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BOUNDARY BEHAVIOR OF μ-HOMEOMORPHISMS

BY

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ABSTRACT

We provide conditions on the complex dilatation of a homeomorphism f of the upper half plane $\mathbb{H}$ into $\mathbb{C}$, which guarantee that $f(\mathbb{H})$ is a proper subset of $\mathbb{C}$ and, in case where $f(\mathbb{H})$ is a Jordan domain, that f has a homeomorphic extension onto $\overline{\mathbb{H}}$.

1. Introduction and preliminaries

Let f be a sense-preserving homeomorphism of the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ into $\mathbb{C}$ and $\mu\colon \mathbb{H} \to \mathbb{C}$ a measurable function with $|\mu| < 1$ a.e. in $\mathbb{H}$. We say that f is a **μ-homeomorphism** or **μ-conformal**, if f is ACL in $\mathbb{H}$, i.e. absolutely continuous on lines, see [A] or [LV], and the complex partial derivatives

$$f_z = \frac{1}{2}(f_x - if_y) \quad \text{and} \quad f_{\bar{z}} = \frac{1}{2}(f_x + if_y)$$

satisfy a.e. the Beltrami equation

$$(B) \qquad f_{\bar{z}} = \mu(z) f_z.$$

The function μ is the **complex dilatation** of f, and it is denoted by $\mu = \mu_f$.

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Note that a μ-homeomorphism in $\mathbb{H}$, if it exists, can be viewed as a conformal embedding of $\mathbb{H}$ endowed with the measurable conformal structure $ds = |dz + \mu(z)d\bar{z}|$ into $\mathbb{C}$. It is well known that if f is conformal in $\mathbb{H}$ (endowed with the euclidian metric), then $f(\mathbb{H})$ is a proper subset of $\mathbb{C}$, and if, in addition, $f(\mathbb{H})$ is a Jordan domain, f has a homeomorphic extension on $\overline{\mathbb{H}}$, the closure of $\mathbb{H}$ in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The same is true for a μ-homeomorphism in $\mathbb{H}$ with $||\mu||_\infty < 1$. If $||\mu||_\infty < 1$ we say that μ is **bounded**.

If μ is measurable and **locally bounded in** $\mathbb{H}$, i.e. $||\mu|A||_\infty < 1$ for every relatively compact subset A in $\mathbb{H}$, a μ-homeomorphism of $\mathbb{H}$ exists, and its image may be either $\mathbb{C}$, see 4.2 below, or a proper subset of $\mathbb{C}$. One may ask under what conditions on μ, with or without the assumption that μ is locally bounded, every μ-homeomorphism f satisfies:

(i) $f(\mathbb{H}) \neq \mathbb{C}$, and

(ii) if, in addition, $f(\mathbb{H})$ is a Jordan domain, then f has a homeomorphic extension on $\overline{\mathbb{H}}$ = the closure of $\mathbb{H}$ in $\overline{\mathbb{C}}$.

A sufficient condition, namely,

$$(1 + |\mu(z)|)/(1 - |\mu(z)|) \leq Q(z) \text{ a.e.}$$

for some function Q of bounded mean oscillation in $\mathbb{H}$ can be derived from David's existence and uniqueness theorem [D]. For an explicit proof see [RSY]. An extension of this result to higher dimensions can be found in [MRSY], where the methods are quite different.

The main results in this note are the following Theorem 1.3 and Theorem 1.4. Some of the ingredients in the proofs of these theorems were developed in [SY]. In Theorem 1.3, the assertion (i) above is obtained for an embedding f of $\mathbb{H}$ into $\mathbb{C}$ under an assumption on the complex dilatation f near a single boundary point. In the latter theorem we assume that μ is locally bounded in $\mathbb{H}$ and that $|\mu(z)| \to 1$ as $z \to E$ for some open set E in $\mathbb{R} = \partial\mathbb{H}$, possibly $E = \mathbb{R}$.

Let $r > 0$ and ρ be measurable a.e. positive in $(0, r)$. We say that ρ is **locally bounded away from** 0, if $\rho|A \geq \rho_0(A) > 0$ a.e. for all relatively compact subset A of $(0, r)$. We say that $\rho(t) \to 0$ a.e. as $t \to 0^+$, if $\rho \to 0$ as $t \to 0^+, t \in (0, r) \setminus A$ for some set A of linear measure zero.

