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INVERTIBILITY OF "LARGE" SUBMATRICES WITH APPLICATIONS TO THE GEOMETRY OF BANACH SPACES AND HARMONIC ANALYSIS

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ABSTRACT

The main problem investigated in this paper is that of restricted invertibility of linear operators acting on finite dimensional l_p-spaces. Our initial motivation to study such questions lies in their applications. The results obtained below enable us to complete earlier work on the structure of complemented subspaces of L_p-spaces which have extremal euclidean distance.

Let A be a real $n \times n$ matrix considered as a linear operator on l_p^n; $1 \leqq p \leqq \infty$. By restricted invertibility of A, we mean the existence of a subset σ of $\{1, 2, \ldots, n\}$ such that $|\sigma| \sim n$ and A acts as an isomorphism when restricted to the linear span of the unit vectors $\{e_i\}_{i \in \sigma}$. There are various conditions under which this property holds. For instance, if the norm $\|A\|_p$ of A is bounded by a constant independent of n and the diagonal of A is the identity matrix, then there exists an index set σ, $|\sigma| \sim n$, for which $R_\sigma A_{|[e_i]_{i \in \sigma}}$ has a bounded inverse (R_σ stands for the restriction map). This is achieved by simply constructing the set σ so that $\|R_\sigma (A - I) R_\sigma\|_p < \frac{1}{2}$.

The case $p = 2$ is of particular interest. Although the problem is purely Hilbertian, the proofs involve besides the space l_2 also the space l_1. The methods are probabilistic and combinatorial. Crucial use is made of Grothendieck's theorem.

The paper also contains a nice application to the behavior of the trigonometric system on sets of positive measure, generalizing results on harmonic density. Given a subset B of the circle $\mathbf{T}$ of positive Lebesgue measure, there exists a subset Λ of the integers $\mathbf{Z}$ of positive density $\operatorname{dens} \Lambda > 0$ such that

$$\left(\int_B |f|^2 du \right)^{1/2} \geqq c \|f\|_2,$$

whenever the support of the Fourier transform $\hat{f}$ of f lies in Λ. The matrices involved here are Laurent matrices.

The problem of restricted invertibility is meaningful beyond the class of l_p-spaces, as is shown in a separate section. However, most of the paper uses specific l_p-techniques and complete results are obtained only in the context of l_p-spaces.

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0. Introduction

The purpose of this paper is to show that, for certain classes of matrices acting as bounded linear operators on euclidean spaces or on more general Banach spaces, it is possible to find "large" submatrices which are invertible. In the present context, invertibility is not considered in an algebraic sense but it rather means that the inverse of the submatrix has a norm bounded by a constant independent of the dimension of the underlying space. Before elaborating on the precise meaning assigned to the expression "large submatrix", we would like to present two examples which illustrate well the concepts discussed in the sequel.

Let $\{e_i\}_{i=1}^n$ denote the unit vector basis of the n-dimensional euclidean space l_2^n and define the operator $S_n: l_2^n \to l_2^n$, by setting $S_n e_i = e_{i+1}$; $1 \leqq i < n$, and $S_n e_n = 0$. The operators $\{S_n\}_{n=1}^\infty$ all have norm one, are nilpotent and, clearly, they are not invertible even in a purely algebraic sense. However, by deleting the last row of the matrix representing S_n, i.e. by restricting S_n to the linear span $[e_i]_{i=1}^{n-1}$ of the first $n-1$ unit vectors, we obtain an isometry whose inverse exists and has norm equal to one.

Even more interesting is the example of the operator $T_n: l_2^n \to l_2^n$, defined by $T_n = I + S_n$. Clearly, $\| T_n \| \leqq 2$ and the spectrum $\sigma(T_n)$ of T_n consists of the point $\lambda = 1$ only. It follows that T_n^{-1} exists but, as a simple computation shows, $\| T_n^{-1} \| \geqq \sqrt{n/2}$, for all n (simply, apply T_n to the vector $x = \sum_{j=1}^n (-1)^{j-1} e_j \in l_2^n$). This situation is not satisfactory from an asymptotical point of view since $\| T_n^{-1} \| \to \infty$, as $n \to \infty$.

Consider now the matrix corresponding to T_n and delete the even-indexed rows and columns. The remaining matrix is actually the identity restricted to the linear span of the odd-indexed unit vectors and thus its inverse has norm equal to one.

The important fact about both these examples is that the well invertible submatrix has rank proportional to the original rank, and the proportion remains fixed in a manner independent of n.

It turns out that this statement is true in general. We prove below that there exists a constant $c = c(M) > 0$ so that, whenever $T: l_2^n \to l_2^n$ is a linear operator of norm $\leqq M$ with $\| Te_i \|_2 = 1$, for all n, then there exists a subset σ of $\{1, 2, \ldots, n\}$ of cardinality $|\sigma| \geqq cn$ for which

$$\left\| \sum_{j \in \sigma} a_j T e_j \right\|_2 \geqq c \left(\sum_{j \in \sigma} |a_j|^2 \right)^{1/2},$$

for all $\{a_j\}_{j \in \sigma}$. In the case when the condition $\| Te_i \|_2 = 1$; $1 \leqq i \leqq n$, is replaced by the existence of 1's on the diagonal of T, one actually obtains a square submatrix of rank $\geqq cn$ which is well invertible.

The above result combined with a theorem of Ruzsa [26] yields an application to harmonic analysis: it follows that every subset B of the circle $\mathbf{T}$, which has positive measure, is a set of isomorphism in L_2, for some family $\{e^{inx}\}_{n\in\Lambda}$ of characters with dens $\Lambda > 0$ in the sense that

$$\| f\chi_B \|_2 \geqq d \| f \|_2,$$

for some constant $d>0$ and every $f \in L_2(\mathbf{T})$, whose Fourier transform is supported by Λ. Surprisingly, a similar statement for $p>2$ fails to be true. Moreover, the subsets B of $\mathbf{T}$, which have this property for some $p>2$, are precisely those for which $\mathbf{T}$ can be covered, up to a negligible set, by a finite number of translates of B.

Another application to infinite dimensional Hilbert spaces consists of the assertion that every Hilbertian system of normalized vectors in a Hilbert space contains a subset of positive upper density which is also Besselian and, therefore, equivalent to an orthonormal system.

Similar invertibility results hold for matrices T acting on l_p^n-spaces, $1 \leqq p \leqq \infty$ as bounded operators provided they have 1's on the diagonal. In the case $p>2$, this condition can be replaced by the requirement that $\| Te_i \|_p = 1$; $1 \leqq i \leqq n$. For $1 \leqq p < 2$, the condition $\| Te_i \|_p = 1$; $1 \leqq i \leqq n$, does not even necessarily imply that T has rank proportional to n. This part of the paper is probably the most difficult.

The invertibility theorem in the case $1 < p \neq 2 < \infty$ has some immediate applications to the geometry of Banach spaces. Namely, it yields the solution to two problems raised by W. B. Johnson and G. Schechtman in [13]. More precisely, it is proved below that any well-complemented n-dimensional subspace of $L_p(0,1)$; $1 < p \neq 2$ whose euclidean distance is maximal (i.e. $\geqq cn^{|1/p-1/2|}$, for some constant $c>0$) contains a well-complemented subspace of dimension k proportional to n which is well isomorphic to l_p^k. Furthermore, it is also shown that any system $\{f_i\}_{i=1}^n$ of functions in $L_p(0,1)$; $1<p<\infty$, which is well equivalent to the unit vector basis of l_p^n, contains in turn a subsystem $\{f_i\}_{i\in\sigma}$ with $|\sigma|$ proportional to n whose linear span is well complemented in $L_p(0,1)$.

In addition to invertibility theorems for "large" submatrices of matrices that act as "bounded" operators on l_p^n, we obtain some unexpected results for "unbounded" operators, too. The extremal case of a linear operator $T: l_\infty^n \to l_\infty^n$ with $\| T \| \leqq M$, for some $M < \infty$, and with 1's on the diagonal, illustrates well this case. The columns of the corresponding matrix of such an operator T are elements of norm $\leqq M$ in l_1^n. Thus, by applying a well-known combinatorial result from [4] or [13], one can find a doubly-stochastic submatrix S_k of $S = T - I$ of size $k \times k$ with $k \sim n/M^2$. In fact, one can even ensure that S_k has norm $< \frac{1}{2}$ in both l_1^k and l_∞^k. This would imply, by an immediate interpolation argument, that $T_k = I + S_k$ is a $k \times k$-submatrix of T whose inverse is of norm $\leqq 2$ in every l_r^k-space, $1 \leqq r \leqq \infty$.

It appears that a somewhat similar result holds for any "bounded" linear operator T on l_p^n; $p > 2$, with 1's on the diagonal or with $\| Te_i \|_p = 1$, for all i. It is proved in the sequel that such an operator is invertible in the above sense (i.e. when it is restricted to a subset of the unit vectors whose cardinality is a fixed percentage of n) not only in l_p^n but also in l_r^n; $1 \leqq r < p$, in spite of the fact that it need not be well bounded in all these spaces. For $1 < p < 2$, exactly the same type of result holds whenever T has 1's on the diagonal and $1 \leqq r \leqq 2$.

It is perhaps interesting to point out that the nature of the invertibility is not necessarily the same for the whole range $1 \leqq r \leqq 2$. For $p \leqq r \leqq 2$, one actually obtains a stronger form of invertibility, namely, a square submatrix which is invertible. In the range $1 \leqq r < p$, as examples below show, this need not be true.

The paper also contains a generalization of the results obtained for matrices acting as "bounded" operators on l_p^n-spaces to the case of operators on spaces with an unconditional basis. In this case, however, we are able to prove only the existence of well invertible submatrices of size $k \times k$ with $k = n^{1-\varepsilon}$, for any $\varepsilon > 0$ given in advance. The next section contains some results of a non-operator nature. We conclude with some polynomial estimates related to some results from [15].

The results presented throughout the paper apply to real as well as to complex spaces and in most of the cases there is no difference whatsoever. The only exception occurs in Section 5 which is based on J. Elton [8] and, therefore, is valid only for real spaces. However, by using A. Pajor [23] instead of [8], one can also extend these results to the complex case.

1. Operators on euclidean spaces

In the first part of this section, we present a theorem on the invertibility of "large" submatrices of matrices with "large" rows which act as bounded linear operators on finite dimensional euclidean spaces.

In the second part, we prove a different version which applies to matrices with "large" diagonal. Actually, this result implies the one for matrices with "large" rows and, in some sense, is more satisfactory since it produces an invertible submatrix of square type. However, the dependence between the rank of the invertible submatrix and that of the original one is best possible in the former and much worse in the latter.

PROPOSITION 1.1. *Let T: $l_2^n \to l_2^n$ be a linear operator such that $\| Te_i \|_2 = 1$; $1 \leqq i \leqq n$. Then*

$$\operatorname{rank} T \geqq n / \| T \|^2.$$

PROOF. Put $k = \operatorname{rank} T$. Then, since the Hilbert–Schmidt norm $\| T \|_{HS}$ of T can be estimated by $\| T \|_{HS} \leqq \| T \| \sqrt{k}$, we get that

$$n = \sum_{i=1}^{n} \| Te_i \|_2^2 = \| T \|_{HS}^2 \leqq \| T \|^2 k,$$

which completes the proof. □

REMARK. The estimate above is sharp. Indeed, if $n = k \cdot m$, for some integers k and m, and T: $l_2^n \to l_2^n$ is defined by

$$Te_{i+jk} = e_i \,; \qquad 1 \leqq i \leqq k, \quad 0 \leqq j < m,$$

then, as is readily verified, $\operatorname{rank} T = k$ and $\| T \| = \sqrt{m}$, i.e.

$$\operatorname{rank} T = n / \| T \|^2.$$

This observation should be compared with the estimate obtained for $|\sigma|$ in the statement of our next result.

THEOREM 1.2. *There is a constant $c > 0$ so that, whenever T: $l_2^n \to l_2^n$ is a linear operator for which $\| Te_i \|_2 = 1$; $1 \leqq i \leqq n$, then there exists a subset σ of $\{1, 2, \ldots, n\}$ of cardinality $|\sigma| \geqq cn / \| T \|^2$ so that*

$$\left\| \sum_{j \in \sigma} a_j Te_j \right\|_2 \geqq c \left(\sum_{j \in \sigma} | a_j |^2 \right)^{1/2},$$

for any choice of scalars $\{a_j\}_{j \in \sigma}$.

The proof requires some preliminary lemmas. The first consists of an inequality of Bernstein (see e.g. [2]) which is quite well known in a more general form than it is stated here.

LEMMA 1.3. *Fix $0 < \delta < 1$ and an integer n, and let $\{\xi_i\}_{i=1}^n$ be a sequence of independent random variables of mean δ over some probability space (Ω, Σ, μ) which take only the values 0 and 1. Then the deviation*

$$D_\gamma = \left\{ \omega \in \Omega ; \left| \sum_{i=1}^{n} \xi_i(\omega) - \delta n \right| \geqq \gamma \right\}$$

satisfies

$$\mu(D_\gamma) \leqq 2e^{-\gamma^2/(2\delta(1-\delta)n+2\gamma/3)}.$$

In particular,

$$\mu(D_{\delta n/2}) \leqq 2e^{-\delta n/10}.$$

The next lemma will be proved by a probabilistic selection.

LEMMA 1.4. *There exists a constant $c_1 > 0$ so that, whenever $T: l_2^n \to l_2^n$ is a linear operator for which $\|Te_i\|_2 = 1$; $1 \leqq i \leqq n$, then there exists a subset σ_1 of $\{1,2,\dots,n\}$ of cardinality $|\sigma_1| \geqq c_1 n/\|T\|^2$ such that*

$$\|P_{[Te_j]_{j\in\sigma_1\sim\{i\}}}(Te_i)\|_2 < 1/\sqrt{2}; \qquad i \in \sigma_1,$$

where $P_{[Te_j]_{j\in\tau}}$ denotes the orthogonal projection from l_2^n onto $[Te_j]_{j\in\tau}$.

PROOF. Take $\delta = 1/8\|T\|^2$ and let $\{\xi_i\}_{i=1}^n$ be a sequence of independent random variables of mean δ over a probability space (Ω, Σ, μ), taking only the values 0 and 1. For each $\omega \in \Omega$, put

$$\sigma(\omega) = \{1 \leqq j \leqq n\,;\ \xi_j(\omega) = 1\}.$$

The variables $\{\xi_i\}_{i=1}^n$ will act as selectors and the set σ_1 will be, essentially speaking, one of the sets $\sigma(\omega)$, for a suitable choice of $\omega \in \Omega$.

