when $T \neq \emptyset$, this is a cocompact lattice in $\mathcal{G}(F)$.

3.3. AUTOMORPHIC REPRESENTATIONS. The group $\mathcal{G}(\mathbb{A})$ acts on the space $L^2(\mathcal{G}(k)\backslash\mathcal{G}(\mathbb{A}))$ by multiplication from the right. The sub-modules are called automorphic representations of $\mathcal{G}(\mathbb{A})$. The closed irreducible sub-modules are said to be discrete, or to belong to the discrete spectrum. Its complement is called the continuous spectrum. If $T \neq \emptyset$ then there is no continuous spectrum. Let $K_\nu = \mathcal{G}(\mathcal{O}_\nu)$ and recall that for every $(g_\nu) \in \mathcal{G}(\mathbb{A})$, $g_\nu \in K_\nu$ for almost all ν. Given irreducible representations $\pi_\nu\colon \mathcal{G}(k_\nu) \to \operatorname{End}(V_\nu)$, with all but finitely many being K_ν-spherical, one defines the restricted tensor product $\pi = \otimes\pi_\nu\colon \mathcal{G}(\mathbb{A}) \to \operatorname{End}(\otimes' V_\nu)$ [Bu].

A fundamental theorem [Bu, Thm. 3.3.3] states that any irreducible automorphic representation of $\mathcal{G}(\mathbb{A})$ is isomorphic to such a restricted tensor product. The representations π_ν in $\pi = \otimes\pi_\nu$ are called the (local) components of π, and since π is irreducible, they are also irreducible. Moreover, π is admissible iff all its components are.

PROPOSITION 3.3: *Assume that $\mathcal{G}(k_\nu)$ is non-compact. If the component π_ν of an irreducible automorphic representation π of $\mathcal{G}(\mathbb{A})$ is trivial, then π is trivial.*

Proof: Let $\pi = \otimes\pi_\nu$ be an automorphic representation acting on $V \leq L^2(\mathcal{G}(k)\backslash\mathcal{G}(\mathbb{A}))$, where π_ν is trivial. For $f \in V$ (assumed to be K-finite, see [Bu, Thm. 3.3.4]), f is $\mathcal{G}(k_\nu)$-invariant from the right and $\mathcal{G}(k)$-invariant from the left. However, by Strong Approximation, $\mathcal{G}(k_\nu)\mathcal{G}(k)$ is dense, so f must be constant everywhere, making π trivial. ∎

Recall that an irreducible representation is $U_{\bar{\imath}}$-spherical if it has a $U_{\bar{\imath}}$-fixed vector. We assume $\mathcal{G}(F)$ is non-compact where $F = k_{\nu_0}$.

PROPOSITION 3.4: *Let π be an irreducible, $U_{\vec{\imath}}$-spherical automorphic representation of $\mathcal{G}(\mathbb{A})$. Then π_{ν_0} is a sub-representation of $L^2(\Gamma_{\vec{\imath}}\backslash\mathcal{G}(F))$. Conversely, if $\rho \leq L^2(\Gamma_{\vec{\imath}}\backslash\mathcal{G}(F))$ is irreducible, then there exists an irreducible $U_{\vec{\imath}}$-spherical automorphic representation π of $\mathcal{G}(\mathbb{A})$ such that π_{ν_0} is isomorphic to ρ.*

The second assertion is seen by lifting a function $f \in V_\rho$ (where V_ρ is the representation space) from $\Gamma_{\vec{\imath}}\backslash\mathcal{G}(F)$ to $\mathcal{G}(k)\backslash\mathcal{G}(\mathbb{A})$ using Corollary 3.2, and taking π to be an irreducible quotient of the (right) $\mathcal{G}(\mathbb{A})$-module generated by f.

3.4. THE CONDUCTOR. For a representation ρ of $\mathcal{G}(k_\nu)$, the **conductor** of ρ, $\mathrm{cond}(\rho) = i$, is defined to be the minimal $i \geq 0$, for which there is a $\mathcal{G}(\mathcal{O}_\nu, P_\nu^i)$-fixed vector in V (such an i exists since the representation is admissible). In particular, $\mathrm{cond}(\rho) = 0$ iff ρ is spherical. Now let π be an irreducible automorphic representation of $\mathcal{G}(\mathbb{A})$. Since almost all the local components are spherical, $\mathrm{cond}(\pi_\nu) = 0$ for almost every ν. We thus let $\mathrm{cond}(\pi)$ be the function $\vec{\imath}\colon \mathcal{V} \to \mathbb{N} \cup \{0\}$ defined by $i_\nu = \mathrm{cond}(\pi_\nu)$ (note that $\vec{\imath}$ is not in $\mathcal{I}$ in general, as we do not assume $i_{\nu_0} = 0$).

