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THE UNIQUENESS OF THE MAXIMAL MEASURE FOR GEODESIC FLOWS ON SYMMETRIC SPACES OF HIGHER RANK

BY

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ABSTRACT

In this paper we show that the geodesic flow on a compact locally symmetric space of nonpositive curvature has a unique invariant measure of maximal entropy. As an application to dynamics we show that closed geodesics are uniformly distributed with respect to this measure. Furthermore, we prove that the volume entropy is minimized at a compact locally symmetric space of nonpositive curvature among all conformally equivalent metrics with the same total volume.

Introduction

In 1969 Margulis [21] constructed for compact manifolds of negative curvature (or more generally for Anosov flows) a measure of maximal entropy, i.e., an invariant measure whose measure theoretic entropy coincides with the topological entropy. The main tools of his construction were the stable and unstable foliations. As an application he obtained precise asymptotic estimates for the growth of the number of periodic trajectories previously only known in the case of constant negative curvature. In 1972 Bowen [4] obtained an equidistribution result for the periodic orbits. More precisely, he showed that if μ_T denotes the flow invariant measure supported on the finite number of periodic orbits of period at most T then μ_T converges in the weak-star topology to a measure of maximal entropy. In 1973 Bowen [5] proved that the measure of maximal entropy is unique and, therefore, his and Margulis' constructions lead to the same measure (see also [12]).

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At a MSRI problem session in 1984, A. Katok [6] conjectured that the geodesic flow on a certain class of compact, nonpositively curved manifolds (rank 1 manifolds) admits a unique invariant measure of maximal entropy. Furthermore, the rank 1 (or hyperbolic) closed geodesics should be uniformly distributed with respect to this measure. In 1998 we [16] confirmed this conjecture (see also [17] for a survey).

In this paper we would like to extend this result to all compact locally symmetric spaces of nonpositive curvature and higher rank, provided they do not have a Euclidean factor. By the rank rigidity theorem of [1] and [7] this essentially implies the uniqueness of the measure of maximal entropy for all compact nonflat manifolds of nonpositive curvature. In the flat case the topological entropy is zero and, therefore, every invariant measure has maximal entropy.

The methods in our proof are completely different from the techniques used in the rank 1 case. Instead, tools developed in [19] will be of central importance.

Description of the results

Let (X, g) be a simply connected Riemannian manifold of nonpositive curvature. Denote for each unit tangent vector $v \in SX$ by c_v the geodesic with initial condition $\dot{c}_v(0) = v$. Then (X, g) is a symmetric space if for each $p \in X$ the geodesic reflection S_p: $X \to X$ given by $S_p(c_v(t) = c_v(-t))$ defines an isometry on X. For symmetric spaces the connected component $G := I_0(X)$ of the isometry group containing the identity acts transitively on X. Furthermore, for each $p \in X$ the isotropy group

$$K := G_p = \{g \in G \mid gp = p\}$$

is a maximal compact subgroup of G. Hence, X is isomorphic to the coset space G/K. Each geodesic is contained in a flat totally geodesic subspace F, whose dimension is equal to the rank of X. A tangent vector $v \in TX$ is called regular if the corresponding geodesic is contained in a unique flat. Denote by **Reg** the set of regular vectors and its complement, called the singular set, by **Sing**. Then for any point p in a maximal flat F the intersection

$$\mathbf{Sing} \cap T_pF$$

constitutes a finite number of hyperplanes. The Weyl chambers (sometimes called spherical Weyl chambers) are the connected components of $S_pF \setminus \mathbf{Sing}$. Each Weyl chamber W is a fundamental domain for the G-action on SX in the following sense. The union of all translations of the closure $\bar{W}$ of W covers SX, i.e.,

$$\bigcup_{g\in G} g\bar{W} = SX.$$

Furthermore, for each Weyl chamber W and $g \in G$ the intersection of gW and W is empty unless g acts as the identity on the unique flat containing W (see prop. 2.12.5 of [9]).