Put $x_i = Te_i$; $1 \leqq i \leqq n$, and notice that

$$\begin{aligned}
\int_\Omega \sum_{i=1}^n \xi_i(\omega)\|P_{[\xi_j(\omega)x_j]_{j\neq i}}(x_i)\|_2^2\,d\mu &= \delta\int_\Omega \sum_{i=1}^n \|P_{[\xi_j(\omega)x_j]_{j\neq i}}(x_i)\|_2^2\,d\mu \\
&\leqq \delta\int_\Omega \sum_{i=1}^n \|P_{[\xi_j(\omega)x_j]_{j=1}^n}T(e_i)\|_2^2\,d\mu \\
&= \delta\int_\Omega \|P_{[\xi_j(\omega)x_j]_{j=1}^n}T\|_{\mathrm{HS}}^2\,d\mu \\
&\leqq \delta\|T\|^2\int_\Omega \sum_{i=1}^n \xi_i(\omega)\,d\mu \\
&= \delta^2 n\|T\|^2.
\end{aligned}$$

In particular,

$$\int_{\Omega\sim D_{\delta n/2}} \sum_{i=1}^n \xi_i(\omega)\|P_{[\xi_j(\omega)x_j]_{j\neq i}}(x_i)\|_2^2\,d\mu \leqq \delta^2 n\|T\|^2$$

which implies that there exists a point $\omega_0 \in \Omega \sim D_{\delta n/2}$ such that

$$\sum_{i \in \sigma(\omega_0)} \| P_{[x_j]_{j \in \sigma(\omega_0) \sim \{i\}}}(x_i) \|^2 \leqq \delta^2 n \| T \|^2$$

and

$$|\sigma(\omega_0)| = \sum_{j=1}^{n} \xi_j(\omega_0) \geqq \delta n/2.$$

Put

$$\sigma_1 = \{ i \in \sigma(\omega_0); \| P_{[x_j]_{j \in \sigma(\omega_0) \sim \{i\}}}(x_i) \|_2 < 2 \| T \| \sqrt{\delta} \}$$

and observe that

$$4 \| T \|^2 \delta \, | \sigma(\omega_0) \sim \sigma_1 | \leqq \delta^2 n \| T \|^2 \leqq 2 \| T \|^2 \delta \, | \sigma(\omega_0) |,$$

i.e.

$$|\sigma_1| \geqq \delta n/4.$$

In view of the choice of δ made in the beginning of the proof, we conclude that

$$\| P_{[x_j]_{j \in \sigma_1 \sim \{i\}}}(x_i) \|_2 < 1/\sqrt{2},$$

for $i \in \sigma_1$, and $|\sigma_1| \geqq n/32 \| T \|^2$. □

THEOREM 1.5. *There exists a constant $c_2 > 0$ such that, whenever $T: l_2^n \to l_2^n$ is a linear operator for which $\| Te_i \|_2 = 1$; $1 \leqq i \leqq n$, then there exists a subset σ_2 of $\{1, 2, \ldots, n\}$ of cardinality $|\sigma_2| \geqq c_2 n / \| T \|^2$ so that*

$$\left\| \sum_{j \in \sigma_2} a_j T e_j \right\|_2 \geqq c_2 \sum_{j \in \sigma_2} |a_j| / \sqrt{|\sigma_2|},$$

for all $\{a_j\}_{j \in \sigma_2}$.

PROOF. Let c_1 and σ_1 be given by Lemma 1.4, and put

$$u_i' = x_i - P_{[x_j]_{j \in \sigma_1 \sim \{i\}}}(x_i); \qquad i \in \sigma_1.$$

Then $\langle x_i, u_j' \rangle = 0$, for $i, j \in \sigma_1$ and $i \neq j$, and also (by the choice of σ_1)

$$\langle x_i, u_i' \rangle = 1 - \| P_{[x_j]_{j \in \sigma_1 \sim \{i\}}}(x_i) \|_2^2 > 1/2; \qquad i \in \sigma_1.$$

It follows that $1 \geqq \| u_i' \|_2 \geqq \frac{1}{2}$; $i \in \sigma_1$, and thus the vectors

$$u_i = u_i' / \| u_i' \|_2$$

satisfy $\langle x_i, u_j \rangle = 0$, for $i, j \in \sigma_1$ and $i \neq j$, and $\langle x_i, u_i \rangle > \frac{1}{2}$, $i \in \sigma_1$.

Consider now the sets of tuples of signs

$$\mathscr{E} = \left\{ (\varepsilon_i)_{i\in\sigma_1} \in \{-1,+1\}^{|\sigma_1|};\ \left\| \sum_{i\in\sigma_1} \varepsilon_i u_i \right\|_2 \leqq 2\sqrt{|\sigma_1|} \right\}.$$

Since

$$\int \left\| \sum_{i\in\sigma_1} \varepsilon_i u_i \right\|_2^2 d\varepsilon = \sum_{i\in\sigma_1} \| u_i \|_2^2 = |\sigma_1|,$$

it follows that

$$|\mathscr{E}| \geqq 3\cdot 2^{|\sigma_1|}/4.$$

By a well-known result of Sauer [27] and S. Shelah [29] (see also [31]), if k satisfies

$$|\mathscr{E}| > \sum_{i=0}^{k-1} \binom{|\sigma_1|}{i}$$

then there exists a subset σ_2 of σ_1 of cardinality k such that, for each tuple $(\varepsilon_i)_{i\in\sigma_2}$, there is an extension $(\varepsilon_i)_{i\in\sigma_1}$ which belongs to $\mathscr{E}$. In our case, we can ensure that $k \geqq |\sigma_1|/2$ and thus

$$|\sigma_2| \geqq c_1 n/2\|T\|^2.$$

In order to complete the proof, for any choice of $\{a_j\}_{j\in\sigma_2}$ write $a_j = b_j + ic_j$ with b_j and c_j real numbers, for all $j\in\sigma_2$, and select signs $(\theta'_j)_{j\in\sigma_2}$ and $(\theta''_j)_{j\in\sigma_2}$ such that $b_j\theta'_j = |b_j|$ and $c_j\theta''_j = |c_j|$; $j\in\sigma_2$. Then let $(\varepsilon'_j)_{j\in\sigma_1}$ and $(\varepsilon''_j)_{j\in\sigma_1}$ be extensions of $(\theta'_j)_{j\in\sigma_2}$, respectively $(\theta''_j)_{j\in\sigma_2}$, which belong to $\mathscr{E}$. It follows that

$$\begin{aligned} 4\sqrt{|\sigma_1|}\cdot\left\| \sum_{j\in\sigma_2} a_j x_j \right\|_2 &\geqq \left| \left\langle \sum_{j\in\sigma_2} a_j x_j,\ \sum_{j\in\sigma_1} (\varepsilon'_j - i\varepsilon''_j) u_j \right\rangle \right| \\ &= \left| \sum_{j\in\sigma_2} ((|b_j| + |c_j|) + i(c_j\varepsilon'_j - b_j\varepsilon''_j))\langle x_j, u_j\rangle \right| \\ &\geqq \sum_{j\in\sigma_2} (|b_j| + |c_j|)\langle x_j, u_j\rangle \\ &> \sum_{j\in\sigma_2} |a_j|/2, \end{aligned}$$

which completes the proof. □

We shall present now two versions for the completion of the proof of Theorem 1.2. After the first draft of the paper was written up, N. J. Kalton suggested to replace the exhaustion argument, appearing below as the first version, by a Maurey–Nikishin factorization type of argument. This is indeed possible and would shorten the proof. However, the proof of the factorization theorem given in [22] is quite complicated and in order to keep the paper as self-contained as possible, we prefer to give here a direct factorization argument which is adapted to the requirement of this proof and is very elementary.

PROOF OF THEOREM 1.2 (first version — an exhaustion argument). Let $T: l_2^n \to l_2^n$ be a linear operator for which $x_i = Te_i$; $1 \leqq i \leqq n$, have all norms equal to one. Let x_2 and σ_2 be given by Lemma 1.5. The proof of 1.2 will be completed once we establish the existence of a subset σ of σ_2 of cardinality $|\sigma| \geqq |\sigma_2|/2$ such that

$$\left\| \sum_{j\in\sigma} a_j x_j \right\|_2 \geqq c_2 \left(\sum_{j\in\sigma} |a_j|^2 \right)^{1/2} \Big/ 4,$$

for any choice of $\{a_j\}_{j\in\sigma}$.

Suppose that this assertion is false. Put $\tau_1 = \sigma_2$ and construct a vector $y_1 = \Sigma_{j\in\tau_1} b_{1,j} x_j$ such that $\|y_1\|_2 < c_2/4$ but $\Sigma_{j\in\tau_1} |b_{1,j}|^2 = 1$.

Assume now that we have already constructed subsets $\tau_1 \supset \tau_2 \supset \cdots \supset \tau_l$ with $|\tau_l| \geqq |\sigma_2|/2$ and vectors $\{y_i\}_{i=1}^l$ such that $y_i = \Sigma_{j\in\tau_i} b_{i,j} x_j$, $\|y_i\|_2 < c_2/4$ and $\Sigma_{j\in\tau_i} |b_{i,j}|^2 = 1$, for $1 \leqq i \leqq l$. Consider then the set

$$\tau_{l+1} = \left\{ j \in \tau_l \,;\, \sum_{i=1}^{l} |b_{i,j}|^2 < 1 \right\}$$

and if $|\tau_{l+1}| < |\sigma_2|/2$ stop the procedure. On the other hand, if $|\tau_{l+1}| \geqq |\sigma_2|/2$ then there exists a vector

$$y_{l+1} = \sum_{j\in\tau_{l+1}} b_{l+1,j} x_j$$

such that $\|y_{l+1}\|_2 < c_2/4$ but $\Sigma_{j\in\tau_{l+1}} |b_{l+1,j}|^2 = 1$.

Suppose that this construction stops after m steps. Then

$$|\tau_{m+1}| < |\sigma_2|/2$$

and, thus, for $j \in \sigma_2 \sim \tau_{m+1}$, we have

$$\sum_{i=1}^{m} |b_{i,j}|^2 \geqq 1$$

with the convention that $b_{i,j} = 0$ for those i and j for which it is not defined (notice that if $j \in \tau_l \sim \tau_{l+1}$, for some $1 \leqq l \leqq m$, then $b_{i,j}$ is defined only for

$1 \leqq i \leqq l$). Hence,

$$m = \sum_{i=1}^{m} \sum_{j \in \tau_i} |b_{i,j}|^2 = \sum_{j \in \sigma_2} \sum_{i=1}^{m} |b_{i,j}|^2$$

$$\geqq \sum_{j \in \sigma_2 - \tau_{m+1}} \sum_{i=1}^{m} |b_{i,j}|^2 \geqq |\sigma_2 \sim \tau_{m+1}|,$$

i.e.

$$m \geqq |\sigma_2|/2.$$

On the other hand, by Lemma 1.5, we have

$$\frac{c_2\sqrt{m}}{4} > \left(\sum_{i=1}^{m} \| y_i \|_2^2\right)^{1/2}$$

$$\geqq \int \left\| \sum_{i=1}^{m} \varepsilon_i y_i \right\|_2 d\varepsilon$$

$$= \int \left\| \sum_{j \in \sigma_2} \left(\sum_{i=1}^{m} \varepsilon_i b_{i,j}\right) x_j \right\|_2 d\varepsilon$$

$$\geqq \frac{c_2}{\sqrt{|\sigma_2|}} \sum_{j \in \sigma_2} \int \left| \sum_{i=1}^{m} \varepsilon_i b_{i,j} \right| d\varepsilon$$

$$\geqq \frac{c_2}{\sqrt{2|\sigma_2|}} \sum_{j \in \sigma_2} \left(\sum_{i=1}^{m} |b_{i,j}|^2\right)^{1/2}.$$

However, the inductive construction described above yields that

$$\sum_{i=1}^{m} |b_{i,j}|^2 \leqq 2,$$

for all $j \in \sigma_2$. Indeed, this is completely clear if $j \in \tau_{m+1}$ while, for $j \in \tau_l - \tau_{l+1}$; $1 \leqq l \leqq m$, we have

$$\sum_{i=1}^{m} |b_{i,j}|^2 = \sum_{i=1}^{l-1} |b_{i,j}|^2 + |b_{l,j}|^2 < 2.$$

It follows that

$$\sqrt{m|\sigma_2|}/2 > \sum_{j \in \sigma_2} \sqrt{2} \cdot \left(\sum_{i=1}^{m} |b_{i,j}|^2\right)^{1/2} \geqq \sum_{j \in \sigma_2} \sum_{i=1}^{m} |b_{i,j}|^2 = m,$$

i.e.

$$|\sigma_2| > 4m.$$

This estimate contradicts, however, the fact that $m \geqq |\sigma_2|/2$. □

PROOF OF THEOREM 1.2 (second version — a factorization argument). Let again T be a linear operator as above and $x_i = Te_i$; $1 \leqq i \leqq n$. By Lemma 1.5, the operator $S\colon X = [x_i]_{i \in \sigma_2} \to l_1^n$, defined by

$$Sx_i = e_i / \sqrt{|\sigma_2|}; \qquad i \in \sigma_2,$$

satisfies $\| S \| \leqq 1/c_2$. The dual S^* of S maps l_∞^n into the Hilbert space X and, thus, its 2-summing norm satisfies

$$\pi_2(S^*) \leqq K_G \| S^* \| \leqq K_G / c_2,$$

where K_G denotes, as usual, the constant of Grothendieck (see e.g. [20] 2.b.7). By Pietsch's factorization theorem [24], there is an operator $U\colon l_2^n \to X$ with $\| U \| \leqq \pi_2(S^*)$ and a diagonal operator $D\colon l_\infty^n \to l_2^n$, defined by $De_i = \lambda_i e_i$; $1 \leqq i \leqq n$, with $\sum_{i=1}^n |\lambda_i|^2 \leqq 1$ so that $S^* = U(D)$. Dualizing this factorization diagram, we conclude that $S = D^*(U^*)$, where $D^* e_i = \lambda_i e_i$; $1 \leqq i \leqq n$. It follows immediately that

$$U^* x_j = e_j / \lambda_j \sqrt{|\sigma_2|}; \qquad j \in \sigma_2.$$

The operator U^* will be a "good" isomorphism on that portion of σ_2 where λ_j's are not too large. To this end, put

$$\sigma = \{ j \in \sigma_2; |\lambda_j| \leqq \sqrt{2/|\sigma_2|} \}$$

and notice that, for any choice of $\{a_j\}_{j \in \sigma}$, we have

$$K_G \Big\| \sum_{j \in \sigma} a_j x_j \Big\|_2 \Big/ c_2 \geqq \Big\| U^* \Big(\sum_{j \in \sigma} a_j x_j \Big) \Big\|_2$$

$$= \Big(\sum_{j \in \sigma} | a_j / \lambda_j \sqrt{|\sigma_2|} |^2 \Big)^{1/2} \geqq \Big(\sum_{j \in \sigma} |a_j|^2 \Big)^{1/2} \Big/ \sqrt{2}.$$

This completes the proof since

$$1 \geqq \sum_{j \in \sigma_2 \sim \sigma} |\lambda_j|^2 \geqq 2 |\sigma_2 \sim \sigma| / |\sigma_2|,$$

i.e.

$$|\sigma| \geqq |\sigma_2|/2. \qquad \square$$

We pass now to the study of matrices acting on finite dimensional euclidean spaces which have 1's on the diagonal.