Remark 3.5: Let $\vec{\imath} = \mathrm{cond}(\pi)$. Then $H = \mathcal{G}(\mathcal{O}_{\nu_0}, P_{\nu_0}^{i_{\nu_0}})U_{\vec{\imath}}$ is the maximal principal congruence subgroup for which π has an H-fixed vector.

The results of this section will be used later for non-simply connected cases, which requires some minor modifications. Let G be a connected, almost simple algebraic group over k. Let $\Pi\colon \mathcal{G} \to G$ be its simply connected cover. Then $\Pi(\mathcal{G}(\mathbb{A})) \trianglelefteq G(\mathbb{A})$ and the quotient is abelian (of finite exponent). In this situation, Proposition 3.3 becomes

PROPOSITION 3.6: *Assume that $G(k_\nu)$ is non-compact. If the component π_ν of an irreducible automorphic representation π of $G(\mathbb{A})$ is one dimensional, then π is one dimensional.*

4. Global automorphic representations

Let G be an almost simple, connected algebraic group defined over k, where k is a global field of arbitrary characteristic. The discrete spectrum of automorphic representations is composed of cuspidal and residual representations. The cuspidal representation space is comprised of functions $f \in L^2(G(k)\backslash G(\mathbb{A}))$ which satisfy $\int_{N(k)\backslash N(\mathbb{A})} f(ng)dn = 0$ for every $g \in G(\mathbb{A})$ and for every N, where N is a unipotent radical of a parabolic subgroup of G. Since the cuspidal condition involves integration from the left, and the action is by right translation, this is a sub-representation space. The other discrete irreducible representations are called residual. Recently, L. Lafforgue has proved the following version of the Ramanujan conjecture:

THEOREM 4.1 ([L], [R]): *Assume k is of positive characteristic and $G = \mathrm{GL}_d$. Let $\pi = \otimes\pi_\nu$ be an irreducible, cuspidal representation with finite central character. For all ν, if π_ν is spherical then π_ν is tempered.*

4.1. THE RESIDUAL SPECTRUM. All the one-dimensional representations are residual, and when $G = \mathrm{GL}_d$ and d is prime these are the only residual representations. If d is not a prime, the other residual representations can be described in terms of the cuspidal representations of smaller rank, as follows: An element $(a_\nu) \in \mathbb{A}$ is invertible only if for almost all ν, $a_\nu \in \mathcal{O}_\nu^\times$. We can thus define an absolute value on $\mathbb{A}^\times$ by $|(a_\nu)|_{\mathbb{A}} = \prod |a_\nu|_{k_\nu}$, which is a finite product. The modular function for parabolic subgroups of $\mathrm{GL}_(\mathbb{A})$ is defined as in the local case (see Equation (2.3), with $|\det(a)|_{\mathbb{A}}$ for each block), and likewise we have a unitary induction from parabolic subgroups, with similar properties to the local case. Let $s > 1$ be a divisor of d, and let π be any cuspidal automorphic representation of $\mathrm{GL}_{d/s}(\mathbb{A})$. The representation

$$T_s(\pi) = \mathrm{Ind}_{P_s(\mathbb{A})}^{\mathrm{GL}_(\mathbb{A})} (|\det|_{\mathbb{A}}^{(1-s)/2}\pi \oplus |\det|_{\mathbb{A}}^{(3-s)/2}\pi \oplus \cdots \oplus |\det|_{\mathbb{A}}^{(s-1)/2}\pi) \tag{4.1}$$

has a unique irreducible sub-representation $J(T_s(\pi))$ (here $P_s(\mathbb{A})$ is the parabolic subgroup of $\mathrm{GL}_d(\mathbb{A})$ associated to the decomposition into s blocks of size d/s).

THEOREM 4.2 ([MW]): *The residual spectrum of $\mathrm{L}^2(\mathrm{GL}_d(k)\backslash\mathrm{GL}_d(\mathbb{A}))$ consists of the representations $J(T_s(\pi))$ for proper divisors $s|d$ and π a cuspidal representation of $\mathrm{GL}_{d/s}(\mathbb{A})$.*

Comparing Equations (2.9) and (4.1), the local ν-component of $J(T_s(\pi))$ is seen to be the (unique) irreducible sub-representation of $M_s(\pi_\nu)$, namely $\mathrm{C}(\pi_\nu)$ which was defined in Subsection 2.5. From Remark 2.14 we then obtain