Now let $X = G/K$ be a symmetric space of nonpositive curvature and without Euclidean factor, i.e., X is a symmetric space of noncompact type. Furthermore, assume that X has higher rank, i.e., $\operatorname{rank} X \geq 2$. Let $\Gamma \subset G$ be a cocompact discrete subgroup (uniform lattice), i.e., $M = X/\Gamma$ is a compact Riemannian manifold. For a fixed $p \in X$, choose a Weyl chamber $W \subset S_pX$. Then, for each $v \in W$ the orbit Gv constitutes a closed set invariant under the geodesic flow (see [9] for a comprehensive treatment of symmetric spaces of nonpositive curvature).

For each v in the interior of W (regular direction) the geodesic flow restricted to $SM_v := \Gamma\backslash Gv \subset SM$ is mixing with respect to the probability measure μ^v induced by the Liouville measure on SM. Furthermore, the projection of each μ^v onto M is the normalized Riemannian volume. The ergodicity has been obtained by Mautner [22] and mixing is a consequence of Moore's ergodic theorem [23]. By homogeneity of the orbit Gv the sum of the positive Lyapunov exponents χ_v is constant on SM_v. This sum is also the measure theoretic entropy $h_{\mu^v}(SM_v)$ of the geodesic flow restricted to SM_v since Pesin's entropy formula asserts that for smooth invariant probability measures their measure theoretic entropy is the average of χ_v. For general invariant probability measures Ruelle's inequality [24] implies that the average of χ_v and, hence, χ_v is an upper bound for the measure theoretic entropy. Therefore, the measure μ^v maximizes the entropy among all invariant measures and by the variational principle, $h_{\mu^v}(SM_v)$ coincides with the topological entropy $h_{\text{top}}(SM_v)$ of the geodesic flow restricted to SM_v.

Furthermore, there is an algebraic description of $h_{\text{top}}(SM_v) = h_{\mu^v}(SM_v) = \chi_v$ as the sum of the positive roots evaluated at v. It attains its maximum at a unique element $v_{\max} \in W$ which we will call the maximal direction [25]. In particular, the topological entropy of the geodesic flow on SM_v is maximal on $SM_{\max} = \Gamma\backslash Gv_{\max}$. Moreover, the topological entropy of the unrestricted geodesic flow on SM coincides with the entropy of the geodesic flow on $SM_{\max}$, and, therefore, the Liouville measure $\mu^{v_{\max}}$ induced on $SM_{\max}$ defines a measure of maximal entropy for the geodesic flow on SM as well.

The main result of this paper asserts:

THEOREM 1: *For each v in the interior of a spherical Weyl chamber, the geodesic flow ϕ^t restricted to SM_v has a unique measure of maximal entropy and is, therefore, given by the induced Liouville measure μ^v.*

The following theorem is a consequence of Theorem 1.

THEOREM 2: *The geodesic flow $\phi^t\colon SM \to SM$ has a unique measure of maximal entropy. This measure is the Liouville measure induced on the maximal set $SM_{\max}$.*

As a corollary we obtain the uniform distribution of closed geodesics in a compact locally symmetric space. Let $P_\epsilon(M)$ be a maximal set of ϵ-separated closed geodesics on M. Two closed geodesics $c_1, c_2\colon \mathbb{R} \to M$ are called ϵ-separated if $d(c_1(t), c_2(t)) > \epsilon$ for some $t \in \mathbb{R}$, where d is the distance function induced by the Riemannian metric. In this set consider the subset of closed geodesics

$$P_\epsilon(T) = \{c \in P_\epsilon(M) \mid \operatorname{per}(c) \leq T\}$$

of period less than T. In his thesis, Spatzier showed [25], using a closing lemma (see also [2] chapter 5 for a published version of Spatzier's result and [9] for a geometric proof of the closing lemma), that the exponential growth rate of $P_\epsilon(T)$ is equal to the topological entropy, i.e.,