THEOREM 1.6. *For every $M<\infty$ and $\varepsilon>0$, there exists a constant $c=c(M,\varepsilon)>0$ such that, whenever $n\geqq 1/c$ and $S\colon l_2^n\to l_2^n$ is a linear operator of norm $\|S\|\leqq M$ whose matrix relative to the unit vector basis has* 0's *on the diagonal, then there exists a subset σ of $\{1,2,\dots,n\}$ of cardinality $|\sigma|\geqq cn$ such that*

$$\|R_\sigma S R_\sigma\|<\varepsilon,$$

where R_σ denotes the orthogonal projection from l_2^n onto the linear span of the unit vectors $\{e_i\}_{i\in\sigma}$.

Theorem 1.6 has the following immediate consequence.

COROLLARY 1.7. *For every $M<\infty$ and $\varepsilon>0$, there exists a constant $d=d(M,\varepsilon)>0$ such that, whenever $n\geqq 1/d$ and $T\colon l_2^n\to l_2^n$ is a linear operator of norm $\|T\|\leqq M$ for which the matrix relative to the unit vector basis has* 1's *on the diagonal, then there exists a subset σ of $\{1,2,\dots,n\}$ of cardinality $|\sigma|\geqq dn$ such that $R_\sigma T R_\sigma$ restricted to $R_\sigma l_2^n$ is invertible and its inverse satisfies*

$$\|(R_\sigma T R_\sigma)^{-1}\|<1+\varepsilon.$$

The proof of Theorem 1.6 requires some preliminary results which in view of further use in the sequel, are presented in a form more general than actually needed in this section.

PROPOSITION 1.8. *There exists a constant $A<\infty$ with the property that, for any $1<r\leqq 2$, $0<\delta<1$ and $\delta^{r'}e^2\leqq\gamma\leqq\delta$, where $r'=r/(r-1)$, one can find an integer n_0 such that, whenever $n\geqq n_0$ and $\{\xi_i\}_{i=1}^n$ is a sequence of independent random variables of mean δ over some probability space (Ω,Σ,μ) taking only the values* 0 *and* 1, *then, with $m=[\gamma n]$, we have*

$$\left\|\sum_{i=1}^n c_i\xi_i\right\|_m=\left(\int_\Omega\left|\sum_{i=1}^n c_i\xi_i(\omega)\right|^m d\mu(\omega)\right)^{1/m}$$

$$\leqq A\cdot\left(\frac{m}{\log(\gamma/\delta^{r'})}\right)^{1/r'}\cdot\|c\|_r,$$

for any choice of $c=\sum_{i=1}^n c_ie_i\in l_r^n$ with $c_i\geqq 0$; $1\leqq i\leqq n$.

PROOF. Fix r, δ and γ as above, and take n large enough so that $n^{1/m}\leqq 2$. Then, for any $c=\sum_{i=1}^n c_ie_i\in l_r^n$ with $\|c\|_r=1$, we have

$$\left\|\sum_{i=1}^n c_i\xi_i\right\|_m=\left(\sum_{1\leqq i_1,i_2,\dots,i_m\leqq n} c_{i_1}c_{i_2}\cdots c_{i_m}\int_\Omega \xi_{i_1}(\omega)\xi_{i_2}(\omega)\cdots\xi_{i_m}(\omega)d\mu(\omega)\right)^{1/m}$$

$$=\left(\sum_{1\leqq i_1,i_2,\dots,i_m\leqq n} c_{i_1}c_{i_2}\cdots c_{i_m}\delta^{h(i_1,i_2,\dots,i_m)}\right)^{1/m},$$

where $h(i_1, i_2, \ldots, i_m)$ denotes the number of distinct integers in the tuple $(i_1, i_2, \ldots, i_m)$. By using Hölder's inequality in an obvious way, we conclude that

$$\left\| \sum_{i=1}^{n} c_i \xi_i \right\|_m \leqq \left\| \sum_{i=1}^{n} \tilde{\xi}_i \right\|_m^{1/r'},$$

where $\{\tilde{\xi}_i\}_{i=1}^n$ is a sequence of independent random variables of mean $\delta^{r'}$ taking only the values 0 and 1.

Since, for each $1 \leqq k \leqq n$, we have

$$\mu\left\{\omega \in \Omega;\ \sum_{i=1}^{n} \tilde{\xi}_i(\omega) = k\right\} = \binom{n}{k} \delta^{r'k}(1-\delta^{r'})^{n-k},$$

it follows, by using Stirling's formula, that

$$\left\| \sum_{i=1}^{n} \tilde{\xi}_i \right\|_m \leqq \left(\sum_{k=1}^{n} k^m \binom{n}{k} \delta^{r'k} \right)^{1/m}$$

$$\leqq n^{1/m} \cdot \max_{1 \leqq k \leqq n} k \cdot (en\delta^{r'}/k)^{k/m}$$

$$\leqq 2m \sup_{\lambda > 0} \lambda (e\delta^{r'}/\lambda\gamma)^{\lambda}.$$

However, the supremum on the right-hand side is attained for $\lambda = \lambda_0$ which satisfies

$$\frac{1}{\log(\gamma/\delta^{r'})} \leqq \lambda_0 < \frac{2}{\log(\gamma/\delta^{r'})}.$$

Hence, one can find a numerical constant $1 \leqq A < \infty$ such that

$$\left\| \sum_{i=1}^{n} \tilde{\xi}_i \right\|_m \leqq A \left(\frac{m}{\log(\gamma/\delta^{r'})} \right),$$

which, of course, completes the proof. □

The proof of Theorem 1.6 requires the use of a variant of the so-called decoupling principle.

This principle can be found in literature, mostly for symmetric matrices. For sake of completeness, we give here a proof of the version needed below.

PROPOSITION 1.9 (a decoupling principle). *Let* (Ω, Σ, μ) *be a probability space and* (Ω', Σ', μ') *an independent copy of* (Ω, Σ, μ). *Fix* $0 < \delta < 1$ *and let* $\{\xi_i\}_{i=1}^n$ *be a sequence of independent bounded random variables of mean* δ *over* (Ω, Σ, μ). *Then, for any double sequence of vectors* $\{x_{i,j}\}_{i,j=1}^n$ *in an arbitrary Banach space* X

such that $x_{i,i}=0$; $1\leqq i\leqq n$, *we have*

$$\int_\Omega \left\| \sum_{i,j=1}^n \xi_i(\omega)\xi_j(\omega)x_{ij}\right\| d\mu(\omega) \leqq 20 \int_\Omega \int_{\Omega'} \left\| \sum_{i,j=1}^n \xi_i(\omega)\xi_j(\omega')x_{i,j}\right\| d\mu'(\omega')d\mu(\omega).$$

PROOF. We shall prove first the statement under the assumption that $\{\xi_i\}_{i=1}^n$ are all of mean 0.

Let $\{\eta_i\}_{i=1}^n$ be a sequence of independent random variables of mean $\frac{1}{2}$ over a probability space $(U,\mathcal{U},\nu)$ taking only the values 0 and 1. Then, for each $1\leqq i\neq j\leqq n$, we have

$$\int_U \eta_i(u)(1-\eta_j(u))d\nu(u)=\tfrac{1}{4}.$$

Hence,

$$\begin{aligned} I &= \int_\Omega \left\| \sum_{i,j=1}^n \xi_i(\omega)\xi_j(\omega)x_{i,j}\right\| d\mu(\omega) \\ &= 4\int_\Omega \left\| \sum_{i,j=1}^n \left[\int_U \eta_i(u)(1-\eta_j(u))d\nu(u)\right] \xi_i(\omega)\xi_j(\omega)x_{i,j}\right\| d\mu(\omega) \\ &\leqq 4\int_U \int_\Omega \left\| \sum_{i,j=1}^n \eta_i(u)(1-\eta_j(u))\xi_i(\omega)\xi_j(\omega)x_{i,j}\right\| d\mu(\omega)d\nu(u). \end{aligned}$$

For each $u\in U$, put

$$\sigma(u)=\{1\leqq i\leqq n;\ \eta_i(u)=1\}$$

and note that

$$I\leqq 4\int_U \int_\Omega \left\| \sum_{i\in\sigma(u)} \sum_{j\notin\sigma(u)} \xi_i(\omega)\xi_j(\omega)x_{i,j}\right\| d\mu(\omega)d\nu(u).$$

However, for each fixed $u\in U$, $\{\xi_i\}_{i\in\sigma(u)}$ are independent from $\{\xi_j\}_{j\notin\sigma(u)}$. Hence,

$$I\leqq 4\int_U \int_\Omega \int_{\Omega'} \left\| \sum_{i\in\sigma(u)} \sum_{j\notin\sigma(u)} \xi_i(\omega)\xi_j(\omega')x_{i,j}\right\| d\mu'(\omega')d\mu(\omega)d\nu(u)$$

which implies the existence of a $u_0\in U$ so that, with the notation $\sigma(u_0)=\sigma$, we get

$$I\leqq 4\int_\Omega \int_{\Omega'} \left\| \sum_{i\in\sigma} \sum_{j\notin\sigma} \xi_i(\omega)\xi_j(\omega')x_{i,j}\right\| d\mu'(\omega')d\mu(\omega).$$

On the other hand, since $\{\xi_i\}_{i=1}^n$ are assumed to be of mean 0, by taking the expectation with respect to the subfield generated by $\{\xi_i(\omega)\}_{i\notin\sigma}$ in Σ and

$\{\xi_j(\omega')\}_{j\in\sigma}$ in Σ', we obtain that

$$J=\int_\Omega\int_{\Omega'}\left\|\sum_{i,j=1}^n \xi_i(\omega)\xi_j(\omega')x_{i,j}\right\|d\mu'(\omega')d\mu(\omega)\geqq I/4.$$

We pass now to the general case where $\{\xi_i\}_{i=1}^n$ are assumed to be of mean $\delta>0$. Then

$$I\leqq\int_\Omega\left\|\sum_{i,j=1}^n(\xi_i(\omega)-\delta)(\xi_j(\omega)-\delta)x_{i,j}\right\|d\mu(\omega)+\delta\int_\Omega\left\|\sum_{i,j=1}^n\xi_i(\omega)x_{i,j}\right\|d\mu(\omega)$$
$$+\delta\int_\Omega\left\|\sum_{i,j=1}^n\xi_j(\omega)x_{i,j}\right\|d\mu(\omega)+\delta^2\left\|\sum_{i,j=1}^n x_{i,j}\right\|.$$

By introducing inside the expression $\|\sum_{i,j=1}^n\xi_i(\omega)\xi_j(\omega')x_{i,j}\|$ the expectation with respect to μ or with respect to μ' or with respect to both μ and μ', we check easily that J exceeds each of the last three terms in the right-hand side of the above inequality. Hence, in view of the result proved for random variables of mean 0, we get that

$$I\leqq 4\int_\Omega\int_{\Omega'}\left\|\sum_{i,j=1}^n(\xi_i(\omega)-\delta)(\xi_j(\omega')-\delta)x_{i,j}\right\|d\mu'(\omega')d\mu(\omega)+3J$$
$$\leqq 7J+4\delta\int_\Omega\left\|\sum_{i,j=1}^n\xi_i(\omega)x_{i,j}\right\|d\mu(\omega)+4\delta\int_{\Omega'}\left\|\sum_{i,j=1}^n\xi_j(\omega')x_{i,j}\right\|d\mu'(\omega')+4\delta^2\left\|\sum_{i,j=1}^n x_{i,j}\right\|$$
$$\leqq 10J. \qquad \square$$

The main and the most difficult part of the argument needed to prove Theorem 1.6 is given in the next proposition, where we show how to select a submatrix $R_\sigma SR_\sigma$ of S, of size proportional to that of S, so that $R_\sigma SR_\sigma$ would have small norm when it maps l_2^n into l_1^n. Again, the result is presented in a more general form than needed.

PROPOSITION 1.10. *There is a constant $D<\infty$ with the property that, for any $0<\delta<1$, one can find an integer $n(\delta)$ so that, whenever $1\leqq p\leqq 2$, $n\geqq n(\delta)$ and S is a linear operator on l_p^n whose matrix $(a_{i,j})_{i,j=1}^n$ relative to the unit vector basis of l_p^n has 0's on the diagonal, then there exists a subset τ of $\{1,2,\dots,n\}$ of cardinality $|\tau|=m=[\delta n]$ for which*

$$\|R_\tau SR_\tau x\|_1\leqq D\|S\|\left(\frac{m}{\log(1/\delta)}\right)^{1/p'}\cdot\|x\|_p,$$

for any $x\in l_p^n$.

PROOF. Fix $1<p\leqq 2$, $0<\delta<1$ and let S and $(a_{i,j})_{i,j=1}^n$ be so that $Se_i=$

$\sum_{j=1}^n a_{i,j}e_j$; $1 \leqq i \leqq n$. Let $\{\xi_i\}_{i=1}^n$ be a sequence of independent random variables of mean δ over some probability space (Ω, Σ, μ) which take only the values 0 and 1, and let (Ω', Σ', μ') be an independent copy of (Ω, Σ, μ).

For a linear operator W on $\mathbf{R}^n$, we shall put

$$|||W||| = \sup\{m^{-1/p'}\|Wx\|_1;\ x \in l_p^n, \|x\|_p \leqq 1\},$$

where again $p' = p/(p-1)$. Note that, for any such W,

$$|||W||| \leqq 2\delta^{-1/p'}\|W\|,$$

where $\|W\|$ denotes the norm of W as an operator from l_p^n into itself.