COROLLARY 4.3: (a) *Every local component of a residual representation is non-tempered.* (b) *If π is an irreducible automorphic representation of GL_d where one of its local components is tempered, then π is cuspidal (and in positive characteristic, all of its spherical components are tempered by Theorem 4.1).*

4.2. THE JACQUET–LANGLANDS CORRESPONDENCE. Let D be a division algebra of degree d over k, and let $D_\nu = D\otimes_k k_\nu$. Then by the Albert–Brauer–Hasse–Noether theorem, $D_\nu \cong \mathrm{M}_d(k_\nu)$ for almost every completion k_ν. Let $G' = D^\times$, which is a form of inner type of $G = \mathrm{GL}_d$. Let T denote the (finite) set of valuations θ such that $D\otimes k_\theta$ is not split. We assume that for every $\theta \in T$, $D\otimes k_\theta$ is a division algebra. There is an injective correspondence, called the local Jacquet–Langlands correspondence, which maps every irreducible, unitary representation ρ' of $G'(k_\theta)$ $(\theta \in T)$ to an irreducible, unitary square-integrable (modulo the center) representation $\rho = \mathrm{JL}_\theta(\rho')$ of $G(k_\theta)$ (see [Ro] or [HT, p. 29] for details). If ϕ is a character of $k_\theta^\times$, then [HT, p. 32]

$$\mathrm{JL}_\theta(\phi \circ \det) = \mathrm{Sp}_d(|\cdot|_{k_\theta}^{(1-d)/2}\phi) \tag{4.2}$$

where Sp_d is defined in Subsection 2.5. Recall by Example 2.15 that $\mathrm{C}_d(|\cdot|_{k_\theta}^{(1-d)/2}\phi)$ is a one-dimensional representation. The global Jacquet–Langlands correspondence maps an irreducible automorphic representation π' of $G'(\mathbb{A})$ to an irreducible automorphic representation $\pi = \mathrm{JL}(\pi')$ of $G(\mathbb{A})$ which occurs in the discrete spectrum (see [HT, p. 195]). If $\nu \notin T$, then

$$\mathrm{JL}(\pi')_\nu \cong \pi'_\nu.$$

Note that the restrictions of $\mathrm{cond}\,\pi$ and $\mathrm{cond}\,\pi'$ to $\mathcal{V} - T$ are equal. The situation in the other local components is as follows: let $\theta \in T$, and consider the component π'_θ of π'. The local Jacquet–Langlands correspondence maps π'_θ to an irreducible square-integrable representation $\mathrm{JL}_\theta(\pi'_\theta)$ of $G(k_\theta)$, which is by Proposition 2.13 a generalized Steinberg representation, of the form $\mathrm{Sp}_s(\psi)$ for some divisor $s|d$ and super-cuspidal representation ψ of $\mathrm{GL}_{d/s}(k_\theta)$. Then $\mathrm{JL}(\pi')_\theta$ is isomorphic to either $\mathrm{Sp}_s(\psi)$ or $\mathrm{C}_s(\psi)$.

THEOREM 4.4 ([HT, p. 196]): *The image of* JL *(for a fixed D) is the set of irreducible automorphic representations π of $\mathrm{GL}_d(\mathbb{A})$ such that π occurs in the discrete spectrum and for every $\theta \in T$ there is a positive integer $s_\theta|d$ and an irreducible super-cuspidal representation ψ_θ of $\mathrm{GL}_{d/s_\theta}(k_\theta)$ such that π_θ is isomorphic to either $\mathrm{Sp}_{s_\theta}(\psi_\theta)$ or $\mathrm{C}_{s_\theta}(\psi_\theta)$.*

Throughout the book [HT], the authors assume characteristic zero. However, see Remark 1.6.

5. Proofs of Theorems 1.2 and 1.3

Let k be a global field of prime characteristic, D a division algebra of degree d over k, G' the algebraic group $D^\times/Z^\times$ where Z is the center, and $G = \mathrm{PGL}_d$.

Let T denote the set of ramified primes, namely valuations θ for which $D_\theta = D \otimes k_\theta$ is non-split. We again assume that for such primes, D_θ is a division algebra. It follows that $G'(k_\theta)$ is compact for $\theta \in T$. The valuation θ extends uniquely to a valuation of D_θ, and we let $\mathcal{O}_{D_\theta}$ denote the ring of integers there.