$$\lim_{T\to\infty} \frac{\log \operatorname{card} P_\epsilon(T)}{T} = h,$$

provided $\epsilon > 0$ is sufficiently small. Consider the invariant measure μ_T defined by

$$\int_{SM} f \, d\mu_T = \frac{\sum_{\{c \in P_\epsilon(T)\}} \frac{1}{\operatorname{per}(c)} \int_0^{\operatorname{per}(c)} f(\dot{c}(s))ds}{\operatorname{card} P_\epsilon(T)},$$

where $\operatorname{per}(c)$ denotes the period of the closed geodesic c and f is a continuous function on SM. As a corollary of the uniqueness of the maximal measure we obtain:

THEOREM 3: *For $\epsilon > 0$ sufficiently small a maximal set of closed geodesics is uniformly distributed with respect to the measure $\mu^{v_{max}}$ of maximal entropy, i.e.,*

$$\mu_T \to \mu^{v_{max}}$$

in the weak-star topology as $T \to \infty$.

Remark: We do not know if one can choose the closed geodesic to be pairwise non-homotopic. As we were told by Ralf Spatzier, it does not follow from his work that the exponential growth rate of non-homotopic closed geodesics is equal to the topological entropy.

The proof of Theorem 3 is a consequence of the uniqueness of the measure of maximal entropy together with Spatzier's result [2] and the following proposition whose proof is essentially given in [16]. There, the proposition is stated for non-homotopic closed geodesics, but all what is used in the proof is that they are separated.

PROPOSITION 4: *Suppose there exists a sequence $T_k \to \infty$ such that*

$$\lim_{T_k \to \infty} \frac{1}{T_k} \log \operatorname{card} P_\epsilon(T_k) = h_{\mathrm{top}}.$$

Then the accumulation points of $\{\mu_{T_k}\}$ with respect to the weak-star topology are measures of maximal entropy.

Closely related to the topological entropy is the volume entropy which is defined for any compact manifold (M, g). Namely, if X is the universal covering of M, $B_r(p)$ the geodesic ball of radius r about $p \in X$, and $\mathrm{vol}(B_r(p))$ the volume with respect to the Riemannian metric g lifted to X, then the volume entropy is given by

$$h(g) = \lim_{r \to \infty} \frac{\log \mathrm{vol}(B_r(p))}{r}.$$

By a result of Manning [20] the volume entropy $h(g)$ is less than or equal to the topological entropy $h_{\mathrm{top}}(g)$. Equality holds if the metric has nonpositive curvature [20] or more generally has no conjugate points [10]. Besson, Courtois and Gallot [3] have shown that compact locally symmetric spaces of negative curvature have minimal entropy among all homotopy equivalent Riemannian manifolds having the same volume. More precisely, they obtained: If (M_0, g_0) is a compact locally symmetric space of negative curvature and of dimension at least three, then for all other homotopy equivalent compact Riemannian manifolds (M, g)

$$h(g) \geq \left(\frac{\mathrm{vol}_{g_0}(M)}{\mathrm{vol}_g(M)} \right)^{1/n} h(g_0).$$

Furthermore, the inequality is strict unless g and g_0 are isometric up to a scaling. This result has been recently extended by Connell and Farb [8] to products of compact locally symmetric spaces of negative curvature, provided the dimension of each factor is at least three.

It is a difficult question if the same holds true for compact locally symmetric spaces of higher rank, which are not products. However, we give an affirmative answer to this question in a conformal class of such a locally symmetric space. In the rank 1 case this is an old result of A. Katok [11] (see also [14] and [18] for a general survey).

THEOREM 5: *Let (M, g_0) be a compact n-dimensional locally symmetric metric of nonpositive curvature and g any other metric conformally equivalent to g_0. Then the inequality*

$$h(g) \geq \left(\frac{\mathrm{vol}_{g_0}(M)}{\mathrm{vol}_g(M)}\right)^{1/n} h(g_0)$$

is strict unless the metrics g and g_0 are homothetic, i.e., they agree up to a constant.