Since $a_{i,i} = 0$; $1 \leqq i \leqq n$, we can use the decoupling principle Proposition 1.9 and get that

$$I = \int_\Omega \left|\left|\left| \sum_{i,j=1}^n \xi_i(\omega)\xi_j(\omega)a_{i,j}e_i \otimes e_j \right|\right|\right| d\mu(\omega)$$

$$\leqq 20 \int_{\Omega'} \int_\Omega \left|\left|\left| \sum_{i\in\tau(\omega)} \sum_{j=1}^n \xi_j(\omega')a_{i,j}e_i \otimes e_j \right|\right|\right| d\mu(\omega)d\mu'(\omega'),$$

where, for each $\omega \in \Omega$,

$$\tau(\omega) = \{1 \leqq i \leqq n;\ \xi_i(\omega) = 1\}.$$

By Lemma 1.3, the subset

$$\tilde{\Omega} = \{\omega \in \Omega;\ |\tau(\omega)| \geqq 2\delta n\}$$

of Ω has measure $\leqq 2e^{-\delta n/10}$. Thus

$$I \leqq 20 \int_{\Omega \sim \tilde{\Omega}} \int_{\Omega'} \left|\left|\left| \sum_{i\in\tau(\omega)} \sum_{j=1}^n \xi_j(\omega')a_{i,j}e_i \otimes e_j \right|\right|\right| d\mu(\omega)d\mu'(\omega') + 40\delta^{-1/p'}\|S\|e^{-\delta n/10}$$

$$\leqq 40 \sup_{\substack{\tau\subset\{1,2,\dots,n\} \\ |\tau|=m}} I(\tau) + 40\delta^{-1/p'}\|S\|e^{-\delta n/10},$$

where

$$I(\tau) = \int_\Omega \left|\left|\left| \sum_{i\in\tau} \sum_{j=1}^n \xi_j(\omega)a_{i,j}e_i \otimes e_j \right|\right|\right| d\mu(\omega).$$

Fix now a subset τ of $\{1,2,\dots,n\}$ of cardinality $|\tau| = m$, take e.g. $\varepsilon = \frac{1}{2}$ and choose an ε-net $\mathcal{F}(\tau)$ in the unit ball of $R_\tau l_p^n$ so that

$$|\mathcal{F}(\tau)| \leqq (2/\varepsilon)^m = 4^m.$$

Since any vector x in the closed unit ball of $R_\tau l_p^n$ can be expressed as a combination $x = \sum_{j=1}^\infty \lambda_j x_j$ with x_j in $\mathcal{F}(\tau)$ and $\lambda_j \geqq 0$, for all j, such that $\sum_{j=1}^\infty \lambda_j \leqq 2$, it follows that

$$\begin{aligned}
I(\tau) &\leqq 2m^{-1/p'} \int_\Omega \max\left\{ \sum_{j=1}^n \left| \sum_{i\in\tau} a_{i,j} b_i \right| \xi_j(\omega);\ x = \sum_{i\in\tau} b_i e_i \in \mathcal{F}(\tau) \right\} d\mu(\omega) \\
&\leqq 2m^{-1/p'} \int_\Omega \left(\sum \left\{ \left| \sum_{j=1}^n \left| \sum_{i\in\tau} a_{i,j} b_i \right| \xi_j(\omega) \right|^m ;\ x = \sum_{i\in\tau} b_i e_i \in \mathcal{F}(\tau) \right\} \right)^{1/m} d\mu(\omega) \\
&\leqq 2m^{-1/p'} |\mathcal{F}(\tau)|^{1/m} \cdot \max\left\{ \left\| \sum_{j=1}^n \left| \sum_{i\in\tau} a_{i,j} b_i \right| \xi_j \right\|_m ;\ x = \sum_{i\in\tau} b_i e_i \in \mathcal{F}(\tau) \right\} \\
&\leqq 8\|S\| m^{-1/p'} \cdot \max\left\{ \left\| \sum_{j=1}^n c_j \xi_j \right\|_m ;\ c = \sum_{j=1}^n c_j e_i \in l_p^n, \|c\|_p \leqq 1 \right\}.
\end{aligned}$$

Thus, by using Proposition 1.8 with $\gamma = \delta$ and $r = p$, we get that

$$I(\tau) \leqq D_0 \|S\| \cdot \left(\log \frac{1}{\delta}\right)^{-1/p'},$$

for a suitable constant D_0, independent of p and n. Hence

$$I \leqq 40 D_0 \|S\| \cdot \left(\log \frac{1}{\delta}\right)^{-1/p'} + 40\delta^{-1/p'} \|S\| e^{-\delta n/10}$$

from which we derive that there exists a point ω_0 in the set

$$\hat{\Omega} = \left\{ \omega \in \Omega;\ m \leqq \sum_{i=1}^n \xi_i(\omega) \leqq 3\delta n/2 \right\}$$

such that

$$\left|\left|\left| \sum_{i,j\in\tau(\omega_0)} a_{i,j} e_i \otimes e_j \right|\right|\right| \leqq 41 D_0 \|S\| \left(\log \frac{1}{\delta}\right)^{1/p'},$$

provided n is large enough relative to δ as to ensure that

$$e^{-\delta n/10} \leqq \frac{D_0}{40} \cdot \left(\frac{\delta}{\log(1/\delta)}\right)^{1/p'}.$$

Since $m \leqq |\tau(\omega_0)| \leqq 3\delta n/2$ we complete the proof by taking as τ any subset of $\tau(\omega_0)$ which has cardinality m. □

We are now prepared to return to Theorem 1.6.

Proof of Theorem 1.6. As in the case of Theorem 1.2, there are two ways to complete the proof: by an exhaustion argument or by a factorization method.

We shall present here only the second alternative. The proof is very similar to that of the factorization argument used to complete the proof of Theorem 1.2.

The starting point is Proposition 1.10 which asserts the existence of a constant D so that, for any $0<\delta<1$, there is a subset τ of $\{1,2,\dots,n\}$, $n \geqq n(\delta)$, with $|\tau| = m = [\delta n]$ for which $W = R_\tau S R_\tau$, considered as an operator from l_2^n into l_1^n, has norm $\leqq D\|S\|(m/\log(1/\delta))^{1/2}$. Since W^* maps l_∞^n into l_2^n it follows, by using Grothendieck's inequality and Pietsch's factorization theorem, that there exist an operator $U: l_2^n \to l_2^n$, with $\|U\| \leqq 1$, and a diagonal operator $V: l_\infty^n \to l_2^n$, defined by $Ve_i = \lambda_i e_i$; $1 \leqq i \leqq n$, with

$$\sum_{i=1}^n |\lambda_i|^2 \leqq (K_G D\|S\|)^2 \cdot m/\log(1/\delta)$$

so that $W^* = U(V)$. By dualizing this factorization diagram, we conclude that $W = V^*(U^*)$, where $V^*e_i = \lambda_i e_i$; $1 \leqq i \leqq n$. Set

$$\sigma = \{1 \leqq i \leqq n; |\lambda_i| \leqq 2K_G D\|S\|/\sqrt{\log(1/\delta)}\}$$

and note that

$$|\sigma| \geqq n(1-\delta/4).$$

Furthermore, for any $x \in l_2^n$, we have

$$\|R_\sigma S R_\sigma x\|_2 = \|R_\sigma W R_\sigma x\|_2 = \|R_\sigma V^* U^* R_\sigma x\|_2$$

$$\leqq \left(\max_{i\in\sigma} |\lambda_i|\right) \cdot \|U^* R_\sigma x\|_2 \leqq 2K_G D\|S\| \cdot \|x\|_2/\sqrt{\log(1/\delta)},$$

i.e.

$$\|R_\sigma S R_\sigma\| \leqq 2K_G D\|S\|/\sqrt{\log(1/\delta)}.$$

Therefore, if $\varepsilon>0$ is given and δ is taken small enough as to ensure that

$$2K_G D\|S\|/\sqrt{\log(1/\delta)} < \varepsilon,$$

then, indeed, $\|R_\sigma S R_\sigma\| < \varepsilon$. □

REMARKS. (1) S. Szarek kindly brought to our attention that B. S. Kashin proved in [15] results of a somewhat similar nature to Theorem 1.6. B. S. Kashin shows in this paper that, whenever S is a norm one operator from l_2^n into l_2^m, then one can select a subset σ of $\{1,2,\dots,m\}$ of cardinality $|\sigma| = n$ such that

$\| R_\sigma S \| \leqq B(\log(m/n))^{-1/2}$, for a suitable $B < \infty$. This result is, of course, interesting when m is much larger than n. Kashin's theorem does not seem to imply Theorem 1.6 directly. However, one can deduce it from his theorem provided it is used in conjunction with the decoupling principle and other arguments given in Proposition 1.10. In Section 8, we shall present some improvements of Kashin's results.

(2) K. Ball and the second-named author obtained previously (unpublished) some weaker version of Corollary 1.7 (e.g. with $|\sigma| \geqq dn^{2/3}$, for some $d > 0$).

(3) Theorem 1.6 implies Theorem 1.2 since $\| Te_i \|_2 = 1$, for all $1 \leqq i \leqq n$, yields that the matrix corresponding to the operator T^*T has 1's on the diagonal. However, the dependence between the cardinality of σ and the norm of T that we obtain in Theorem 1.6 is of the form $|\sigma| \geqq dne^{-\|T\|^2}$, for a suitable $d > 0$. This, of course, is much worse than the estimate given by Theorem 1.2.

2. Applications to harmonic analysis and Hilbertian systems

The natural extension of the notion of "cardinality proportional to n" to an infinite setting is that of positive density or upper density. Recall that, for a set Λ of integers, the upper density $\overline{\text{dens}}\,\Lambda$ and the lower density $\underline{\text{dens}}\,\Lambda$ of Λ are defined as $\overline{\lim}_{n\to\infty}$, respectively $\underline{\lim}_{n\to\infty}$, of the sequence

$$\frac{|\Lambda \cap \{1,2,\dots,n\}|}{n}; \qquad n = 1,2,\dots .$$

If $\overline{\text{dens}}\,\Lambda = \underline{\text{dens}}\,\Lambda$ then their common value $\text{dens}\,\Lambda$ is called the asymptotic density or simply the density of Λ.

The first part of this section is devoted to the study of some questions concerning the characters on the circle. The notation related to this notion will be the standard one. Throughout this section, the circle is denoted by $\mathbf{T}$ while ν stands for the normalized measure on $\mathbf{T}$. For $1 \leqq p \leqq \infty$ and Λ a subset of the integers, we shall denote the closed linear span of the characters $\{e^{inx}\}_{n\in\Lambda}$ in $L_p(\mathbf{T}, \nu)$ by $L_p^\Lambda(\mathbf{T}, \nu)$.

We start with a result which asserts that, for any subset B of the circle $\mathbf{T}$ with $\nu(B) > 0$, there exists a subset Λ of the integers of positive density such that $L_2^\Lambda(\mathbf{T}, \nu)$ contains no function vanishing a.e. on B. This result solves a question raised by W. Schachermayer.

Quite surprisingly, the situation differs completely for $p > 2$. This follows from a characterization of those subsets B of the circle which have the above property in $L_p(\mathbf{T}, \nu)$; $p > 2$.

The last part of this section contains an extension of the theorem in $L_2(\mathbf{T}, \nu)$ to the more general case of Hilbertian systems.

In order to simplify the statements of some of the results presented in the sequel, we introduce the following definition.

DEFINITION 2.1. A subset B of $\mathbf{T}$ is called a set of isomorphism in L_p; $1 \leqq p < \infty$, for some family of characters of positive density if there exist a constant $d > 0$ and a subset Λ of the integers with $\operatorname{dens} \Lambda > 0$ such that

$$\|f \cdot \chi_B\|_p \geqq d \cdot \|f\|_p,$$

whenever $f \in L_p^\Lambda(\mathbf{T}, \nu)$.

THEOREM 2.2. *Every subset B of the circle $\mathbf{T}$ of positive measure is a set of isomorphism in L_2, for some family of characters of positive density. More precisely, there exists a constant $c > 0$ so that, for any $B \subset \mathbf{T}$, one can find a subset Λ of the integers with* $\operatorname{dens} \Lambda \geqq c\nu(B)$, *for which*

$$\|f\|_2 \geqq \|f \cdot \chi_B\|_2 \geqq c \cdot \sqrt{\nu(B)} \cdot \|f\|_2,$$

whenever $f \in L_2^\Lambda(\mathbf{T}, \nu)$.

PROOF. Suppose that $\nu(B) > 0$; otherwise, Theorem 2.2 lacks content. Let T be the operator acting on $L_2(\mathbf{T}, \nu)$ which is defined by

$$T(f) = f \cdot \chi_B / \sqrt{\nu(B)}; \qquad f \in L_2(\mathbf{T}, \nu),$$

and note that $\|T\| = 1/\sqrt{\nu(B)}$.

By Theorem 1.2, there exists a $c > 0$ such that, for each n, there is a subset σ_n of $\{1, 2, \ldots, n\}$ of cardinality

$$|\sigma_n| \geqq cn / \|T|_{[e^{ijx}]_{j=0}^{n-1}}\|^2 \geqq cn\nu(B)$$

for which

$$\|\chi_B f\|_2 \geqq c\sqrt{\nu(B)}\|f\|_2,$$

whenever the Fourier transform of f is supported by σ_n.

Consider now the family $\mathcal{H}$ of all finite subsets σ of the integers for which

$$\|\chi_B f\|_2 \geqq c\sqrt{\nu(B)}\|f\|_2,$$

whenever the Fourier transform of f is supported by σ. The family $\mathcal{H}$ is homogeneous in the sense of [26] Definition 3.1, i.e., for every $\sigma \in \mathcal{H}$, all the subsets and translations of σ belong to $\mathcal{H}$. For each n, put

$$d_n(\mathcal{H}) = \max\{|\sigma \cap \{1, 2, \dots, n\}|/n; \sigma \in \mathcal{H}\}$$

and note that

$$d(\mathcal{H}) = \lim_{n \to \infty} d_n(\mathcal{H})$$

exists since $\{d_n(\mathcal{H})\}_{n=1}^{\infty}$ is a non-increasing sequence. Since clearly $\sigma_n \in \mathcal{H}$, for all n, we easily conclude that

$$d(\mathcal{H}) > c\nu(B).$$

Now, by I. Z. Ruzsa [26] Theorem 4, there exists a set Λ of integers whose finite subsets all belong to $\mathcal{H}$ and

$$\operatorname{dens} \Lambda = d(\mathcal{H}).$$

This, of course, completes the proof in view of the definition of $\mathcal{H}$. □

REMARK. The use of Ruzsa's result to pass from the finite setting to a density statement was pointed out by Y. Peres. Our original proof yielded only upper density.

As we have mentioned above, Theorem 2.2 fails for $p > 2$. This is an immediate consequence of the following result.

THEOREM 2.3. *Let $p > 2$. A subset B of $\mathbf{T}$ is a set of isomorphism in L_p, for some family of characters of positive density, if and only if $\mathbf{T}$ is the union of finitely many translates of B, up to a set of measure zero.*

The proof of Theorem 2.3 requires two preliminary lemmas.

LEMMA 2.4. *There exists a constant $C < \infty$ such that, whenever $t \in \mathbf{T}$, $\varepsilon > 0$, Γ is a subset of integers for which $|1 - e^{int}| < \varepsilon$ if $n \in \Gamma$ and $f \in L_p^{\Gamma}(\mathbf{T}, \nu)$; $1 \leqq p < \infty$, then*

$$\|f - f_t\|_p \leqq C\varepsilon \|f\|_p,$$

where $f_t(x) = f(x + t)$ denotes the translate of f by t.