The group $G'(\mathcal{O}_\theta)$ depends on the specific embedding $G'(k)\hookrightarrow \mathrm{GL}_r(k)$, namely, $G'(k_\theta)$ is the subgroup of $\mathrm{GL}_r(k_\theta)$ defined by the equations defining $G'(k)$, and $G'(\mathcal{O}_\theta) = G'(k_\theta) \cap \mathrm{GL}_r(\mathcal{O}_\theta)$. For most of our applications the precise embedding is irrelevant ($G'(\mathcal{O}_\theta)$ is well defined up to commensurability anyway). However, for Theorem 1.3(b), we need the embedding to satisfy

$$G'(\mathcal{O}_\theta) \supseteq k_\theta^\times \mathcal{O}_{D_\theta}^\times / k_\theta^\times, \tag{5.1}$$

where both groups are viewed as subgroups of $G'(k_\theta) = (D \otimes_k k_\theta)^\times / k_\theta^\times$, which is embedded in $\mathrm{GL}_r(k_\theta)$ for some r.

This condition is in fact satisfied by a natural embedding. Let E be a cyclic extension of dimension d over k, which is unramified at every $\theta \in T$ (the existence of E is guaranteed by Grunwald's theorem for function fields [AT, Chap. 10]). From the Albert–Brauer–Hasse–Noether theorem it follows that E is a splitting field of D, making D a cyclic division algebra. Moreover, there is an element $z \in D$ such that $D = E[z]$ and conjugation by z is an automorphism of E, generating $\mathrm{Gal}(E/k)$.

Let $e_1, \ldots, e_d$ be an integral basis of E/k (with respect to every $\theta \in T$). Then, for every valuation $\theta \in T$, $\mathcal{O}_{E_\theta} = \sum \mathcal{O}_\theta e_i$, where $E_\theta = E \otimes_k k_\theta$ and $\mathcal{O}_{E_\theta}$ is its ring of integers. Now, z can be chosen so that $\mathcal{O}_{D_\theta} = \mathcal{O}_{E_\theta}[z] = \sum_{i,j} \mathcal{O}_\theta e_i z^j$. The left regular representation of D via the basis $\{e_i z^j\}$ defines an embedding $D^\times \to \mathrm{GL}_{d^2}(k)$ which sends $\mathcal{O}_{D_\theta}^\times$ into $\mathrm{GL}_{d^2}(\mathcal{O}_\theta)$ (and central elements to scalar matrices). Composing this with the adjoint representation of $\mathrm{PGL}_{d^2}(k)$ (into $\mathrm{GL}_{d^4}(k)$), we obtain an embedding of $G'(k) = D^\times / k^\times$ which satisfies (5.1).

Lemma 5.1: *For $\theta \in T$, $\mathcal{O}_{D_\theta}^\times$ is normal in $D_\theta^\times$, and $D_\theta^\times / k_\theta^\times \mathcal{O}_{D_\theta}^\times \cong \mathbb{Z}/d$.*

Proof: The uniformizer ϖ of k_θ is a uniformizer for E_θ as well, and (by choosing the generator $\sigma \in \mathrm{Gal}(E_\theta/k_\theta)$ appropriately) we may assume $D_\theta = E_\theta[z]$ where $z^d = \varpi$ and conjugation by z induces σ. Since z normalizes $\mathcal{O}_{D_\theta}^\times$, this is a normal subgroup of $D_\theta^\times$.

The elements of value zero in $\mathcal{O}_{D_\theta}$ are invertible there, so every element of $D_\theta^\times$ is of the form cz^i for some $c \in \mathcal{O}_{D_\theta}^\times$ and an integer i. Such an element is equivalent to z^i in $D_\theta^\times / k_\theta^\times \mathcal{O}_{D_\theta}^\times$, and $z^d = \varpi \in k_\theta^\times$. Finally, z^i induces a non-trivial automorphism on E for every $0 < i < d$, so the order of z modulo the center is equal to d. ∎

By our assumption (5.1), the lemma implies that $G'(k_\theta)/G'(\mathcal{O}_\theta)$ is a quotient of $\mathbb{Z}/d$.

For $\vec{\imath} \in \mathcal{I}$ set $\Gamma_{\vec{\imath}} = G'(R_0, I_{\vec{\imath}})$, as in Equation (3.8). For $\vec{\imath}, \vec{\jmath} \in \mathcal{I}$, we say that $\vec{\imath} \leq_T \vec{\jmath}$ if $i_\nu \leq j_\nu$ for every $\nu \in \mathcal{V} - T$.

PROPOSITION 5.2: *Let $\vec{\imath} \in \mathcal{I}$. The complex $\Gamma_{\vec{\imath}} \backslash \mathcal{B}$ is Ramanujan iff every spherical infinite-dimensional ν_0-component of an irreducible automorphic discrete representation π' of $G'(\mathbb{A})$ with* $\text{cond}(\pi') \leq_T \vec{\imath}$ *is tempered.*

Proof: This follows immediately from Propositions 1.5 and 3.4 (and Remark 3.5). ∎

We can now prove the theorems stated in the Introduction.