Proof of the theorems

Let v be a vector in the interior of a fixed Weyl chamber W, $SM_v := \Gamma \backslash Gv$ be the closed subset of SM invariant under the geodesic flow ϕ^t on SM and $\mu := \mu^v$ be the Liouville measure induced on SM_v.

For each flow invariant measure ν on SM_v we denote by $h_\nu(SM_v)$ the measure theoretic entropy and by $h_{\mathrm{top}}(SM_v)$ the topological entropy of the geodesic flow ϕ^t restricted to SM_v. First we remark that it is enough to prove uniqueness of the measure of maximal entropy among the ergodic measures on SM_v. This follows from the fact that by the ergodic decomposition theorem we can decompose each invariant measure ν into the average of ergodic measures ν_y, i.e.,

$$\nu = \int_E \nu_y dm(y),$$

where m is a probability measure on the set E of all ergodic ϕ^t-invariant measures on SM_v. Since entropy is an affine function on the set of probability measures on SM_v,

$$h_\nu(SM_v) = \int_E h_{\nu_y}(SM_v) dm(y).$$

If $h_\nu(SM_v) = h_{\mathrm{top}}(SM_v)$ the variational principle implies $h_{\nu_y}(SM_v) = h_{\mathrm{top}}(SM_v)$ for m almost all $y \in E$. Therefore, the uniqueness in the class of ergodic measures yields uniqueness in general.

Let Λ^+ be the set of positive roots α: $T_pF \to \mathbb{R}$, associated to the Weyl chamber $W \subset T_pF$ contained in the tangent space of a maximal flat F. As was shown in [25] (see also [9]), the sum

$$\chi_v = \sum_{\alpha\in\Lambda^+} \alpha(v) m_\alpha$$

is equal to the sum of the positive Lyapunov exponents of $\phi^t\colon SM_v \to SM_v$, where m_α is the multiplicity of the root α. As explained above it is equal to the topological entropy $h_{\text{top}}(SM_v)$. Now consider an ergodic invariant measure on SM_v such that $h_\nu(SM_v) = h_{\text{top}}(SM_v)$. Therefore, Ruelle's entropy inequality [24]

$$h_\nu(SM_v) \le \int_{SM_v} \chi_v d\nu$$

becomes an equality and by a well-known result of Ledrappier and Young [19] the conditionals ν_x on the strong unstable foliation W^u in SM_v are smooth and, hence, absolutely continuous to the conditionals μ_x of μ.

We recall that the conditionals ν_x are defined as follows (see [19], page 513).

Let ξ be a measurable partition subordinate to W^u, i.e., a measurable partition such that for all $x \in SM_v$ we have

(a) $\xi(x) \subset W^u(x)$,

(b) $\xi(x)$ contains a neighborhood of x open in the submanifold topology of $W^u(x)$.

Then there exists a family of probability measures ν_x on $\xi(x)$ such that for all measurable sets $A \subset SM_v$,

$$x \to \nu_x(A) = \nu_x(A \cap \xi(x))$$

is measurable with respect to the σ-algebra B_ξ generated by elements of ξ. Furthermore, ν decomposes with respect to ξ, i.e.,

$$\nu(A) = \int \nu_x(A) d\nu(x).$$

The conditional measures ν_x are up to a set of ν-measure zero uniquely characterized by those properties. If ν is a measure of maximal entropy ν_x is absolutely continuous for ν-almost all x to μ_x, where μ_x are the conditionals of the Liouville measure $\mu := \mu^v$ restricted to SM_v. Furthermore, the Radon Nikodym derivative $\rho = d\nu_x/d\mu_x$ has the property ([19], page 533) that

$$\frac{\rho(y)}{\rho(x)} = \frac{\prod_{j=1}^{\infty} \operatorname{Jac} D\phi|E^u(\phi^{-j}y)}{\prod_{j=1}^{\infty} \operatorname{Jac} D\phi|E^u(\phi^{-j}x)},$$

where $\operatorname{Jac} D\phi|E^u(x)$ is the Jacobian determinant of $\phi = \phi^1$ restricted to $E^u(x) = T_xW^u(x)$. Since the Jacobian is constant on SM_v, $\rho(y) = \rho(x)$ for all