PROOF. Recall the classical fact that $1 - e^{ix}$ is a function of spectral synthesis, i.e. it can be approximated in the space $A(\mathbf{T})$ of the absolutely convergent Fourier series by functions which vanish in a neighborhood of $x = 0$. More precisely, there exists a constant $C<\infty$ so that, for every $\varepsilon>0$, one can find a function $F_\varepsilon \in A(\mathbf{T})$ for which

(i) $F_\varepsilon(x)=0$, whenever $x\in\mathbf{T}$ satisfies $|1-e^{ix}|<\varepsilon$,

and

(ii) $\|1-e^{ix}-F_\varepsilon(x)\|_{A(\mathbf{T})}<C\varepsilon$.

A simple proof of this assertion, originally due to N. Wiener, can be found, e.g., in [16].

The above properties of F_ε imply that if we put

$$1-e^{ix}-F_\varepsilon(x)=\sum_{j=-\infty}^{+\infty} a_j e^{ijx},$$

then

$$\sum_{j=-\infty}^{+\infty} |a_j|<C\varepsilon,$$

and, furthermore, that

$$1-e^{ix}=\sum_{j=-\infty}^{+\infty} a_j e^{ijx},$$

whenever $|1-e^{ix}|<\varepsilon$.

Fix now $t\in\mathbf{T}$ and $\varepsilon>0$, and let Γ be a subset of the integers for which $|1-e^{int}|<\varepsilon$; $n\in\Gamma$. Then, for every $f\in L_p^\Gamma(\mathbf{T},\nu)$, we have

$$\begin{aligned}\|f-f_t\|_p &= \Big\|\sum_{n\in\Gamma}\hat f(n)(1-e^{int})e^{inx}\Big\|_p\\ &= \Big\|\sum_{j=-\infty}^{+\infty} a_j\sum_{n\in\Gamma}\hat f(n)e^{in(x+jt)}\Big\|_p\\ &= \Big\|\sum_{j=-\infty}^{+\infty} a_j f_{jt}\Big\|_p\\ &\leqq \sum_{j=-\infty}^{+\infty}|a_j|\,\|f_{jt}\|_p\\ &< C\varepsilon\|f\|_p. \end{aligned}$$

□

LEMMA 2.5. *Fix $\gamma>0$ and a positive integer r, and let B be a subset of the circle $\mathbf{T}$ such that*

$$\nu\Big(\bigcup_{k=1}^{r}(B+t_k)\Big)<1,$$

for any choice of $\{t_k\}_{k=1}^r$ *in* $\mathbf{T}$. *Then there is an integer* l *with the property that, for every sequence* $\{t_k\}_{k=1}^r$ *of points in* $\mathbf{T}$, *one can find a dyadic interval* J *so that*

$$\nu(J) = 2^{-l} \quad \text{and} \quad \nu((J + t_k) \cap B) < \gamma \cdot 2^{-l},$$

for all $1 \leqq k \leqq r$. *Furthermore, the assertion remains valid if* l *is replaced by any other integer larger than it.*

Proof. We need first some additional notation: for any vector

$$\bar{t} = (t_1, t_2, \ldots, t_r)$$

in $\mathbf{T}^r$ and any dyadic subinterval I of $\mathbf{T}$, we put

$$\varphi_I(\bar{t}) = \nu\left(I \cap \bigcup_{k=1}^{r} (B - t_k)\right) \Big/ \nu(I).$$

It follows from the Lebesgue density theorem that, for any $\bar{t} \in \mathbf{T}^r$, one can find a dyadic interval $I(\bar{t})$ so that

$$\varphi_{I(\bar{t})}(\bar{t}) < \gamma.$$

Moreover, since $\mathbf{T}^r$ is compact and, for each I as above, φ_I is clearly a continuous function, one can choose finitely many dyadic intervals $\{I_h\}_{h=1}^H$ so that

$$\min_{1 \leqq h \leqq H} \varphi_{I_h}(\bar{t}) < \gamma,$$

for all $\bar{t} \in \mathbf{T}^r$. Suppose that $\nu(I_h) = 2^{-l(h)}$, for some positive integer $l(h)$; $1 \leqq h \leqq H$, and let l be any integer larger than $\max_{1 \leqq h \leqq H} l(h)$. Each of the intervals I_h can be split into a union,

$$I_h = \bigcup_{i \in \Delta_h} I_{h,i},$$

of mutually disjoint dyadic intervals $\{I_{h,i}\}_{i \in \Delta_h}$ of length 2^{-l} and

$$\varphi_{I_h}(\bar{t}) = \sum_{i \in \Delta_h} (\nu(I_{h,i}) / \nu(I_h)) \varphi_{I_{h,i}}(\bar{t}); \qquad \bar{t} \in \mathbf{T}^r.$$

Hence

$$\gamma > \min_{|J| = 2^{-l}} \varphi_J(\bar{t}) \geqq \min_{|J| = 2^{-l}} \max_{1 \leqq h \leqq r} 2^l \nu((J + t_k) \cap B),$$

which clearly completes the proof. □

PROOF OF THEOREM 2.3. Suppose first that B is a subset of the circle for which there exist points $\{t_k\}_{k=1}^m$ on $\mathbf{T}$ such that

$$\nu\left(\bigcup_{k=1}^{m}(B+t_k)\right)=1.$$

Then, for any function $f\in L_p(\mathbf{T},\nu)$, we have

$$\|f\|_p\leqq\sum_{k=1}^{m}\|f_{t_k}\cdot\chi_B\|_p\leqq m\cdot\|f\chi_B\|_p+\sum_{k=1}^{m}\|f-f_{t_k}\|_p.$$

Let C be the constant given by Lemma 2.4, take $\varepsilon=1/2Cm$ and put

$$\Lambda(\varepsilon)=\left\{n\in\mathbf{Z};\ \max_{1\leqq k\leqq m}|1-e^{int_k}|<\varepsilon\right\}.$$

In view of Lemma 2.4, whenever $f\in L_p^{\Lambda(\varepsilon)}(\mathbf{T},\nu)$, then

$$\|f\|_p\leqq m\|f\cdot\chi_B\|_p+Cm\varepsilon\|f\|_p,$$

i.e.

$$\|f\chi_B\|_p\geqq\|f\|_p/2m,$$

and the proof of the "if" part will be completed provided we show that $\operatorname{dens}\Lambda(\varepsilon)>0$. To this end, consider the group homomorphism $\psi:\mathbf{Z}\to\mathbf{T}^m$, defined by

$$\psi(n)=(e^{int_1},e^{int_2},\ldots,e^{int_m});\qquad n\in\mathbf{Z}.$$

The fact that $\Lambda(\varepsilon)$ is of positive density is a consequence of the compactness of $\mathbf{T}^m$ which implies that the range of ψ can be covered by open sets of the form

$$G_j=\left\{\bar{x}=(x_1,x_2,\ldots,x_m)\in\mathbf{T}^m;\ \max_{1\leqq k\leqq m}|e^{ijt_k}-e^{ix_k}|<\varepsilon\right\};\qquad j\in\Delta,$$

with Δ being a finite set of integers. This yields that

$$\mathbf{Z}=\bigcup_{j\in\Delta}\Gamma_j,$$

where

$$\Gamma_j=\{n\in\mathbf{Z};(nt_1,nt_2,\ldots,nt_m)\in G_j\};\qquad j\in\Delta.$$

However, as is easily verified, for each $j\in\Delta$, the set Γ_j is a j-translate of $\Delta(\varepsilon)$, i.e. $\mathbf{Z}$ is a finite union of translates of $\Delta(\varepsilon)$. This implies that $\operatorname{dens}\Lambda(\varepsilon)>0$, thus completing the proof of the "if" part.

In order to prove the coverse, we assume now that $B \subset \mathbf{T}$, $\Lambda \subset \mathbf{Z}$ and $c > 0$ are so that

(i) $\operatorname{dens} \Lambda > c$,

and

(ii) $\|f\chi_B\|_p \geqq c\|f\|_p$; $f \in L_p^\Lambda(\mathbf{T}, \nu)$,

and, moreover, that

$$\nu\left(\bigcup_{k=1}^{m} (B + t_k)\right) < 1,$$

for any choice of $\{t_k\}_{k=1}^m$ in $\mathbf{T}$; $m = 1, 2, \ldots$.

Take $\tau = (c^2/2^{13})^{p/(p-2)}$, $r = [2^9/\tau^3] + 1$ and $\gamma = c^{2p}/r \cdot 2^{13}$, and let l be an integer satisfying the assertion of Lemma 2.5, for the above values of r and γ, and so that the set

$$\Lambda_0 = \{n \in \Lambda; |n| \leqq 2^l\}$$

has cardinality $|\Lambda_0| > c \cdot 2^l$.

Consider now the function

$$F(x) = \sum_{n \in \Lambda_0} e^{inx}; \qquad x \in \mathbf{T},$$

and choose a maximal system $\{t_k\}_{k=1}^m$ of 2^{-l-1}-separated points in $\mathbf{T}$ such that

$$|F(t_k)| \geqq \tau \cdot 2^l; \qquad 1 \leqq k \leqq m.$$

For each $1 \leqq k \leqq m$, put

$$W_k = \{x \in \mathbf{T}; |x - t_k| < \tau \cdot 2^{-l-5}\},$$

and note that, whenever $x \in W_k$, we have

$$\begin{aligned} |F(x)| &\geqq |F(t_k)| - |F(t_k) - F(x)| \\ &\geqq \tau \cdot 2^l - \sum_{n \in \Lambda_0} |e^{int_k} - e^{inx}| \\ &\geqq \tau \cdot 2^l - |\Lambda_0| 2^l \tau \cdot 2^{-l-5} \\ &> \tau \cdot 2^{l-1}. \end{aligned}$$

Hence

$$2^{l+2} \geqq |\Lambda_0| = \int_{\mathbf{T}} |F(x)|^2 d\nu \geqq \sum_{k=1}^{m} \int_{W_k} |F(x)|^2 d\nu \geqq m\tau^3 \cdot 2^{l-7},$$

from which it follows that

$$m \leqq 2^9 \cdot \tau^{-3} < r.$$

This means that, by Lemma 2.5, there exists a point $t \in \mathbf{T}$ such that

$$\nu((I + t + t_k) \cap B) < \gamma \cdot 2^{-l},$$

for all $1 \leqq k \leqq m$, where I denotes the interval $[-2^{-l-1}, 2^{-l-1}]$. Put

$$V = \mathbf{T} \sim \bigcup_{k=1}^{m} (I + t + t_k)$$

and note that the translate F_t of F satisfies

$$\begin{aligned} \| F_t \cdot \chi_B \|_p^p &\leqq \int_V |F_t(x)|^p d\nu + \sum_{k=1}^{m} \int_{(I+t+t_k)\cap B} |F_t(x)|^p d\nu \\ &\leqq \int_V |F_t(x)|^p d\nu + m \| F_t \|_\infty^p \cdot \gamma \cdot 2^{-l}. \end{aligned}$$

However, whenever $x \in V$, then $x - t$ is 2^{-l-1}-separated from all the points $\{t_k\}_{k=1}^m$. Hence, by the maximality of this system, $|F(x-t)| < \tau \cdot 2^l$, i.e. $|F_t(x)| < \tau \cdot 2^l$. It follows that

$$\begin{aligned} \| F_t \cdot \chi_B \|_p^p &< (\tau \cdot 2^l)^{p-2} \int_V |F_t(x)|^2 d\nu + m |\Lambda_0|^p \gamma \cdot 2^{-l} \\ &\leqq \tau^{p-2} \cdot 2^{l(p-2)} |\Lambda_0| + m |\Lambda_0|^p \gamma 2^{-l} \\ &\leqq (\tau^{p-2} + m\gamma) \cdot 2^{l(p-1)+2p}, \end{aligned}$$

which, by (ii) and the fact that $F \in L_p^{\Lambda_0}(\mathbf{T}, \nu) \subset L_p^{\Lambda}(\mathbf{T}, \nu)$, yields that

$$c \| F \|_p \leqq \| F_t \cdot \chi_B \|_p \leqq (\tau^{1-2/p} + (r\gamma)^{1/p}) \cdot 2^{l/p'+2},$$

where $p' = p/(p-1)$.

On the other hand, if J denotes, e.g., the interval $[-2^{-l-10}, 2^{-l-10}]$ then, by Hölder's inequality, we get that

$$c 2^{-10} < 2^{-l-10} |\Lambda_0| < \left| \int_J F(x) d\nu \right| \leqq \| F \|_p \cdot |J|^{1/p'} < \| F \|_p \cdot 2^{-l/p'}.$$

By combining these inequalities, we obtain

$$c^2 < 2^{12}(\tau^{1-2/p} + (r\gamma)^{1/p}),$$

which, in view of the fact that $\tau^{1-2/p} = c^2 \cdot 2^{-13}$, implies that

$$c^2 \cdot 2^{-13} < (r \cdot \gamma)^{1/p}.$$

This contradicts the choice of γ, thus proving the converse. □

COROLLARY 2.6. *If $B \subset \mathbf{T}$ is a set of isomorphisms in L_p; $p > 2$, for some family of characters of positive density, then its closure $\bar{B}$ has non-empty interior.*

COROLLARY 2.7. *There exists a subset of the circle with positive measure which, for each $p > 2$, is a set of isomorphism for no family of characters of positive density.*

We return now to the study of systems of vectors in Hilbert space and present a generalization of Theorem 2.2. First, we point out another way of expressing the fact that the operator T, defined in the proof of Theorem 2.2, is bounded. Namely, it can be asserted that the vectors $\Psi_n(x) = \chi_B e^{inx}$; $n \in \mathbf{Z}$, satisfy the estimate

$$\left\| \sum_{n=-\infty}^{+\infty} a_n \Psi_n \right\| \leqq \left(\sum_{n=-\infty}^{+\infty} |a_n|^2 \right)^{1/2},$$

for any choice of $\{a_n\}_{n=-\infty}^{+\infty}$. This leads naturally to the notion of Hilbertian systems.

DEFINITION 2.8. A normalized system of vectors $\{x_n\}_{n=1}^{\infty}$ in a Banach space X is called Hilbertian provided there exists a constant $M < \infty$ such that

$$\left\| \sum_{n=1}^{\infty} a_n x_n \right\| \leqq M \left(\sum_{n=1}^{\infty} |a_n|^2 \right)^{1/2},$$

for any choice of $\{a_n\}_{n=1}^{\infty}$. If the reverse inequality holds, i.e. if

$$\left(\sum_{n=1}^{\infty} |a_n|^2 \right)^{1/2} \leqq M \left\| \sum_{n=1}^{\infty} a_n x_n \right\|,$$

again, for every choice of $\{a_n\}_{n=1}^{\infty}$, then we say that $\{x_n\}_{n=1}^{\infty}$ is a Besselian system.