Let E be an open set in $\mathbb{R}$ and U a neighborhood of E in $\mathbb{C}$. We say that $|\mu(x, y)| \to 1$ **a.e. uniformly in** U as $y \to 0^+, x \in E$, if U has a subset A with $m_2(A) = 0$ such that for all $x \in E$,

$$\lim_{y \to 0^+} |\mu(x, y)| = 1, \quad (x, y) \in U \setminus A, \tag{1.1}$$

and there exists $C > 1$ such that for all (x_1, y) and (x_2, y) in $U \setminus A$,

$$\frac{1}{C} \leq \frac{1-|\mu(x_1,y)|}{1-|\mu(x_2,y)|} \leq C. \tag{1.2}$$

1.3. THEOREM: *Let f be an embedding of $\mathbb{H}$ into $\mathbb{C}$ and $a \in \mathbb{R}$. Suppose that a has a neighborhood U in $\mathbb{C}$ such that f is ACL in $U^+ = U \cap \mathbb{H}$ with a locally bounded complex dilatation $\mu = \mu_f$ in U^+. If*

(a) *$|\mu(x,y)| \to 1$ a.e. uniformly in U^+ as $y \to 0^+, x \in U \cap \mathbb{R}$,*

(b) *$\arg \mu$ is continuously differentiable with bounded partial derivatives in U^+,*

(c) *$|\arg \mu| \leq 2\theta_0 < \pi$ in U^+,*

then

(i) *$f(\mathbb{H}) \neq \mathbb{C}$,*

(ii) *if, in addition, $f(\mathbb{H})$ is a Jordan domain, then f has a homeomorphic extension on $\mathbb{H} \cup (U \cap \mathbb{R})$.*

1.4. THEOREM: *Let E be an open set in $\mathbb{R}$, possibly $E = \mathbb{R}$, $\mu\colon \mathbb{H} \to \mathbb{C}$ a locally bounded measurable function, and $f\colon \mathbb{H} \to \mathbb{C}$ a μ-homeomorphism. Suppose that every point $a \in E$ has a neighborhood $U, U \subset \mathbb{C}$ such that conditions (a)–(c) of Theorem 1.3 hold with θ_0 depending on U.*

If $f(\mathbb{H})$ is a Jordan domain, and if f is quasiconformal in every domain $D, D \subset \mathbb{H}$ whose closure in $\overline{\mathbb{C}}$ is contained in $\overline{\mathbb{H}} \setminus E$, then f has a homeomorphic extension on $\overline{\mathbb{H}}$.

Suppose that μ satisfies the assumptions of Theorem 1.4 with $E = \mathbb{R}$, and that f is a μ-homeomorphism of $\mathbb{H}$ onto a Jordan domain. Then, by Theorem 1.4 and in view of the fact that $f(\mathbb{H})$ is a Jordan domain in $\mathbb{C}$, it follows that f has a homeomorphic extension on $\mathbb{H} \cup \mathbb{R}$; however, f need not have a homeomorphic extension on $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{R} \cup \{\infty\}$ as illustrated in 4.1.

Throughout the paper, for a point $a \in \mathbb{R}$ and $r > 0$, $I = I(a,r)$ denotes the interval $(a-r, a+r)$, $S = S(a,r) = \{(x,y) : |x-a| < r, |y| < r\}$ denotes the square centered at a with side of the length $2r$, and $S^+ = S^+(a,r) = S \cap \mathbb{H}$.

2. Preliminaries

2.1. DEFORMATION OF THE COMPLEX DILATATION. Let μ be a locally bounded complex-valued measurable function in $\mathbb{H}$, and let g be an embedding of $\mathbb{H}$ into $\mathbb{C}$, and suppose that g is locally quasiconformal. The **deformation**

$g * \mu$ of μ which is **induced by** g is defined in $g(\mathbb{H})$ by

$$g * \mu(w) = \frac{A\mu - B}{\bar{A} - \bar{B}\mu} \circ g^{-1}(w), \quad w \in g(\mathbb{H}), \tag{2.2}$$

where $A = \partial g$ and $B = \bar{\partial} g$ are the complex partial derivatives of g.