Proof of Theorems 1.2 and 1.3(a): Write the given ideal of R_0 as $I = I_{\vec{\imath}}$ for $\vec{\imath} \in \mathcal{I}$ (see Equation (3.2)). Let π' be an irreducible discrete automorphic representation of $G'(\mathbb{A})$ with $\text{cond}\,\pi' \leq_T \vec{\imath}$, and assume $\rho = \pi'_{\nu_0}$ is spherical and infinite-dimensional. By Proposition 5.2, $\Gamma(I)\backslash\mathcal{B}$ is Ramanujan iff in all such cases π'_{ν_0} is tempered. By the Jacquet–Langlands correspondence, there is an irreducible automorphic sub-representation π of $\mathrm{L}^2(G(k)\backslash G(\mathbb{A}))$ such that $\pi_\nu = \pi'_\nu$ for every $\nu \notin T$. In particular, $\pi_{\nu_0} = \pi'_{\nu_0}$. Assume d is prime; then all the infinite-dimensional automorphic representations of $G(\mathbb{A})$ are cuspidal, so π is cuspidal. By Lafforgue's Theorem 4.1, the components of a cuspidal representation are tempered. Therefore, $\rho = \pi_{v_0}$ is tempered, and Theorem 1.2 is proved. Now let d be arbitrary, and assume $i_\theta = 0$ for some $\theta \in T$ (namely I is prime to θ). Thus, π'_θ has a $G'(\mathcal{O}_\theta)$-fixed vector. By Lemma 5.1, $G'(\mathcal{O}_\theta)$ is normal in $G'(k_\theta)$, and π'_θ is an irreducible representation of the cyclic quotient, so it is one-dimensional. Write $\pi'_\theta = \phi \circ \det$ for a suitable character $\phi\colon k_\theta^\times \to \mathbb{C}$ (of order d), where here det stands for the reduced norm of $G'(k_\theta)$. By Equation (4.2) we have that $\mathrm{JL}_\theta(\pi'_\theta) = \mathrm{JL}_\theta(\phi \circ \det) = \mathrm{Sp}_d(\psi)$ for the character $\psi = |\cdot|^{(1-d)/2}\phi$ of $k_\theta^\times$. By Example 2.15, $\mathrm{C}_d(\psi)$ is one-dimensional. As mentioned in Subsection 4.2, π_θ is isomorphic to either $\mathrm{Sp}_s(\psi)$ or $\mathrm{C}_s(\psi)$, but π_θ cannot be one-dimensional (by Remark 3.6). Therefore, $\pi_\theta = \mathrm{Sp}_s(\psi)$, which is square-integrable (Proposition 2.13) and, in particular, tempered. Now, Corollary 4.3(a) implies that π is cuspidal, and by Theorem 4.1, $\pi_{\nu_0} = \rho$ is tempered too. ∎

Proof of Theorem 1.3(b): By Proposition 5.2, we need to find an irreducible sub-representation π' of $\mathrm{L}^2(G'(k)\backslash G'(\mathbb{A}))$ such that π'_{ν_0} is spherical and non-tempered. Then π'_{ν_0} would be a sub-representation of $\mathrm{L}^2(G'(R_0, I)\backslash G(F))$ for some $I \lhd R_0$, and $G'(R_0, I)\backslash\mathcal{B}$ would not be Ramanujan. We use the following result, which is a variant of a special case of [V, Thm. 2.2].

PROPOSITION 5.3: *Let $T = \{\theta_1, \dots, \theta_t\}$ and $\nu_1 \notin T$ be valuations of k. For $i = 1, \dots, t$, let ψ_i be a super-cuspidal representation of $\mathrm{PGL}_m(k_{\theta_i})$, where $m > 1$ is fixed. Then, there exists an automorphic cuspidal representation π of $\mathrm{PGL}_m(\mathbb{A})$, such that $\pi_{\theta_i} = \psi_i$ for $i = 1, \dots, t$, and $\pi_{\nu'}$ is spherical for every valuation $\nu' \notin T \cup \{\nu_1\}$.*