It turns out that Theorem 2.2 can be extended to any Hilbertian system in an arbitrary Hilbert space. However, instead of positive density we can prove the corresponding statement only with positive upper density.

THEOREM 2.9. *There is a constant $d > 0$ such that, for every Hilbertian system $\{x_n\}_{n=1}^{\infty}$ in a Hilbert space X with constant M, for some $M < \infty$, there exists a set Λ of integers with $\overline{\text{dens}}\, \Lambda \geqq d/M^2$ so that $\{x_n\}_{n \in \Lambda}$ is also Besselian with constant d^{-1}. In particular, $\{x_n\}_{n \in \Lambda}$ is Md^{-1}-equivalent to an orthonormal system.*

PROOF. For simplicity, we shall suppose that the underlying space X is l_2. Let $\{e_n\}_{n=1}^\infty$ denote the unit vectors in l_2 and let R_n and R'_n be the orthogonal projections from l_2 onto $[e_i]_{i=1}^n$, respectively $[e_i]_{i=n+1}^\infty$.

The hypothesis that $\{x_n\}_{n=1}^\infty$ is a Hilbertian system with constant $M<\infty$ can be translated into the fact that the linear operator T on l_2, defined by $Te_n = x_n$, for all n, has norm $\leqq M$.

Let $c>0$ be the constant given by Theorem 1.2 and let $\{\tau_n\}_{n=1}^\infty$ be a sequence of positive reals such that

$$\tau = \left(\sum_{n=1}^\infty \tau_n^2\right)^{1/2} < c/2M.$$

We shall now construct two increasing sequences of integers $\{q_n\}_{n=1}^\infty$ and $\{r_n\}_{n=1}^\infty$ in the following way. Put $q_1 = 1$ and choose r_1 so that

$$\| R'_{r_1}|_{[x_j]_{j=q_1+1}^{2q_1}} \| < \tau_1/2.$$

Note that

$$\lim_{q\to\infty} \| R_{r_1}|_{[x_j]_{j=q+1}^{2q}} \| = 0$$

since, otherwise, one could construct a sequence $\{\eta_m\}_{m=1}^\infty$ of mutually disjoint subsets of the integers and a sequence of vectors of the form $u_m = \sum_{j\in\eta_m} c_j x_j$ so that $\|u_m\|_2 = 1$ but $\|R_{r_1}(u_m)\|_2 \geqq \alpha$, for some $\alpha>0$ and all m. Since w-$\lim_{m\to\infty} u_m = 0$ we can assume, by passing to a subsequence if necessary, that $\{u_m\}_{m=1}^\infty$ is an orthonormal system. This would imply that we have

$$\alpha^2 k \leqq \sum_{m=1}^k \|R_{r_1}(u_m)\|_2^2 \leqq \|R_{r_1}\|_{HS}^2 \leqq r_1,$$

for all k; and, hence, contradiction. It follows that we can find a $q_2 > q_1$ so that

$$\| R_{r_1}|_{[x_j]_{j=q_2+1}^{2q_2}} \| < \tau_2/2.$$

Then we choose an $r_2 > r_1$ so that

$$\| R'_{r_2}|_{[x_j]_{j=q_2+1}^{2q_2}} \| < \tau_2/2.$$

Continuing so, we construct, by induction, two increasing sequences of integers $\{q_n\}_{n=1}^\infty$ and $\{r_n\}_{n=1}^\infty$, for which

$$\| R_{r_{n-1}}|_{[x_j]_{j=q_n+1}^{2q_n}} \| < \tau_n/2 \quad\text{and}\quad \| R'_{r_n}|_{[x_j]_{j=q_n+1}^{2q_n}} \| < \tau_n/2,$$

for all n.

Fix now n and apply Theorem 1.2 to the operator $T_{|\{e_j\}_{j=q_n+1}^{2q_n}}$. It follows that there exists a subset σ_n of $\{q_n+1, q_n+2, \ldots, 2q_n\}$ so that $|\sigma_n| \geqq cq_n/M^2$ and

$$\left\| \sum_{j\in\sigma_n} a_j x_j \right\|_2 \geqq c\left(\sum_{j\in\sigma_n} |a_j|^2 \right)^{1/2},$$

for any choice of $\{a_j\}_{j\in\sigma_n}$.

Consider the set $\Lambda = \bigcup_{n=1}^{\infty} \sigma_n$ and observe that, for each n, we have

$$\frac{|\Lambda \cap \{1,2,\ldots,2q_n\}|}{2q_n} \geqq \frac{|\sigma_n|}{2q_n} \geqq \frac{c}{2M^2},$$

i.e., $\overline{\text{dens}}\,\Lambda \geqq c/2M^2$. Furthermore, for any choice of $\{a_j\}_{j=1}^{\infty}$ such that $\sum_{n=1}^{\infty}\sum_{j\in\sigma_n}|a_j|^2 = 1$, we have

$$\begin{aligned} S &= \left\| \sum_{n=1}^{\infty} \sum_{j\in\sigma_n} a_j x_j \right\|_2 \\ &\geqq \left\| \sum_{n=1}^{\infty} (R_{r_n} - R_{r_{n-1}}) \sum_{j\in\sigma_n} a_j x_j \right\|_2 - \sum_{n=1}^{\infty} \left\| (R_{r_{n-1}} + R'_{r_n}) \sum_{j\in\sigma_n} a_j x_j \right\|_2 \\ &\geqq \left(\sum_{n=1}^{\infty} \left\| (R_{r_n} - R_{r_{n-1}}) \sum_{j\in\sigma_n} a_j x_j \right\|_2^2 \right)^{1/2} - \sum_{n=1}^{\infty} \tau_n \left\| \sum_{j\in\sigma_n} a_j x_j \right\|_2 \\ &\geqq \left(\sum_{n=1}^{\infty} \left\| (R_{r_n} - R_{r_{n-1}}) \sum_{j\in\sigma_n} a_j x_j \right\|_2^2 \right)^{1/2} - \tau M. \end{aligned}$$

It follows that

$$\begin{aligned} (S + c/2)^2 &\geqq \sum_{n=1}^{\infty} \left\| \sum_{j\in\sigma_n} a_j x_j \right\|_2^2 - \sum_{n=1}^{\infty} \left\| (R_{r_{n-1}} + R'_{r_n}) \sum_{j\in\sigma_n} a_j x_j \right\|_2^2 \\ &\geqq c^2 \sum_{n=1}^{\infty} \sum_{j\in\sigma_n} |a_j|^2 - \tau^2 M^2 \\ &> 3c^2/4, \end{aligned}$$

i.e.

$$S \geqq (\sqrt{3}-1)c/2.$$

This, of course, completes our proof. □

REMARK. It is not true in general that, for any Besselian system or even basis $\{x_n\}_{n=1}^{\infty}$, there exists a set Λ of integers with $\overline{\text{dens}}\,\Lambda > 0$ so that $\{x_n\}_{n\in\Lambda}$ is also Hilbertian. Indeed, fix $0 < \alpha < \frac{1}{2}$ and consider the vectors

$$f_n = c(\alpha)e^{inx}/|x|^{\alpha}; \qquad |n| = 0, 1, 2, \ldots,$$

where $c(\alpha)$ is chosen so that the norm of f_n in $L_2(-\pi,\pi)$ is equal to one, for all n. It is known that these vectors form (under the above condition that $0<\alpha<\frac{1}{2}$) a conditional basis in $L_2(-\pi,\pi)$ (cf. [1]; see also [20]). The system $\{f_n\}_{n=-\infty}^{+\infty}$ is clearly Besselian.

Suppose now that there exists a set Λ of integers with $\lambda=\overline{\operatorname{dens}}\,\Lambda>0$ such that $\{f_n\}_{n\in\Lambda}$ is also Hilbertian with constant M, for some $M<\infty$. Take $\beta=\lambda(2-\alpha)/4(1-\alpha)$ and choose an integer k for which the set $\Lambda_k=\Lambda\cap\{1,2,\dots,k\}$ satisfies $|\Lambda_k|\geqq\lambda k/2$. Then, by the Cauchy–Schwartz inequality, we have that

$$\int_0^{\beta/k}\Big|\sum_{n\in\Lambda_k}f_n(x)\Big|\,dx\leqq\Big\|\sum_{n\in\Lambda_k}f_n\Big\|_2\cdot\sqrt{\beta/k}\leqq M\sqrt{\beta}.$$

On the other hand,

$$\begin{aligned}\int_0^{\beta/k}\Big|\sum_{n\in\Lambda_k}f_n(x)\Big|\,dx&\geqq c(\alpha)\left(\frac{|\Lambda_k|\beta^{1-\alpha}}{(1-\alpha)k^{1-\alpha}}-\sum_{n\in\Lambda_k}\int_0^{\beta/k}\frac{|e^{inx}-1|}{x^\alpha}\,dx\right)\\&\geqq c(\alpha)\left(\frac{\lambda\beta^{1-\alpha}k^\alpha}{2(1-\alpha)}-2\sum_{n\in\Lambda_k}\int_0^{\beta/k}\frac{\sin(nx/2)}{x^\alpha}\,dx\right)\\&\geqq c(\alpha)k^\alpha\left(\frac{\lambda\beta^{1-\alpha}}{2(1-\alpha)}-\frac{\beta^{2-\alpha}}{2-\alpha}\right).\end{aligned}$$

It follows, by taking into account the choice of β, that

$$\frac{c(\alpha)\lambda\beta^{1/2-\alpha}}{4(1-\alpha)}\,k^\alpha\leqq M,$$

which, of course, is contradictory if k is sufficiently large.

3. Operators on l_p^n-spaces; $1\leqq p\leqq\infty$

The main result of this section asserts that any matrix T with 1's on the diagonal which acts as a "bounded" linear operator on l_p^n, for some $1\leqq p\leqq\infty$ and some n, contains a square submatrix of rank proportional to n (the proportion being determined by the norm of T) which is well invertible.

The cases $p=1$ and $p=\infty$ are, essentially speaking, known (cf. [4] and [13]) though not exactly in the formulation given above. The proof in these two cases uses a combinatorial lemma which asserts that any $n\times n$ matrix contains in turn a submatrix of size proportional to that of the original one such that, for each row, the sum of the absolute values of the off diagonal elements is reduced to one half of what it originally was. The case $p=2$ has already been presented in Section 1.

In the case $p > 2$, the invertibility results hold also for matrices with "large" rows (rather than "large" diagonal, as above). This is no longer true for $1 \leqq p < 2$. In this case, the rank of a $n \times n$ matrix with "large" rows need not even be proportional to n. An example of when the rank is as small as possible, i.e. of order of magnitude $n^{2/p'}$; $p' = p/(p-1)$, is given by the natural projection from l_p^n; $1 < p < 2$, onto a well-complemented Hilbertian subspace of maximal dimension.

We begin with our first result.

THEOREM 3.1. *For every* $1 \leqq p \leqq \infty$, $M < \infty$ *and* $\varepsilon > 0$, *there exists a constant* $c = c(p, M, \varepsilon) > 0$ *such that, whenever* $n \geqq 1/c$ *and* S *is a linear operator on* l_p^n *of norm* $\|S\| \leqq M$ *for which the matrix relative to the unit vector basis has* 0's *on the diagonal, then there exists a subset* σ *of* $\{1, 2, \ldots, n\}$ *of cardinality* $|\sigma| \geqq cn$ *so that*

$$\|R_\sigma S R_\sigma\| < \varepsilon,$$

where R_σ *denotes the natural projection from* l_p^n *onto the linear span of the unit vectors* $\{e_i\}_{i \in \sigma}$.

As an immediate consequence, we obtain our main invertibility result.

COROLLARY 3.2. *For every* $1 \leqq p \leqq \infty$, $M < \infty$ *and* $\varepsilon > 0$, *there exists a constant* $d = d(p, M, \varepsilon) > 0$ *such that, whenever* $n \geqq 1/d$ *and* T *is a linear operator on* l_p^n *of norm* $\|T\| \leqq M$ *for which the matrix relative to the unit vector basis has* 1's *on the diagonal, then there exists a subset* σ *of* $\{1, 2, \ldots, n\}$ *of cardinality* $|\sigma| \geqq dn$ *so that* $R_\sigma T R_\sigma$ *restricted to* $R_\sigma l_p^n$ *is invertible and its inverse satisfies* $\|(R_\sigma T R_\sigma)^{-1}\| < 1 + \varepsilon$.

The proof of Theorem 3.1 requires some preliminary results. The first asserts that any bounded linear operator on an l_p^n-space is also bounded on l_2^n provided that it is restricted to a suitable set of unit vectors of "large" cardinality. This result is, in fact, a direct consequence of a theorem from W. B. Johnson and L. Jones [14]. We prefer, however, to give a direct proof based on an exhaustion argument.

In order to distinguish between the different norms of the same operator, we shall denote the norm of an operator S on l_p^n by $\|S\|_p$. Also, as usual, K_G denotes the constant of Grothendieck.

PROPOSITION 3.3. *For every* $1 \leqq p \leqq \infty$ *and every linear operator* S: $l_p^n \to l_p^n$, *there exists a subset* η *of* $\{1, 2, \ldots, n\}$ *such that* $|\eta| \geqq n/2$ *and*

$$\|R_\eta S R_\eta\|_2 \leqq 4 K_G \|S\|_p.$$

PROOF. We first observe that it suffices to prove the result for $2 < p \leqq \infty$ since, otherwise, we pass to the adjoint S^* of S.