2.3. LEMMA: *Let μ and g be as in 2.1. Then*

(i) *$g * \mu$ is measurable and locally bounded in $g(\mathbb{H})$.*

(ii) *If h is a locally quasiconformal embedding of $g(\mathbb{H})$ into $\mathbb{C}$ with complex dilatation $\mu_h = g * \mu$ a.e. in $g(\mathbb{H})$, then $f = h \circ g$ is locally quasiconformal with complex dilatation $\mu_f = \mu$ a.e. in $\mathbb{H}$.*

Proof: (i) The functions $A, \bar{A}, B, \bar{B}$ and μ are measurable in $\mathbb{H}$ and hence so is $(A\mu - B)/(\bar{A} - \bar{B}\mu)$. Now, g is locally quasiconformal, and thus preserves measurable sets. Consequently, $g * \mu$ is measurable in $g(\mathbb{H})$. Furthermore, since g is locally quasiconformal, and μ is locally bounded, so is $\mu_g = B/A$ and hence, by a simple estimate, it follows that $g * \mu$ is locally bounded in $g(\mathbb{H})$.

(ii) Since g and h are locally quasiconformal embeddings, so is f. Therefore, for almost all z in $\mathbb{H}$, g is differentiable at z and h at $g(z)$, and at these points an application of the chain rule yields

$$\mu_f = \frac{\bar{A}(\mu_h \circ g) - B}{\bar{B}(\mu_h \circ g) + A}. \tag{2.4}$$

If we now set $\mu_h = g * \mu$ and apply (2.4), we get $\mu_f = \mu$ a.e. in $\mathbb{H}$, as asserted.

2.5. LEMMA: *Let a be a point in $\mathbb{R}$, $r > 0$ and $\mu(x, y)$ a locally bounded measurable function in the rectangle $S^+ = S^+(a, r)$. If $|\mu(x, y)| \to 1$ a.e. uniformly in S^+ as $y \to 0^+, x \in (a - r, a + r)$, then there are measurable functions $\rho(y)$: $(0, r) \to \mathbb{R}_+$ and M: $S^+ \to \mathbb{R}$, such that $\rho(y)$ is a.e. positive and locally bounded away from 0 in $(0, r)$, and $\rho(y) \to 0$ a.e. in R_+ as $y \to 0^+$, and M satisfies*

$$\frac{1}{C} \le M(x, y) \le C \quad \text{a.e. in } S^+ \tag{2.6}$$

for some $C > 1$, and such that for a.e. $(x, y) \in S^+$,

$$1 - |\mu(x, y)| = \rho(y) M(x, y). \tag{2.7}$$

Proof: Recall that by (1.1), there is a set A in S^+ with $m_2(A) = 0$ such that $|\mu(x, y)| \to 1$ as $y \to 0^+$, $(x, y) \in S^+ \setminus A$. For $x \in E$, let $A(x) = A \cap l(x)$, where $l(x)$ is the vertical line, which contains the point $(x, 0)$. Then, by Fubini's theorem, $m_1(A(x)) = 0$ for a.e. $x \in (a-r, a+r)$.

Let x_1 be a point in $(a-r, a+r)$ such that $|\mu(x_1, y)| < 1$ for a.e. $(x_1, y) \in l(x_1)$, $|\mu(x_1, y)| \to 1$ as $y \to 0^+$ and such that $m_1(A(x_1)) = 0$. Set $\rho(y) = 1 - |\mu(x_1, y)|$. Then ρ is a measurable function in an interval $(0, r)$. Clearly, $\rho(y) \to 0$ a.e. in $(0, r)$ as $y \to 0^+$, and ρ is a.e. positive in $(0, r)$. Then, ρ has a measurable extension on $\mathbb{R}$, denoted again by ρ, which is positive a.e. in $\mathbb{R}_+$. Furthermore, ρ is locally bounded in $(0, r)$, since μ is locally bounded in $\mathbb{H}$.

For $(x, y) \in S^+$, set

$$M(x, y) = \frac{1 - |\mu(x, y)|}{1 - |\mu(x_1, y)|}.$$

Then the assertion of the Lemma follows by (1.2).