$y \in \xi(x)$, and since ν_x and ν_y are probability measures, $\rho \equiv 1$. Let A be the subset of SM_v such that for each continuous function $f \in C^0(SM_v)$ the time average coincides with space average, i.e.,

$$A = \left\{ x \in SM_v \mid \lim_{T\to\infty} \frac{1}{T} \int_0^T f(\phi^t x)dt = \int f d\nu \right\}.$$

By the ergodicity of ν, A is a set of full measure and the decomposition

$$\nu(A) = \int \nu_x(A) d\nu$$

implies that $\nu_x(A) = 1$ for ν-almost all $x \in SM_v$. Hence, for ν-almost all $x \in SM_v$, we have

$$\int f d\nu = \int_{\xi(x)} \lim_{T\to\infty} \frac{1}{T} \int_0^T f(\phi^t(p))dt d\nu_x(p)$$

for all $f \in C^0(SM_v)$. Furthermore, the dominated convergence theorem and Fubini's theorem imply

$$\int f d\nu = \lim_{T\to\infty} \frac{1}{T} \int_0^T \int_{\xi(x)} f \circ \phi^t(p) d\nu_x(p) dt.$$

As we have seen by the considerations above $d\nu_x = d\mu_x$ for ν-almost all x and, therefore, the equation

$$\int f d\nu = \lim_{T\to\infty} \frac{1}{T} \int_0^T \int_{\xi(x)} f \circ \phi^t(p) d\mu_x(p) dt$$

holds for ν-almost all x. Therefore, the proof of Theorem 1 is complete if we show that for all $x \in X$

$$\int_{\xi(x)} f \circ \phi^t(p) d\mu_x(p) \mapsto \int f d\mu \tag{1}$$

as $t \to \infty$. For that it will be of importance that μ is mixing.

PROPOSITION 6: *Let $U \subset W^u(x_0)$ be a relatively compact open neighborhood of x_0 in $W^u(x_0)$. Then for $f \in C^0(SM_v)$*

$$\int_U f \circ \phi^t(p) d\mu_{x_0}(p) \mapsto \int_{SM_v} f d\mu,$$

as $t \to \infty$, where μ_{x_0} is the normalized Riemannian volume on the open subset $U \subset W^u(x_0)$.

Proof: Since the unstable foliation W^u is transversal to the center stable foliation W^{cs} and both foliations are smooth, we can choose to a given x_0 a constant $\delta_0 > 0$ such that the cartesian product of the balls $W^{cs}_{\delta_0}(x_0)$ and $W^u_{\delta_0}(x_0)$ of radius δ_0 about x_0 with respect to the induced Riemannian metric on the corresponding leaves is diffeomorphic to an open neighborhood of x_0 in SM_v. More specifically, there exists a further constant $\delta_1 > 0$ such that the map

$$B\colon W^{cs}_{\delta_0}(x_0) \times W^u_{\delta_0}(x_0) \to SM_v$$

with $(x,y) \mapsto W^u_{\delta_1}(x) \cap W^{cs}_{\delta_1}(y)$ is well defined and determines a diffeomorphism onto its image. Hence, for each $x \in W^{cs}_{\delta_0}(x_0)$ the holonomy map

$$H_{x_0,x}\colon W^u_{\delta_0}(x_0) \to W^u_{\delta_1}(x)$$

with $y \mapsto B(x,y)$ determines a diffeomorphism onto its image as well. Note that it suffices to prove the proposition for small open neighborhoods and, therefore, we can assume that U is contained in $W^u_{\delta_0}(x_0)$. Choose a measurable partition ξ subordinate to W^u such that for all $x \in W^{cs}_{\delta_0}(x_0)$ the sets $\xi(x)$ are given by $H_{x_0,x}(U)$.