Proof: Here we let G denote the group PGL_m. Let f_{θ_i} be matrix coefficients of ψ_i, and let U_{θ_i} denote the (compact and open) support. For $\nu \notin T \cup \{\nu_1\}$ let $U_\nu = G(\mathcal{O}_\nu)$, and choose an open compact subgroup U_{ν_1} of $G(k_{\nu_1})$ such that $U = \prod U_\nu \subseteq G(\mathbb{A})$ intersects $G(k)$ only in the identity. For $\nu \neq \theta_1, \dots, \theta_t$, let f_ν be the characteristic function of U_ν. Let $f = \otimes f_\nu \in \mathrm{L}^2(G(\mathbb{A}))$. Define an operator R_f: $\mathrm{L}^2(G(k)\backslash G(\mathbb{A})) \to \mathrm{L}^2(G(k)\backslash G(\mathbb{A}))$ by

$$R_f\varphi(g) = \int_{G(\mathbb{A})} f(g^{-1}x)\varphi(x)dx.$$

The image of R_f is in the discrete spectrum. Let π be an irreducible representation in the image; then $\pi_{\theta_i} = \psi_i$ and, in particular, π is cuspidal. Moreover, $f_{\nu'}$ is a fixed vector of $\pi_{\nu'}$ so these are spherical for every $\nu' \notin T \cup \{\nu_1\}$. It remains to show that $R_f \neq 0$:

$$R_f\varphi(g) = \int_{G(k)\backslash G(\mathbb{A})} K_f(g, x)\varphi(x)dx$$

where $K_f(g, x) = \sum_{\gamma \in G(k)} f(g^{-1}\gamma x)$, which is a finite sum since f is compactly supported. But $K_f(1, 1) = f(1) + \sum_{1 \neq \gamma \in G(k)} f(\gamma) = 1$, showing that $K_f \neq 0$ and $R_f \neq 0$. ∎

For T we take the usual set of places in which D remains a division algebra, and we choose an arbitrary $\nu_1 \notin T \cup \{\nu_0\}$. Now pick any proper divisor s of d. For every $i = 1, \dots, t$ choose a super-cuspidal representation ψ_i of $\mathrm{PGL}_{d/s}(k_{\theta_i})$, and let π be the representation of $\mathrm{PGL}_{d/s}(\mathbb{A})$ given by Proposition 5.3; in particular, π_{ν_0} is spherical. Then let $\tilde{\pi} = T_s(\pi)$, as in Equation (4.1), and let $\overline{\pi} = J(\tilde{\pi})$ be its unique irreducible sub-representation. By Proposition 4.2, $\overline{\pi}$ is in the residual spectrum, and in particular $\overline{\pi}_{\nu_0}$ is spherical (since $\nu_0 \neq \nu_1$) and non-tempered (Corollary 4.3(a)). Now, for every $i = 1, \dots, t$, $\overline{\pi}_{\theta_i} = \mathrm{C}_s(|\det|_F^{(1-s)/2}\psi_i)$ (see the remark preceding Corollary 4.3), so by Theorem 4.4, $\overline{\pi}$ is in the image of the Jacquet–Langlands correspondence, corresponding to a representation $\overline{\pi}'$ of $G'(\mathbb{A})$ where $G' = D^\times/Z^\times$. But $\overline{\pi}_{\nu_0} = \overline{\pi}'_{\nu_0}$, so this component is spherical and not tempered. ∎