2.8. Lemma: *Let a be a point in $\mathbb{R}$, U a neighborhood of a in $\mathbb{C}$, and θ be a continuously differentiable function with bounded partial derivatives in $U^+ = U \cap \mathbb{H}$ such that $|\theta(z)| \le \theta_0 < \pi/2$ for all $Z \in U^+$. Then*

(i) *there is a square $S = S(a, r)$, $r > 0$, $S \subset U$ such that the equation*

$$\frac{\partial \xi}{\partial x} \sin\theta - \frac{\partial \xi}{\partial y} \cos\theta = 0 \tag{2.9}$$

has a unique solution $\xi(x, y)$ in S^+, which has a continuous extension on $S^+ \cup I$, $I = I(a-r, a+r)$, satisfying the initial condition

$$\xi(x, 0) = x, \quad x \in I; \tag{2.10}$$

(ii) *the mapping*

$$g_1(x, y) = \begin{cases} (\xi(x, y), y), & \text{for } (x, y) \in S^+ \\ (x, 0), & \text{for } (x, y) \in I \end{cases} \tag{2.11}$$

is an embedding of $S^+ \cup I$ into $\mathbb{C}$, which is a C^1-diffeomorphism of S^+ into $\mathbb{H}$ and identity on I.

Proof: Consider the differential equation

$$\frac{dx}{dy} = -m(x, y) := -\tan\theta, \quad x + iy \in U^+. \tag{2.12}$$

In view of the assumptions on θ, $m(x, y)$ is continuously differentiable with bounded derivatives in U^+, and hence has a continuous extension on $\overline{U^+}$, and the coefficient function $m(x, y)$ is Lipschitz in U^+ with respect to y. Therefore, there exists $\delta_0 > 0$ such that for every $\xi \in (a - \delta_0, a + \delta_0)$, (2.12) has a unique solution $x = \varphi(\xi, y)$, $0 \le y \le \delta_0$, which satisfies the initial condition

$$\varphi(\xi, 0) = \xi. \tag{2.13}$$

Since $m(x, y)$ is bounded in U^+, there exists $\delta \in (0, \delta_0)$ such that the mapping $z = G(\zeta), z = x + iy, \zeta = \xi + i\eta$ which is given by

$$\begin{cases} x = \varphi(\xi, \eta) \\ y = \eta \end{cases} \tag{2.14}$$

is well defined in the closed rectangle

$$\overline{Q} = \{(\xi, \eta) : |\xi - a| \le \delta, 0 \le \eta \le \delta\}.$$

Clearly $G|[a - \delta, a + \delta] = \mathrm{id}$, and by the classical existence and uniqueness theorem for first order ordinary differential equations, G is an embedding of $\overline{Q}$ into $\mathbb{C}$. Since $m \in C^1$, it follows that $\varphi(\xi, \eta)$ and hence also G is continuously differentiable in $Q = \operatorname{int} \overline{Q}$, see [H, Theorem 3.1.2]. Furthermore, see [H, 3.1.16],

$$\frac{\partial x}{\partial \xi} = \exp\Big(- \int_0^\eta \frac{\partial m}{\partial x}(\varphi(\xi, \eta), s) ds \Big) > 0,$$

and thus the Jacobian J_G of G satisfies

$$J_G = \frac{\partial(x, y)}{\partial(\xi, \eta)} = \frac{\partial x}{\partial \xi} > 0, \tag{2.15}$$

in Q. Therefore, G is a sense preserving homeomorphism in $\overline{Q}$, and $G(Q) \subset \mathbb{H}$. Then, there exists $r > 0$ such that $\overline{S^+}(a, r) \subset G(\overline{Q})$.

Let $g_1 = G^{-1}|\overline{S^+}(a, r)$. Then g_1 is a sense preserving homeomorphism which is a diffeomorphism, and hence locally quasiconformal in $S^+(a, r)$, and g_1 maps $S^+(a, r)$ into Q. Furthermore, by (2.13),

$$g_1|(a - r, a + r) = \mathrm{id}. \tag{2.16}$$

Now, (2.14) and the fact that $x = \varphi(\xi, y)$ is a solution of (2.12) imply

$$\frac{\partial \xi}{\partial x} \sin\theta - \frac{\partial \xi}{\partial y} \cos\theta = 0. \tag{2.17}$$

Indeed,

$$\begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} = \begin{pmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{pmatrix}^{-1} = \begin{pmatrix} x_\xi & -m \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/x_\xi & m/x_\xi \\ 0 & 1 \end{pmatrix}.$$

Hence, $\xi_y / \xi_x = m = \tan\theta$, which yields (2.17).