For $0 < \delta \le \delta_0$ consider the box

$$B_\delta = \bigcup_{x \in W^{cs}_\delta(x_0)} \xi(x). \tag{2}$$

Using the decomposition of the measure μ with respect to the partition ξ

$$\int_{SM_v} \int_{\xi(x)} f d\mu_x d\mu = \int_{SM_v} f d\mu$$

for all $f \in L^1(SM_v)$, where

$$\mu_x = \frac{\mu_{W^u(x)}}{\mu_{W^u(x)}(\xi(x))}$$

is the normalized Riemannian measure $\mu_{W^u(x)}$ on $W^u(x)$. Since μ is mixing, we obtain for all $f, g \in L^1(SM_v)$

$$\begin{aligned}\lim_{t\to\infty} \int_{SM_v} f \circ \phi^t \cdot g d\mu &= \lim_{t\to\infty} \int_{SM_v} \int_{\xi(x)} f \circ \phi^t(y) g(y) d\mu_x(y) d\mu(y)\\ &= \int_{SM_v} f d\mu \cdot \int_{SM_v} g d\mu.\end{aligned}$$

Therefore,

$$\lim_{t\to\infty} \frac{\int_{SM_v} \int_{\xi(x)} f \circ \phi^t(y) g(y) d\mu_x(y) d\mu(x)}{\int_{SM_v} g d\mu} = \int_{SM_v} f d\mu,$$

provided $\int_{SM_v} g d\mu \neq 0$. We apply this to a continuous function $f\colon SM_v \to \mathbb{R}$ and the characteristic functions $g = \chi_{B_\delta}$. Since ξ is a partition, the support of $x \mapsto \int_{\xi(x)} f \circ \phi^t(y)\chi_{B_\delta}(y)d\mu_x$ is contained in B_δ. Hence

$$\lim_{t\to\infty} \frac{\int_{B_\delta}\int_{\xi(x)} f\circ\phi^t(y)d\mu_x(y)d\mu(x)}{\mu(B_\delta)} = \int_{SM_v} f d\mu.$$

The proposition is now a simple consequence of the continuity of

$$x \to \int_{\xi(x)} f\circ\phi^t(y)d\mu_x$$

at x_0, whose proof is provided in the next lemma. ∎

Lemma 7: *Let $f\colon SM_v \to \mathbb{R}$ be a continuous function. Then for each $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\left| \int_{\xi(x)} f\circ\phi^t(y)d\mu_x(y) - \int_{\xi(x_0)} f\circ\phi^t(y)d\mu_{x_0}(y)\right| < \epsilon$$

for all $x \in B_\delta$, where B_δ is the box defined in (2).

Proof: For all $x \in B_\delta$ the holonomy map $H_{x_0,x}\colon \xi(x_0) \to \xi(x)$ is smooth and, therefore, absolutely continuous with respect to the smooth conditional measure μ_x of the stable foliation. Therefore, $(H^{-1}_{x_0,x})_*\mu_x = q_x \cdot d\mu_{x_0}$ and $\|q_x - 1\| \to 0$ as $x \to x_0$. For $x \in D_\delta = W^{cs}_\delta(x_0)$ consider

$$\begin{aligned} A := &\int_{\xi(x)} f\circ\phi^t(y)d\mu_x(y) - \int_{\xi(x_0)} f\circ\phi^t(y)d\mu_{x_0}(y)\\ = &\int_{\xi(x_0)} f\circ\phi^t\circ H_{x_0,x}(y)q_x(y)d\mu_{x_0}(y) - \int_{\xi(x_0)} f\circ\phi^t(y)d\mu_{x_0}(y). \end{aligned}$$