Suppose now that the assertion is false for some operator $S: l_p^n \to l_p^n$. Then, as in the exhaustion arguments presented in Section 1, we can construct subsets

$$\tau_1 \supset \tau_2 \supset \cdots \supset \tau_m$$

of $\{1, 2, \ldots, n\}$ with $|\tau_m| \geqq n/2$ and vectors $\{y_i\}_{i=1}^m$ in l_2^n such that $\|y_i\|_2 = 1$, $R_{\tau_i} y_i = y_i$,

$$\|R_{\tau_i} S y_i\|_2 > 4K_G \|S\|_p, \qquad 1 \leqq i \leqq m,$$

and if $y_i = \sum_{j \in \tau_i} b_{i,j} e_j$; $1 \leqq i \leqq m$, then the set

$$\tau_{m+1} = \left\{ j \in \tau_m \,;\, \sum_{i=1}^m |b_{i,j}|^2 < 1 \right\}$$

is of cardinality $|\tau_{m+1}| < n/2$ (with the convention that $b_{i,j} = 0$ when $j \notin \tau_i$). Note also that

$$m = \sum_{i=1}^m \|y_i\|_2^2 \geqq \sum_{j \notin \tau_{m+1}} \sum_{i=1}^m |b_{i,j}|^2 \geqq n/2.$$

By using Grothendieck's inequality in the form presented in [17] (see also [21] 1.f.14) with the convention $1/p = 0$ when $p = \infty$, it follows that

$$\begin{aligned} 4K_G \|S\|_p m^{1/2} &< \left(\sum_{i=1}^m \|Sy_i\|_2^2 \right)^{1/2} \\ &= \left\| \left(\sum_{i=1}^m |Sy_i|^2 \right)^{1/2} \right\|_2 \\ &\leqq n^{1/2-1/p} \left\| \left(\sum_{i=1}^n |Sy_i|^2 \right)^{1/2} \right\|_p \\ &\leqq K_G \|S\|_p n^{1/2-1/p} \left\| \left(\sum_{i=1}^m |y_i|^2 \right)^{1/2} \right\|_p \\ &= K_G \|S\|_p n^{1/2-1/p} \left\| \sum_{j=1}^n \left(\sum_{i=1}^m |b_{i,j}|^2 \right)^{1/2} e_j \right\|_p . \end{aligned}$$

However, the above procedure yields that

$$\sum_{i=1}^m |b_{i,j}|^2 \leqq 2,$$

for all $1 \leqq j \leqq n$, and, therefore, we conclude that

$$2 < n^{-1/p} \cdot \left\| \sum_{j=1}^{n} e_j \right\|_p ,$$

which is contradictory. □

Proposition 3.3 together with the Riesz–Thorin interpolation theorem yield immediately the following result.

COROLLARY 3.4. *For every* $2 \leqq r < p$ *or* $2 \geqq r > p \geqq 1$ *and every linear operator* S *on* l_p^n, *there exists a subset* η *of* $\{1,2,\dots,n\}$ *such that* $|\eta| \geqq n/2$ *and*

$$\| R_\eta S R_\eta \|_r \leqq 4K_G \| S \|_p .$$

Corollary 3.4 cannot be improved beyond the range $2 \leqq r < p$ or $2 \geqq r > p \geqq 1$. This fact is illustrated by two examples which will be presented in Section 5.

PROPOSITION 3.5. *There is a constant* $K < \infty$ *with the property that, for any* $0 < \delta < 1/e^{e^2}$, *one can find an integer* $n(\delta)$ *such that, whenever* $n \geqq n(\delta)$, $1 < r < 2$ *and* S *is a linear operator on* l_r^n, *then there exists a subset* η *of* $\{1,2,\dots,n\}$ *of cardinality* $|\eta| = [\delta n]$, *for which*

$$\| R_\eta S x \|_1 \leqq K(\delta^2 n/(r'-2))^{1/r'} \cdot \| Sx \|_r ,$$

if $x \in l_r^n$ *has support of cardinality* $\leqq [\delta^2 n]$.

PROOF. Fix δ, r and S, as above, and put $h = [\delta^2 n]$. Then, for each subset σ of $\{1,2,\dots,n\}$ of cardinality $|\sigma| = h$, take $\varepsilon = \frac{1}{2}$ and select an ε-net $\mathcal{G}(\sigma)$ in the unit sphere of $[Se_i]_{i \in \sigma}$, considered as a subspace of l_r^n, such that

$$|\mathcal{G}(\sigma)| \leqq (2/\varepsilon)^h = 4^h .$$

Put

$$\mathcal{G} = \bigcup \{ \mathcal{G}(\sigma); \sigma \subset \{1,2,\dots,n\}, |\sigma| = h \}$$

and let $\{\xi_i\}_{i=1}^n$ be, as usual, a sequence of independent random variables of mean δ over some probability space (Ω, Σ, μ) which take only the values 0 and 1. Then, with

$$m = \left[\delta^2 \left(\log \frac{1}{\delta} \right) n \right] + 1,$$

we have

$$J = \int_\Omega \max \left\{ \sum_{i=1}^n |c_i| \, \xi_i(\omega);\ c = \sum_{i=1}^n c_i e_i \in \mathcal{G} \right\} d\mu(\omega)$$

$$\leqq \left(\sum \left\{ \left\| \sum_{i=1}^n |c_i| \, \xi_i \right\|_m^m ;\ c = \sum_{i=1}^n c_i e_i \in \mathcal{G} \right\} \right)^{1/m} .$$

However, by Proposition 1.8, there exists a constant $A < \infty$ so that

$$\left\| \sum_{i=1}^{n} |c_i| \xi_i \right\|_m \leqq A \cdot \left(\frac{m}{(r'-2)\log(1/\delta)} \right)^{1/r'},$$

for any choice of $c = \sum_{i=1}^n c_i e_i$ in l_r^n with $\|c\|_r = 1$. Note also that, by Stirling's formula,

$$|\mathcal{G}| \leqq \binom{n}{h} 4^h \leqq (4en/h)^h \leqq (8e/\delta^2)^h,$$

i.e.

$$\log|\mathcal{G}| \leqq \delta^2 n \left(\log 8e + 2\log\frac{1}{\delta} \right) \leqq 5\delta^2 n \log\frac{1}{\delta} \leqq 5m.$$

Consequently,

$$J \leqq |\mathcal{G}|^{1/m} A \left(\frac{m}{(r'-2)\log(1/\delta)} \right)^{1/r'} \leqq 2Ae^5(\delta^2 n/(r'-2))^{1/r'}.$$

Observe now that one can find a point ω_0 in the set

$$\hat{\Omega} = \left\{ \omega \in \Omega;\ \delta n/2 \leqq \sum_{i=1}^{n} \xi_i(\omega) \leqq 3\delta n/2 \right\}$$

such that

$$\sum_{i=1}^{n} |c_i| \xi_i(\omega_0) \leqq 2Ae^5(\delta^2 n/(r'-2))^{1/r'},$$

for all $c = \sum_{i=1}^n c_i e_i \in \mathcal{G}$. The proof can be now completed by taking as η any subset of $\tau = \{1 \leqq i \leqq n;\ \xi_i(\omega_0) = 1\}$ which has cardinality $[\delta n/2]$ (note that $|\tau| \geqq \delta n/2$). □

Proposition 3.6. *For any $0 < \delta < 1/e^{e^2}$ and $M < \infty$, one can find a constant $d = d(\delta, M) > 0$ such that, whenever $1 < r < p < 2$ and S is a linear operator on l_p^n of norm $\|S\|_p \leqq M$ whose matrix relative to the unit vector basis has 0's on the diagonal, then there exist a constant D_r, depending only on r, and subsets η_1 and τ of $\{1, 2, \ldots, n\}$ such that $|\eta_1| = [dn]$, $|\tau| = [\delta dn]$, $\eta_1 \supset \tau$ and*

$$\|R_\tau Sx\|_1 \leqq D_r(\delta^2 dn)^{1/r'}(\|R_{\eta_1} Sx\|_r + \|x\|_r),$$

for all $x \in [e_i]_{i \in \tau}$.

Proof. Fix δ, M and S, as above. By Corollary 3.4, there exists a subset η_0 of $\{1, 2, \ldots, n\}$ so that $|\eta_0| \geqq n/2$ and

$$\|R_{\eta_0} S R_{\eta_0}\|_2 \leqq 4K_G M.$$

Thus, by Theorem 1.6, one can find a constant $d = d(\delta, M) > 0$ and a subset η_1 of η_0 such that $n_1 = |\eta_1| = [dn]$ and

$$\| R_{\eta_1} S R_{\eta_1} \|_2 < \delta^2.$$

Furthermore, by applying Proposition 3.5 to the operator $R_{\eta_1} S R_{\eta_1}$ with r satisfying the condition $1 < r < p$, we deduce the existence of a constant K_r, depending only on r, and of a subset τ of η_1, for which $|\tau| = [\delta d n]$ and

$$\| R_\tau S x \|_1 \leqq K_r (\delta^2 n_1)^{1/r'} \| R_{\eta_1} S x \|_r,$$

whenever $x \in R_{\eta_1} l_r^n$ has support of cardinality $\leqq [\delta^2 n_1]$.

The main difficulty encountered in the present proof derives from the fact that $|\tau|$ is larger, by $1/\delta$, than the cardinality of the support of x, for which the above inequality is valid. In order to overcome this problem, for $x = \Sigma_{i \in \tau} x_i e_i \in [e_i]_{i \in \tau}$, we put

$$\tau_x = \{ i \in \tau ; |x_i| < \| x \|_r / (\delta^2 n_1)^{1/r} \}, \qquad y = R_{\tau_x} x \quad \text{and} \quad z = x - y.$$

Then $z = \Sigma_{i \in \tau \sim \tau_x} x_i e_i$ satisfies

$$\| x \|_r^r \geqq \| z \|_r^r \geqq \| x \|_r^r | \tau \sim \tau_x | / \delta^2 n_1,$$

i.e.

$$| \tau \sim \tau_x | \leqq \delta^2 n_1,$$

and we are allowed to apply the above inequality to z. It follows that

$$\| R_\tau S x \|_1 \leqq \| R_\tau S y \|_1 + \| R_\tau S z \|_1 \leqq | \tau |^{1/2} \| R_\tau S y \|_2 + K_r (\delta^2 n_1)^{1/r'} \| R_{\eta_1} S z \|_r.$$

However, by the estimate for the norm of $R_{\eta_1} S R_{\eta_1}$ as an operator on l_2^n, we obtain that

$$\| R_\tau S y \|_2 < \delta^2 \| y \|_2 = \delta^2 \cdot \left(\sum_{i \in \tau_x} | x_i |^2 \right)^{1/2} < \delta^2 | \tau |^{1/2} \| x \|_r / (\delta^2 n_1)^{1/r}.$$

We also have that

$$\| R_{\eta_1} S z \|_r \leqq \| R_{\eta_1} S x \|_r + \| R_{\eta_1} S y \|_r$$

and

$$\| R_{\eta_1} S y \|_r \leqq n_1^{1/r - 1/2} \cdot \| R_{\eta_1} S y \|_2 \leqq \delta^{2/r'} \| x \|_r.$$

By substituting these estimates in the inequality above, we get that

$$\| R_\tau S x \|_1 \leqq \delta^2 | \tau | \| x \|_r / (\delta^2 n_1)^{1/r} + K_r (\delta^2 n_1)^{1/r'} [\| R_{\eta_1} S x \|_r + \delta^{2/r'} \| x \|_r]$$

$$\leqq (\delta + K_r \delta^{2/r'}) (\delta^2 n_1)^{1/r'} \| x \|_r + K_r (\delta^2 n_1)^{1/r'} \| S x \|_r$$

which proves the assertion. □

In general, an operator from an L_∞-space to an L_q-space; $1<q<\infty$, need not be q-absolutely summing. The next result gives a condition that ensures this fact.

PROPOSITION 3.7. *Let τ be a subset of $\{1,2,\dots,n\}$ and $1<r<p<2$, and suppose that W is a linear operator on l_p^n which satisfies the condition*

$$\|R_\tau Wx\|_1 \leqq C\cdot(\|Wx\|_r+\|x\|_r),$$

for some constant $C<\infty$ and all $x\in[e_i]_{i\in\tau}$. Then there exists a constant A_r, depending only on r, such that $R_\tau W^*R_\tau$ is p'-absolutely summing when it is considered as an operator from l_∞^n into $l_{p'}^n$ and

$$\pi_{p'}(R_\tau W^*R_\tau\colon l_\infty^n\to l_{p'}^n)\leqq A_r\cdot C\|W\|_p n^{1/r-1/p}.$$

PROOF. Put $W_\tau=R_\tau WR_\tau$, take vectors $\{u_i\}_{i=1}^h$ in l_∞^n so that $(\Sigma_{i=1}^h|u_i|^{p'})\leqq 1$ (coordinatewise) and choose elements $\{v_i\}_{i=1}^h$ in l_p^n, for which

$$\sum_{i=1}^h\|v_i\|_p^p\leqq 1\quad\text{and}\quad\left(\sum_{i=1}^h\|W_\tau^*u_i\|_{p'}^{p'}\right)^{1/p'}=\sum_{i=1}^h\langle W_\tau^*u_i,v_i\rangle.$$

Let now $\{\varphi_i\}_{i=1}^h$ be a sequence of p-stable independent random variables over a probability space (Ω,Σ,μ) which are normalized in $L_1(\Omega,\Sigma,\mu)$. Then

$$\begin{aligned}\left|\sum_{i=1}^h\langle W_\tau^*u_i,v_i\rangle\right|&=\left|\sum_{i=1}^h\langle u_i,W_\tau v_i\rangle\right|\\&\leqq\left\|\left(\sum_{i=1}^h|W_\tau v_i|^p\right)^{1/p}\right\|_1\\&=\left\|\int_\Omega\left|\sum_{i=1}^h\varphi_i(\omega)W_\tau v_i\right|d\mu(\omega)\right\|_1\\&=\int_\Omega\left\|\sum_{i=1}^h\varphi_i(\omega)W_\tau v_i\right\|_1d\mu(\omega)\\&\leqq C\int_\Omega\left(\left\|\sum_{i=1}^h\varphi_i(\omega)Wv_i\right\|_r+\left\|\sum_{i=1}^h\varphi_i(\omega)v_i\right\|_r\right)d\mu(\omega)\\&\leqq C\left(\left\|\left(\int_\Omega\left|\sum_{i=1}^h\varphi_i(\omega)Wv_i\right|^rd\mu(\omega)\right)^{1/r}\right\|_r\right.\\&\qquad\left.+\left\|\left(\int_\Omega\left|\sum_{i=1}^h\varphi_i(\omega)v_i\right|^rd\mu(\omega)\right)^{1/r}\right\|_r\right)\\&\leqq A_rC\left(\left\|\left(\sum_{i=1}^h|Wv_i|^p\right)^{1/p}\right\|_r+\left\|\left(\sum_{i=1}^h|v_i|^p\right)^{1/p}\right\|_r\right),\end{aligned}$$

where A_r denotes the norm of φ_1 in $L_r(\Omega,\Sigma,\mu)$. It follows that

$$\left(\sum_{i=1}^{h} \| W_\tau^* u_i \|_{p'}^{p'}\right)^{1/p'} \leqq A_r C n^{1/r-1/p}\left(\left\|\left(\sum_{i=1}^{h} | W v_i |^p\right)^{1/p}\right\|_p + \left\|\left(\sum_{i=1}^{h} | v_i |^p\right)^{1/p}\right\|_p\right),$$

and this completes the proof. □

PROOF OF THEOREM 3.1. First, note that there is no loss of generality in assuming that $1 \leqq p \leqq 2$ since in the case $2 < p \leqq \infty$ we can consider the adjoint operator S^* instead of S.