2.18. LEMMA: *Let $r_1 > 0$, $I_1 = (a - r_1, a + r_1)$ and $S_1 = S(a, r_1)$ and let ρ be an a.e. positive measurable function in $(0, r_1)$, such that ρ is a.e. locally bounded away from 0 in $(0, r_1)$ and such that $\rho(t) \to 0$ a.e. in $(0, r_1)$ as $t \to 0^+$. Then the mapping $g_2\colon S_1^+ \cup I_1 \to \mathbb{H} \cup I_1$, which is defined by*

$$g_2(\xi, \eta) = (\xi, R(\eta)), \tag{2.19}$$

where

$$R(\eta) = \int_0^{\eta} \rho(t)dt \tag{2.20}$$

is an embedding, is locally quasiconformal in S_1^+ and identity on I_1.

Proof: Since ρ is measurable and a.e. positive in $(0, r_1)$, g_2 is injective, and hence an embedding in S_1^+. Clearly, g_2 is an ACL and hence g_2 is a.e. differentiable in $S_1^+ \cup J$. Since ρ is locally bounded away from 0 in $(0, r_1)$, and $\mu_{g_2} = (1-\rho)/(1+\rho)$ a.e. in S_1^+, it follows that g_2 is locally quasiconformal in S_1^+. Obviously, $g_2|I_1 = \mathrm{id}$.

3. Proof of the main results

3.1. Proof of Theorem 1.3: Let a, f and U be as in the theorem. Set $\theta(z) = \arg\mu(z)/2$. Then θ satisfies the assumptions of Lemma 2.8. Let $r > 0$, $S = S(a, r)$, $I = I(a, r)$, and g_1 be as in Lemma 2.8. Then g_1 is identity on I and maps S^+ homeomorphically into $\mathbb{H}$. By Lemma 2.5, there are measurable functions $\rho\colon (0, r) \to \mathbb{R}_+$ and $M\colon S^+ \to \mathbb{R}$ such that for a.e. $(x, y) \in S^+$,

$$\mu(x, y) = e^{i \arg\mu(x,y)}[1 - \rho(y) \cdot M(x, y)] \tag{3.2}$$

and such that (2.6) and (2.7) hold. Next, choose $r' > 0$ such that $S'^+ \subset g_1(S^+)$, where $S' = S(a, r')$ and let $I' = S' \cap \mathbb{R}$. Choose $r_1 \in (0, r')$ such that for $S_1 = S(a, r_1)$, $^+_1 \subset g_1(S')$.

Now, let $A = \partial g_1/\partial z, B = \partial g_1/\partial \bar{z}$ and

$$C = Ae^{i\theta} - Be^{-i\theta}. \tag{3.3}$$

Then

$$\begin{gathered} A = \frac{1}{2}[\xi_x + 1 - i\xi_y], \quad B = \frac{1}{2}[\xi_x - 1 + i\xi_y], \\ \operatorname{Im} C = \xi_x \sin\theta - \xi_y \cos\theta, \end{gathered} \tag{3.4}$$

and thus, in view of (2.9), C is real.

By (3.3), $|C| \geq |A| - |B| > 0$, and since $|A|^2 - |B|^2 = J_{g_1} > 0$, it follows that $C \neq 0$, and thus, by straightforward computations,

$$e^{i\theta}\Big(\frac{A}{C} + \frac{\bar{B}}{\bar{C}}\Big) = \frac{J_{g_1}}{C\bar{C}} = J_{g_1}|C|^{-2} > 0. \tag{3.5}$$

Let $\mu_1 = g_1 * \mu$ be the deformation of the dilatation μ, which is induced by g_1. By setting in (2.2) the expression for μ which is given in 3.2, one obtains that a.e. in S_1^+

$$\mu_1 = \frac{e^{2i\theta}(1-\rho M)A - B}{\bar{A} - e^{-2i\theta}(1-\rho M)\bar{B}} \circ G = \frac{C - e^{i\theta}\rho M A}{\bar{C} + e^{i\theta}\rho M \bar{B}} \circ G. \tag{3.6}$$