Then

$$\begin{aligned} |A| \le &\int_{\xi(x_0)} |f\circ\phi^t\circ H_{x_0,x}(y)q_x(y) - f\circ\phi^t(y)q_x(y)|d\mu_{x_0}(y)\\ &+ \int_{\xi(x_0)} |f\circ\phi^t(y)q_x(y) - f\circ\phi^t(y)|d\mu_{x_0}(y)\\ \le &\sup_{y\in\xi(x_0)} |q_x(y)| \int_{\xi(x_0)} |f\circ\phi^t\circ H_{x_0,x}(y) - f\circ\phi^t(y)|d\mu_{x_0}(y)\\ &+ \|f\|_\infty \int_{\xi(x_0)} |q_x(y) - 1|d\mu_{x_0}(y). \end{aligned}$$

Since $y \in \xi(x_0)$ and $H_{x_0,x}(y) \in \xi(x)$ are on the same center stable leaf,

$$d_1(\phi^t \circ H_{x_0,x}(y), \phi^t(y)) \leq d_1(H_{x_0,x}(y), (y)),$$

where $d_1(v,w) = \max_{t\in[0,1]} d(c_v(t), c_w(t))$ and d is the distance induced by the Riemannian metric on G/K. The uniform continuity and the fact that $||q_x - 1|| \to 0$ for $x \to x_0$ implies that for each $\epsilon > 0$ one can choose $D_\delta \subset D_{\delta_0} = W^{cs}_{\delta_0}(x_0)$ such that

$$\int_{\xi(x_0)} |f \circ \phi^t \circ H_{x_0,x}(y) - f \circ \phi^t(y)| d\mu_{x_0}(y) \leq \frac{\epsilon}{2 \cdot \sup_{y\in\xi(x_0)} |q_x(y)|}$$

and

$$\int_{\xi(x_0)} |q_x(y) - 1| d\mu_{x_0}(y) \leq \frac{\epsilon}{2 \cdot ||f||_\infty}$$

for all $x \in B_\delta$. This proves the lemma. ∎

Now we show that Theorem 2 is a simple consequence of Theorem 1. Let ν be an invariant measure of maximal entropy for the geodesic flow on SM. Then by Ruelle's inequality, we have

$$h_\nu(SM) = h_{\text{top}}(SM) \leq \int_{SM} \chi(w) d\nu.$$

Since the sum $\chi(w)$ of the positive Lyapunov exponents is equal to χ_v for $w \in SM_v$ and $\chi(w) < \chi_{v_{\max}} = h_{\text{top}}(SM)$ unless $w \in SM_{v_{\max}}$, the measure ν must have full support on $SM_{v_{\max}}$, i.e., its complement must have measure zero. By Theorem 1 this implies $\nu = \mu^{v_{\max}}$.

For the proof of Theorem 5 we need the following result which we proved in [18].

THEOREM 8: *Let (M, g_0) be a compact manifold of nonpositive curvature and μ a ϕ_{g_0} invariant probability measure. Then, for any other metric g the estimate*

$$h(g) \geq \frac{h_\mu(g_0)}{\int_{(SM)_{g_0}} ||v||_g d\mu}$$

holds.

Now we are ready to prove Theorem 5.

Proof: Let (M, g_0) be a compact locally symmetric space of nonpositive curvature and $g = fg_0$ a metric conformally equivalent to g. Let μ be the measure of maximal entropy for the geodesic flow with respect to g_0. Since the projection of μ on M is the normalized Riemannian volume, we obtain using Jensen's inequality

$$\begin{aligned}\int_{(SM)_{g_0}} \|v\|_g d\mu &= \frac{1}{\mathrm{vol}_{g_0}(M)} \int_M f^{1/2} d \underset{g_0}{\mathrm{vol}} \\ &\leq \left(\frac{1}{\mathrm{vol}_{g_0}(M)} \int_M f^{n/2} d \underset{g_0}{\mathrm{vol}}\right)^{1/n} \\ &= \left(\frac{\mathrm{vol}_g(M)}{\mathrm{vol}_{g_0}(M)}\right)^{1/n}.\end{aligned}$$

Note that Jensen's inequality is strict unless f is constant. Hence, this yields Theorem 5. ∎

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