6. Outer forms

Theorem 1.2 (especially when compared to Theorem 1.3(b)) may suggest that if d is an odd prime, then every finite quotient of the Bruhat–Tits building $\mathcal{B} = \mathcal{B}_d(F)$ is Ramanujan, where F is a local field. Indeed, if Y is such a finite quotient of $\mathcal{B}$, then the fundamental group $\Gamma_1 = \pi_1(Y)$ acts on $\mathcal{B}$, the universal cover of Y, and $Y = \Gamma_1\backslash\mathcal{B}$. By a well known result of Tits, $\operatorname{Aut}(\mathcal{B})$ is $G = \mathrm{PGL}_d(F)$, up to compact extension. It seems likely that Γ_1 has a subgroup of finite index Γ which is contained in G, and the corresponding finite cover of Y can be obtained as $\Gamma\backslash G/K$. Now, by Margulis' arithmeticity theorem [Ma2], Γ is an arithmetic lattice of G. A well known conjecture of Serre [Se2] asserts that arithmetic lattices of G (where $d \geq 3$) satisfy the congruence subgroup property. This essentially means that every finite index subgroup is a congruence subgroup. If Γ is of inner type, our Theorem 1.2 applies to it, and shows that the quotients are really Ramanujan. However, there are other arithmetic subgroups (see, for example, the classification of the k-forms of GL_d in [Se1, III.1.4]). The outer forms of PGL_d all come from the following general construction: let k be a global field, k'/k a quadratic separable extension, and A a k'-central simple algebra with an involution $u \mapsto u^*$ which induces the non-trivial automorphism of k'/k on the center of A. Let $\mathrm{N}_{k'/k}$ denote the norm map. The algebraic group $G' = \{u \in A\colon uu^* = 1\}/Z$ (where $Z = \operatorname{Ker}(\mathrm{N}_{k'/k})$ is the center) gives a form of PGL_d. Now, if d is a prime, A may be either a division algebra, or the matrix algebra $\mathrm{M}_d(k')$. The second case corresponds to Hermitian forms [PR], i.e. G' is $\mathrm{PGU}_d(q,k') = \{a \in \mathrm{M}_d(k')\colon q(a(v)) = q(v)\}/Z$ of operators preserving the Hermitian form $q\colon (k')^d \to k'$. In this situation, the involution on A is $a \mapsto b^{-1}\bar{a}^t b$, where b is a skew-symmetric matrix representing q. But, if $\operatorname{char} k = p > 0$, every Hermitian form over k represents 0 if $d \geq 3$. Indeed, this is known to be true for local fields [Sc, Sec. 4.2] and by the Hasse Principal [Sc, Sec. 4.5], this is also true for k. Now in order to form a cocompact arithmetic lattice Γ in $\mathrm{PGL}_d(F)$, the form G' should be anisotropic (i.e. have k-rank zero), but if q represents 0 over k, the k-rank is greater than zero. Thus, there are no arithmetic lattices of Hermitian form type if $d \geq 3$ and F is of positive characteristic (the situation is different for characteristic zero, see below). On the other hand, the case when A is a division algebra is possible (e.g. the cyclic algebra $A = \mathbb{F}_{q^d}(t)[z|z^d = t]$, where z induces the Frobenius automorphism on $\mathbb{F}_{q^d}$, is a division algebra with center $k' = \mathbb{F}_q(t)$, and has an involution defined by $z^* = z^{-1}$ and $\alpha^* = \alpha$ for $\alpha \in \mathbb{F}_{q^d}$, which is non-trivial on k'). We do not know if Theorem 1.2 is valid in this case, but if it is true then together with Serre's conjecture this would imply the remarkable possibility that if $\operatorname{char} F > 0$ and $d \geq 3$ is a prime, then all the finite quotients of $\mathcal{B}_d(F)$ are Ramanujan. We leave it, however, as an open problem. For $d = 2$, i.e. $\mathrm{PGL}_2(F)$, all arithmetic lattices are of inner type, as the Dynkin diagram of A_1 does not have graph automorphisms, so Theorem 1.2 applies for all lattices (a result which has been proved before by Morgenstern [Mo]). Still we have

PROPOSITION 6.1: *If $d = 2$, for every nonarchimedean local field F, of any characteristic, $\mathrm{PGL}_2(F)$ has cocompact (arithmetic) lattices, such that the quotient $\Gamma \backslash \mathcal{B}_2(F)$ of the tree $\mathcal{B}_2(F)$ is not Ramanujan.*

Proof: The group $\mathrm{PGL}_2(F)$ has cocompact (arithmetic) lattices, and these are virtually free (cf. [Se3]). Let Γ be a free cocompact lattice in $\mathrm{PGL}_2(F)$, so $\Gamma' = [\Gamma, \Gamma]$ is of infinite index in Γ. Let Γ_n be a sequence of finite index subgroups of Γ, such that $\bigcap \Gamma_n = \Gamma'$. By [Lu1, Sec. 4.3], the graphs $\Gamma_n \backslash \mathcal{B}$ are not expanders, let alone Ramanujan graphs. Of course, in light of Theorem 1.2 (or [Mo]) for positive characteristic, and [Lu1, Thm. 7.3.1] (see also [JL]) for zero characteristic, almost all the Γ_n are non-congruence subgroups. ∎

LEMMA 6.2: *Let F be a local nonarchimedean field of characteristic zero. For every $d \geq 2$, there exists a number field k with a quadratic extension k' such that $k \subseteq k' \subseteq F$, and an anisotropic Hermitian form q of dimension d over k'/k.*

Proof: Let p be the prime such that $\mathbb{Q}_p \subseteq F$. Choose a natural number $\delta > 0$ such that $-\delta$ is a quadratic residue modulo p if p is odd (e.g. $\delta = p - 1$), and take $\delta = 7$ if $p = 2$. Let $k = \mathbb{Q}$ and $k' = \mathbb{Q}[\sqrt{-\delta}]$, and let $u \mapsto \bar{u}$ denote the non-trivial automorphism of k'/k. Let $q(u_1, \dots, u_d) = u_1\bar{u}_1 + \cdots + u_d\bar{u}_d$. Writing $u_i = x_i + \sqrt{-\delta} y_i$ for $x_i, y_i \in \mathbb{Q}$, we have that