Since $C \neq 0$ and real and $\rho(\eta) \to 0$ a.e. as $\eta \to 0$,

$$\mu_1 = \Big[\frac{C}{\bar{C}}(1 - e^{i\theta}\Big(\frac{A}{C} + \frac{\bar{B}}{\bar{C}}\Big)M\rho + \rho \cdot o(\rho))\Big] \circ G \tag{3.7}$$

holds a.e. for a smaller rectangle, which we denote again by $S_1^+, S_1 = S(a, r_1)$. Thus

$$\mu_1(\zeta) = 1 - \rho(\eta)M_1(\zeta) \quad \text{a.e. in } S_1, \tag{3.8}$$

where, in view of (3.5),

$$M_1 = MJ_{g_1}|C|^{-2} + o(\rho)$$

is a real-valued measurable function, which satisfies

$$\frac{1}{R_1} \leq |M_1| \leq R_1 \tag{3.9}$$

for some $R_1 > 0$.

Next, let g_2 be as in Lemma 2.18 and let $\mu_2 = g_2 * \mu_1$ be the deformation of μ_1, which is induced by $g_2|S_1^+$. Since μ_1 is locally bounded in S_1^+ and g_2 is locally quasiconformal, it follows, by Lemma 2.3, that μ_2 is measurable and locally bounded in $g_2(S_1^+)$.

We now compute and estimate μ_2. Let $A = (g_2)_\zeta$ and $B = (g_2)_{\bar{\zeta}}$. Then

$$A = \frac{1}{2} + \frac{\rho(\eta)}{2} \quad \text{and} \quad B = \frac{1}{2} - \frac{\rho(\eta)}{2}$$

a.e. in S_1^+. To obtain μ_2 we plug $\mu_1 = 1 - \rho(\eta)M_1(\zeta)$ of (3.8), in (2.2), and get

$$\mu_2 \circ g_2 = \frac{(1+\rho)(1-\rho M_1) - 1 + \rho}{1 + \rho - (1-\rho)(1-\rho M_1)} = \frac{2 - M_1}{2 + M_1} + o(\rho), \quad \text{a.e. in} S_2^+$$

as $\rho \to 0$. In view of (3.9) and the fact that $\rho(\eta) \to 0$ a.e. as $\eta \to 0^+$, there is $r_2 \in (0, r_1)$ such that for $S_2 = S(a, r_2), S_2^+ \subset g_2(S_1^+)$, and such that $|\mu_2|_\infty < 1$ in S_2^+. Then a quasiconformal mapping $\varphi : S_2^+ \to \mathbb{C}$ with $\mu_\varphi = \mu_2$ a.e. in S_2^+ exists.

Let $V = g_1^{-1} \circ g_2^{-1}(S_2^+)$ and let $J = \overline{V} \cap \mathbb{R}$. Then $J = \overline{I}_2$ where $I_2 = I(a, r_2)$. Also, by Lemma 2.3, for a.e. $z \in V$,

$$\mu_f(z) = \mu(z) = \mu_{\varphi \circ g_2 \circ g_1}.$$

Hence, there exists a conformal mapping h of $\varphi \circ g_2 \circ g_1(V) = \varphi(S_2^+)$ onto $f(V)$.

The cluster set of f on J is the same as the cluster set of $h \circ \varphi$ on I_2. The latter meets $\mathbb{C}$ since $h \circ \varphi$ is quasiconformal. It thus follows that the cluster set of f on $\mathbb{R}$ meets $\mathbb{C}$, and therefore $f(\mathbb{H}) \neq \mathbb{C}$, and (i) follows.

Suppose now that $f(\mathbb{H})$ is a Jordan domain. Then $f(V)$ is a Jordan domain, and since $h \circ \varphi$ is quasiconformal, it has a homeomorphic extension on $S_2^+ \cup J$. Now $g_1|I_1 = g_2|I_2 = \mathrm{id}$, hence $f = (h \circ \varphi) \circ (g_2 \circ g_1)$ has a homeomorphic extension on $U \cap J$.

The same argument can be applied to any other point in $U \cap \mathbb{R}$, from which we can conclude that if $f(\mathbb{H})$ is a Jordan domain, then f has a homeomorphic extension on $\mathbb{H} \cup (U \cap \mathbb{R})$, and thus (ii) follows.

3.2. Proof of Theorem 1.4: Let E, μ and f be as in the theorem. If $E = \emptyset$, then f is quasiconformal in H, and the assertion follows by the classical qc theory.

Suppose that $E \neq \emptyset$. Then E is a countable union of disjoint open intervals I. Fix one of these intervals, say I. Then, by Theorem 1.3, every point in I has an interval, which is contained in I where f extends homeomorphically, and since $f(H)$ is a Jordan domain, f has a homeomorphic extension on $\mathbb{H} \cup I$, and consequently on $\mathbb{H} \cup E$.

We now show that f has a continuous extension on $\mathbb{H} \setminus E$. Let b be a point in $(\mathbb{R} \cup \{\infty\}) \setminus E$. Since $\overline{\mathbb{C}}$, and hence $\overline{f(\mathbb{H})}$, is compact, the cluster set of f at b is not non-empty. It suffices to show that it is non-degenerate. Suppose that it is not non-degenerate. Then there are sequences x_n and x_n' in $\mathbb{H}$ which tend to b such that $f(x_n)$ and $f(x_n')$ tend to different limits, say w and w', respectively. Let K and K' be disjoint arcs in $\overline{f(\mathbb{H})}$, the first one containing all points $f(x_n)$ and the other one all $f(x_n')$ and such that each of these two arcs lies in $f(\mathbb{H})$ except for its end point, which is w and w' respectively. Now choose a domain D with $\overline{D} \subset \mathbb{H} \cup \{b\}$, which contains $f^{-1}(K) \cup f^{-1}(K')$, and such that the family Γ of all paths in D which join $f^{-1}(K)$ and $f^{-1}(K')$ has infinite modulus, $M(\Gamma) = \infty$. The path family $f(\Gamma)$ is a subfamily of all paths joining K and K'

in $\overline{\mathbb{C}}$, and the latter has finite modulus, since K and K' are disjoint continua. Therefore the modulus $M(f(\Gamma))$ of $f(\Gamma)$ is finite, which is impossible since f is quasiconformal in D and $M(\Gamma) = \infty$.

Suppose now that J is a non-degenerate component of $\mathbb{R} \cup \{\infty\} \setminus E$. Choose a domain $D, D \subset \mathbb{H}$ such that $J \subset \partial D \cap \mathbb{R}$. Since f is quasiconformal in D, and $f(J)$ is a free boundary arc in $f(D)$, it follows that f has a homeomorphic extension on the union of H and the interior (in the R topology) of J, and thus (in view of the existence of a continuous extension at each boundary point) on $\mathbb{H} \cup J$.

Recalling that $f(D)$ is a Jordan domain, we obtain that f has a homeomorphic extension on $\mathbb{H}$.

4. Examples

4.1. In the following example, f and μ satisfy the conditions of Theorem 1.3 with $E = \mathbb{R}$. Here f maps $\mathbb{H}$ homeomorphically onto itself, and it has a homeomorphic extension onto $\mathbb{H} \cup \mathbb{R}$, as asserted in the theorem, but not onto $\overline{\mathbb{H}}$. More precisely, here $f(\mathbb{H} \cup \mathbb{R}) = \mathbb{H} \cup \mathbb{R}_+ \cup \{\infty\}$, and the cluster set of f at ∞ is the closed ray $[-\infty, 0]$.

The mapping f in this example is defined by

$$f(x,y) = e^{x+2i\arctan y^2}, \quad y > 0.$$

Then the complex dilatation of f is given by

$$\mu(x,y) = \frac{y^2 - 4y + 1}{y^2 + 4y + 1}$$

and

$$1 - |\mu| = 1 - \mu = \frac{4y}{y^2 + 4y + 1}.$$

Therefore $1 - |\mu(x,y)| = \rho(y) \cdot M(x,y)$, where

$$\rho(y) = 4y \quad \text{and} \quad M(x,y) = \frac{1}{y^2 + 4y + 1}.$$

Then condition (a) holds at every point $x \in \mathbb{R}$ with $C = 1$ in (1.2). Also $\arg \mu = 0$, and thus conditions (b) and (c) hold too.

4.2. In the following example f is a μ-homeomorphism in $\mathbb{H}$ with μ satisfying conditions (a) and (b) in Theorem 1.3, but not condition (c). In this example f maps $\mathbb{H}$ onto $\mathbb{C}$.