$$q(u_1, \dots, u_d) = x_1^2 + \cdots + x_d^2 + \delta(y_1^2 + \cdots + y_d^2),$$

which does not represent zero even over $\mathbb{R}$. ∎

Proof of Theorem 1.4: Let k'/k be the quadratic extension and q the anisotropic Hermitian form as in the lemma, and let $G' = \mathrm{PGU}_d(q)$ and $G = \mathrm{PGL}_d$. Then $G'(F) \cong G(F)$, because $k' \subseteq F$, so $k' \otimes_k F = F \times F$ and

$$G'(F) = \{(a, b) \in \mathrm{GL}_d(F) \times \mathrm{GL}_d(F) \colon b = a^*\}/Z^\times = G(F).$$

Choose $\Gamma = G'(R_0)$ (where R_0 is as defined in Equation (1.2)) and ν_0 is the valuation on F. This is a cocompact lattice of $G'(F)$ since $G'(k) = \mathrm{PGU}_d(q, k)$

has rank zero. Moreover, if we let q_1 denote the sum of the first $d-1$ terms in a diagonal form of q, then q_1 does not represent zero, and setting $H' = \mathrm{PGU}_d(q_1)$, $H'(F) = \mathrm{PGU}_d(q_1, F)$ embeds in $G'(F)$ as $(d-1) \times (d-1)$ matrices (and is isomorphic to $H(F) = \mathrm{PGL}_{d-1}(F)$ for the same reasons as for q). We deduce that $\Lambda = \Gamma \cap H'(F)$ is a cocompact lattice in $H'(F) = H(F)$. By Proposition 1.5 it remains to find a spherical non-tempered sub-representation ρ of $\mathrm{L}^2(\Gamma_I \backslash G(F))$ for a congruence subgroup Γ_I. Now, since $\Lambda \backslash H(F)$ is compact,

$$\mathrm{L}^2(\mathrm{GL}_{d-1}(F)\backslash \mathrm{GL}_d(F)) = \mathrm{L}^2(H(F)\backslash G(F)) \subseteq \mathrm{L}^2(\Lambda \backslash G(F)).$$

The group $\mathrm{GL}_d(F)$ acts on $V \oplus V^*$ where $V = F^d$ and V^* is the dual space. Fixing $e_1 \in V$, the stabilizer of $e_1 \oplus e_1^*$ is isomorphic to $\mathrm{GL}_{d-1}(F)$, so $\mathrm{L}^2(\mathrm{PGL}_{d-1}(F)\backslash \mathrm{PGL}_d(F)) \subseteq \mathrm{L}^2(V \oplus V^*) \cong \mathrm{L}^2(V \otimes F^2)$. Now, using the action of $\mathrm{GL}_d \times \mathrm{GL}_2$ on $V \otimes F^2$, one can prove that $\mathrm{L}^2(V \otimes F^2)$ is the direct integral of $\rho' \otimes \rho$ over tempered $\rho \in \widehat{\mathrm{GL}_2}$ (the unitary dual), where ρ' is the representation of GL_d obtained by inducing $\rho \otimes \mathrm{id}_{d-2}$ from GL_2. In particular, ρ' is not tempered if $d \geq 4$ (since id_s is non-tempered if $s \geq 2$). Thus, $\mathrm{L}^2(\mathrm{PGL}_{d-1}(F)\backslash \mathrm{PGL}_d(F))$ has spherical non-tempered sub-representations. We thank R. Howe for this argument. For an ideal $I \lhd R_0$, let $\Lambda_I = \Lambda\Gamma(I)$. The Λ_I have finite index in Γ and so are cocompact in $G(F)$. Moreover, $\bigcap_I \Lambda_I = \Lambda$. Now, $\mathrm{L}^2(\Lambda \backslash G(F))$ is weakly contained in $\bigcup_I \mathrm{L}^2(\Lambda_I \backslash G(F))$ [BLS1], [BLS2], so for some $I \lhd R_0$, $\mathrm{L}^2(\Lambda_I \backslash \mathrm{PGL}_d(F))$ contains a spherical non-tempered sub-representation (which is discrete since Λ_I is cocompact). It follows that $\Lambda_I \backslash \mathcal{B}(F)$ as well as $\Gamma(I)\backslash \mathcal{B}$ are non-Ramanujan. ∎

A final remark is in order: so far, all the Ramanujan complexes constructed were quotients of $\tilde{A}_{d-1}(F)$ where F is an arbitrary local field of positive characteristic. For characteristic zero the problem is still open, except for $d = 2$. Of course, one hopes eventually to define and construct Ramanujan complexes as quotients of the Bruhat–Tits buildings of other simple groups as well.

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