The mapping f in this example is defined by

$$f(x,y) = x + i\log y, \quad y > 0.$$

Then the complex dilatation of f is given by

$$\mu(x,y) = \frac{y-1}{y+1},$$

and

$$1 - |\mu| = \frac{2y}{y+1} = \rho(y) \cdot M(x,y)$$

where

$$\rho(y) = 2y,\ M(x,y) = \frac{1}{y+1},\ \text{and}\ \arg\mu(x,y) \to \pi \text{ as } y \to 0^+.$$

Thus (a) and (b) hold, and (c) fails.

4.3. In the following example f is a μ-homeomorphism in $\mathbb{H}$ with a complex dilatation μ satisfying conditions (a) and (b) in Theorem 1.3, but not condition (c). Here f maps $\mathbb{H}$ onto itself, but f has no injective, and hence no homeomorphic extension on $\mathbb{H} \cup \mathbb{R}$.

The mapping f in this example is defined by

$$f(x,y) = xy + iy, \quad y > 0.$$

Then

$$\mu(x,y) = \frac{-1+y+ix}{1+y-ix},$$

$$1 - |\mu| = \frac{1-|\mu|^2}{1+|\mu|} = \frac{4y}{(y+1)^2+x^2} \cdot \frac{1}{1+|\mu|} = \rho(y) \cdot M(x,y)$$

where

$$\rho(y) = 4y \quad \text{and} \quad M(x,y) = \frac{4y}{(y+1)^2+x^2} \cdot \frac{1}{1+|\mu|}.$$

Note that μ satisfies conditions (a) and (b) in Theorem 1.3, and $\mu(x,y) \to -1$ as $y \to 0^+$, and thus condition (c) fails. Here for every $x \in \mathbb{R}$, $f(x,y) \to 0$ as $y \to 0^+$.

4.4. In the following example f is a μ-homeomorphism in $\mathbb{H}$ with μ satisfying conditions (b) and (c) in Theorem 1.3, but not condition (a). Here f maps $\mathbb{H}$ onto itself, but f has no limit at $x = 0$, and hence has no homeomorphic extension on $\mathbb{H} \cup \mathbb{R}$.

The mapping f in this example is defined as follows. For $y > 0$, let $f(x, y) = u(x, y) + iy^2$, where for $x \geq 0$

$$u(x,y) = \begin{cases} x/y & \text{if } x \leq y/2, \\ x/y + \frac{y-1}{2}(x/y - 1/2)^2 & \text{if } y/2 \leq x \leq 3y/2, \\ 1 + x - y & \text{if } x > 3y/2, \end{cases}$$

and for $y > 0$ and $x < 0$

$$u(x, y) = -u(-x, y).$$

By checking the values of $u(x, y)$ and its partial derivatives at each of the five sectors which are defined by $|x| < y/2$, $y/2 < |x| < 3y/2$ and $|x| > 3y/2$ and their limits at points on the lines $|x| = y/2$ and $|x| = 3y/2$, $y > 0$, it is not hard to verify that f is a C^1-homeomorphism in $\mathbb{H}$.

One can compute $\arg \mu(z)$ and $\partial \arg \mu(z)/\partial y$ in $\mathbb{H}$ (say with the aid of a tool like MAPLE) and verify that there is $r > 0$ such that conditions (b) and (c) hold in each of the rectangles $S^+(x, r), x \in \mathbb{R}$.

Obviously, for $x \in \mathbb{R}$, $|\mu(x, y)| \to 1$ as $y \to 0^+$. Simple computations show that near every point $x > 0$ in $\mathbb{R}$, and therefore near every point $x < 0$ in $\mathbb{R}$,

$$1 - |\mu(x, y)|^2 = \frac{8y}{(1 + 2y)^2 + 1}.$$

Hence, near each of these points

$$1 - |\mu(x, y)| = O(y), \quad \text{as } y \to 0^+.$$

By similar computations one obtains that for points (x, y) in the middle sector $|x| < y/2$,

$$1 - |\mu(x, y)|^2 = \frac{8y^4}{x^2 + y^2(1 + 2y^2)^2},$$

and thus

$$1 - |\mu(x, y)| = o(y) \quad \text{as } y \to 0^+$$

in the middle sector. Therefore, condition (a) fails at the point 0. Note that condition (a) holds near every other point in $\mathbb{R}$.

As noted above, f maps $\mathbb{H}$ onto itself. Clearly, the cluster set of f at 0 is the line segment $|x| \leq 1, y = 0$. Hence f has no homeomorphic extension on $\mathbb{H} \cup \mathbb{R}$.

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