The Case $p = 1$. As we have already mentioned in the introduction to this section, the case $p = 1$ is actually known, though not exactly in the formulation of Theorem 3.1. In this case, the matrix $(a_{i,j})_{i,j=1}^n$ of S relative to the unit vector basis of l_1^n satisfies

$$\sum_{j=1}^{n} | a_{i,j} | \leqq \| S \|_1; \qquad 1 \leqq i \leqq n.$$

Thus, by [4] or [13], for each $\varepsilon > 0$, there exists a subset σ_ε of $\{1, 2, \ldots, n\}$ of cardinality $| \sigma_\varepsilon | \geqq n\varepsilon^2/16 \| S \|_1^2$ such that

$$\sum_{j \in \sigma_\varepsilon} | a_{i,j} | \leqq \varepsilon; \qquad i \in \sigma_\varepsilon.$$

The Case $1 < p < 2$. Fix $0 < \delta < 1/e^{\varepsilon^2}$ and $1 < r < p$ so that $2/r' > 1/p'$, and let $S: l_p^n \to l_p^n$ be a linear operator whose matrix has 0's on the diagonal.

Let $d = d(\delta, \| S \|_p) > 0$, $D_r < \infty$, $\tau \subset \eta_1 \subset \{1, 2, \ldots, n\}$ be given by Proposition 3.6 so that $n_1 = | \eta_1 | = [dn]$, $| \tau | = [\delta dn]$ and

$$\| R_\tau S x \|_1 \leqq C(\| S x \|_r + \| x \|_r),$$

for all $x \in [e_i]_{i \in \tau}$, where $C = D_r(\delta^2 dn)^{1/r'}$.

Since the operator $W = R_{\eta_1} S R_{\eta_1}$ satisfies the conditions of Proposition 3.7, it follows that

$$\begin{aligned}\pi_{p'}(R_\tau S^* R_\tau : l_\infty^n \to l_{p'}^n) &\leqq A_r C \| S \|_p n_1^{1/r-1/p} \\ &\leqq 2A_r D_r \| S \|_p \delta^{2/r'-1/p'} | \tau |^{1/p'},\end{aligned}$$

where A_r is a constant depending on r only. Thus, by the Pietsch factorization theorem [24], there exists an operator $U: l_{p'}^n \to l_{p'}^n$ with $\| U \|_{p'} \leqq 1$ and a diagonal operator $V: l_\infty^n \to l_{p'}^n$, defined by $Ve_i = \lambda_i e_i$; $i \in \tau$, and $Ve_i = 0$; $i \notin \tau$, so that $R_\tau S^* R_\tau = UV$ and

$$\left(\sum_{i \in \tau} | \lambda_i |^{p'}\right)^{1/p'} \leqq K | \tau |^{1/p'},$$

where $K = 2A_r D_r \| S \|_p \delta^{2/r'-1/p'}$.

Consider now the set

$$\sigma = \{ i \in \tau ; |\lambda_i| < 2^{1/p'} K \}$$

and note that

$$|\sigma| \geqq |\tau|/2.$$

Moreover, since $R_\sigma S R_\sigma = R_\sigma V^* U^* R_\sigma$ and $V^* e_i = \lambda_i e_i$; $i \in \tau$, it follows easily that, for any $x \in l_p^n$, we have that

$$\begin{aligned} \| R_\sigma S R_\sigma x \|_p &= \| R_\sigma V^* U^* R_\sigma x \|_p \\ &\leqq \left(\max_{i \in \sigma} |\lambda_i| \right) \cdot \| U^* R_\sigma x \|_p \\ &\leqq 2K \| x \|_p . \end{aligned}$$

Hence, if $\varepsilon > 0$ is given and δ is chosen appropriately then $\| R_\sigma S R_\sigma \|_p < \varepsilon$. This completes the proof since the case $p = 2$ has already been considered in Theorem 1.6. □

We pass now to the study of operators on l_p^n-spaces whose matrices have "large" rows rather than "large" diagonal.

We first give an estimate for the rank of such an operator.

Proposition 3.8. *Let $p > 2$ and let T: $l_p^n \to l_p^n$ be a linear operator such that $\| T e_i \|_p = 1$; $1 \leqq i \leqq n$. Then*

$$\operatorname{rank} T \geqq n / \| T \|^2 .$$

Proof. If $k = \operatorname{rank} T$ than it is well known that the 2-summing norm $\pi_2(T)$ of T satisfies (cf. [24])

$$\pi_2(T) \leqq \| T \| \cdot \sqrt{k}.$$

Hence

$$\begin{aligned} \sqrt{n} &= \left(\sum_{j=1}^{n} \| T e_j \|_p^2 \right)^{1/2} \\ &\leqq \pi_2(T) \sup \left\{ \left(\sum_{j=1}^{n} |x^*(e_j)|^2 \right)^{1/2} ; x^* \in l_{p'}^n, \| x^* \|_{p'} \leqq 1 \right\} \\ &\leqq \| T \| \cdot \sqrt{k}, \end{aligned}$$

which completes the proof. □

Actually, an operator with "large" rows on l_p^n; $p>2$, has also a "large" permutation of the diagonal when it is restricted to some set of unit vectors which is proportional to n.

PROPOSITION 3.9. *For every $p>2$ and $M<\infty$, there exists a $c=c(p,M)>0$ so that, whenever T is a linear operator on l_p^n of norm $\|T\|\leqq M$ for which $\|Te_i\|_p=1$; $1\leqq i\leqq n$, then there exist a subset η of $\{1,2,\dots,n\}$ and a one-to-one mapping π from η into $\{1,2,\dots,n\}$ such that*

$$|\eta|\geqq cn \quad \textit{and} \quad |e^*_{\pi(i)}Te_i|\geqq c,$$

for all $i\in\eta$, where $\{e_i^*\}_{i=1}^n$ denote the unit vectors in $l_{p'}^n$.

PROOF. Put $x_i=Te_i$; $1\leqq i\leqq n$, and observe that, by Grothendieck's inequality,

$$\left\|\left(\sum_{i=1}^n|x_i|^2\right)^{1/2}\right\|_p\leqq K_G\|T\|n^{1/p}.$$

Then a simple interpolation argument shows that

$$n^{1/p}=\left\|\left(\sum_{i=1}^n|x_i|^p\right)^{1/p}\right\|_p\leqq\left\|\left(\sum_{i=1}^n|x_i|^2\right)^{1/2}\right\|_p^{2/p}\cdot\left\|\max_{1\leqq i\leqq n}|x_i|\right\|_p^{1-2/p},$$

i.e.

$$\left\|\max_{1\leqq i\leqq n}|x_i|\right\|_p\geqq dn^{1/p},$$

where $d=1/(K_G\cdot\|T\|)^{2/(p-2)}$. Put

$$\sigma=\left\{1\leqq l\leqq n;\, e_l^*\left(\max_{1\leqq i\leqq n}|x_i|\right)\geqq d/2^{1/p}\right\}$$

and observe that

$$\left\|\chi_\sigma\cdot\max_{1\leqq i\leqq n}|x_i|\right\|_p>d(n/2)^{1/p}.$$

Now split σ into mutually disjoint subsets $\{\sigma_j\}_{j=1}^n$ so that

$$e_l^*\left(\max_{1\leqq i\leqq n}|x_i|\right)=e_l^*|x_j|;\qquad l\in\sigma_j,\quad j=1,2,\dots,n.$$

Put

$$\eta=\{1\leqq j\leqq n;\,\sigma_j\neq\emptyset\}$$

and note that, for each $j \in \eta$, there exists an element $\pi(j) \in \sigma_j$ such that $\pi(j) \neq \pi(k)$, whenever $j, k \in \eta$ and $j \neq k$. Finally, observe that

$$\begin{aligned} d^p n/2 &\leqq \Big\| \sum_{j \in \eta} \chi_{\sigma_j} \cdot \max_{1 \leqq i \leqq n} |x_i| \Big\|_p^p \\ &= \sum_{j \in \tau} \Big\| \chi_{\sigma_j} \cdot \max_{1 \leqq i \leqq n} |x_i| \Big\|_p^p \\ &= \sum_{j \in \eta} \| \chi_{\sigma_j} x_j \|_p^p \\ &\leqq \| T \|^p \, |\eta|, \end{aligned}$$

i.e.

$$|\eta| \geqq d^2 n / 2 \| T \|^p. \qquad \square$$

Corollary 3.10. *For every $p \geqq 2$ and $M < \infty$, there exists a $d = d(p, M) > 0$ such that, whenever T is a linear operator on l_p^n of norm $\|T\| \leqq M$ for which $\|Te_i\|_p = 1$; $1 \leqq i \leqq n$, then there exists a subset σ of $\{1, 2, \ldots, n\}$ of cardinality $|\sigma| \geqq dn$ so that*

$$\Big\| \sum_{j \in \sigma} a_j T e_j \Big\|_p \geqq d \Big(\sum_{j \in \sigma} |a_j|^p \Big)^{1/p},$$

for any choice of scalars $\{a_j\}_{j \in \sigma}$.

Proof. For the case $p = 2$, the matrix corresponding to the operator T^*T has 1's on the diagonal and the proof can be completed by using Corollary 3.2. If, on the other hand, $p > 2$ then it follows from Proposition 3.9 that $\pi^{-1} T_{|[e_i]_{i \in \eta}}$, where π^{-1} is the corresponding permutation operator, has "large" diagonal, and again the proof can be completed by using Corollary 3.2. $\square$

The results obtained above for matrices with "large" rows acting on l_p^n-spaces with $p > 2$ are, in general, false in the case of $1 \leqq p < 2$. Even the rank of such an operator need not be proportional to n.

Proposition 3.11. *Let $1 < p < 2$ and let T: $l_p^n \to l_p^n$ be a linear operator such that $\|Te_i\|_p = 1$; $1 \leqq i \leqq n$. Then*

$$\operatorname{rank} T \geqq n^{2/p'} / \| T \|^2.$$

Proof. The argument is identical to that used in order to prove Proposition 3.8. If $k = \operatorname{rank} T$ then, again,

$$\begin{aligned} n^{1/2} &= \left(\sum_{i=1}^{n} \| Te_i \|_p^2 \right)^{1/2} \\ &\leqq \pi_2(T) \sup \left\{ \left(\sum_{i=1}^{n} | x^*(e_i) |^2 \right)^{1/2} ; x^* \in l_{p'}^n, \| x^* \|_{p'} \leqq 1 \right\} \\ &\leqq \| T \| \cdot k^{1/2} \cdot n^{1/2 - 1/p'}, \end{aligned}$$

which yields our assertion. □

The estimate from below for the rank of T given by Proposition 3.11 is asymptotically sharp, as is shown by the following example. The case $p = 1$ is absolutely trivial; it is easily seen that there are operators on l_1^n with $\| Te_i \|_1 = 1$; $1 \leqq i \leqq n$, and rank $T = 1$.

PROPOSITION 3.12. *For every $1 < p < 2$, there is a constant M_p and, for each integer n, there exists a linear operator T: $l_p^n \to l_p^n$ of norm $\leqq M_p$ such that $\| Te_i \|_p = 1$; $1 \leqq i \leqq n$, but*

$$\operatorname{rank} T = [n^{2/p'}].$$

Essentially speaking, this property is shared by all the orthogonal projections from l_p^n onto well-complemented Hilbert subspaces of l_p^n having maximal dimension.

PROOF. Fix $1 < p < 2$ and n. For sake of simplicity of notation, we shall work with the function space L_p^n instead of l_p^n. By a result from [3], there exists a constant $C_p < \infty$ such that, for each n, the space $L_{p'}^n$ contains a subspace H_m of dimension $m = [n^{2/p'}]$ for which $d(H_m, l_2^m) \leqq C_p$. By B. Maurey [22] Theorem 76, there exists a constant $C'_p < \infty$, an element $g \in L_r^n$ with $\| g \|_r = 1$, where r satisfies

$$\frac{1}{2} = \frac{1}{p'} + \frac{1}{r},$$

and a linear operator from gH_m, considered as a subspace of L_2^n, into $L_{p'}^n$ such that $\| S \| \leqq C'_p$ and

$$S(gx) = x; \qquad x \in H_m.$$

By Hölder's inequality, we have

$$\| gz \|_2 \leqq \| g \|_r \cdot \| z \|_{p'} = \| z \|_{p'}; \qquad z \in L_{p'}^n.$$

On the other hand, if $x \in H_m$ then we also have

$$\| x \|_{p'} \leqq \| S \| \cdot \| gx \|_2 \leqq C'_p \| gx \|_2,$$

i.e.

$$\| gx \|_2 \leqq \| x \|_{p'} \leqq C'_p \| gx \|_2,$$

for all $x \in H_m$. In a similar way, we can show that

$$\| gy \|_p \leqq \| y \|_2 \leqq C'_p \| gy \|_p,$$

for all $y \in gH_m$, considered as a subspace of L_2^n.

Let R be the orthogonal projection from L_2^n onto its subspace gH_m and denote by M_g the operator acting as "multiplication" by g. Then

$$P = M_{g^{-1}} R M_g$$

defines clearly a projection from $L_{p'}$ onto H_m and, by the above estimates,

$$\| Pz \|_{p'} = \| g^{-1} R(gz) \|_{p'} \leqq C'_p \| R(gz) \|_2 \leqq C'_p \| gz \|_2 \leqq C'_p \| z \|_{p'},$$

for all $z \in L_{p'}^n$, i.e. $\| P \| \leqq C'_p$. By duality, we conclude that P^* is a projection from L_p^n onto its subspace $g^2 H_m$, and, moreover, that $P^* = M_g R M_{g^{-1}}$.

Now, observe that

$$n^{2/p'}/2 \leqq m = \| R \|_{\mathrm{HS}}^2 = n \sum_{j=1}^n \| Re_j \|_2^2$$

$$\leqq (C'_p)^2 n \sum_{j=1}^n \| gRe_j \|_p^2 = (C'_p)^2 n \sum_{j=1}^n \| P^*(ge_j) \|_p^2$$

since the unit vectors $\{e_j\}_{j=1}^n$ have norm equal to $n^{-1/2}$, when considered in L_2^n.

Suppose that

$$g = \sum_{j=1}^n g_j e_j.$$

Then $(\sum_{j=1}^n | g_j |^r / n)^{1/r} = \| g \|_r = 1$ from which it follows, by using Hölder's inequality, that

$$n \leqq 2(C'_p)^2 \sum_{j=1}^n | g_j |^2 \cdot \| P^*(n^{1/p} e_j) \|_p^2$$

$$\leqq 2(C'_p)^2 \left(\sum_{j=1}^n | g_j |^r \right)^{2/r} \cdot \left(\sum_{j=1}^n \| P^*(n^{1/p} e_j) \|_p^{p'} \right)^{2/p'}$$

$$\leqq 2(C'_p)^2 n^{2/r} \cdot \left(\sum_{j=1}^n \| P^*(n^{1/p} e_j) \|_p^{p'} \right)^{2/p'}.$$

Hence

$$n \leqq 2^{p'/2}(C'_p)^{p'} \cdot \sum_{j=1}^{n} \| P^*(n^{1/p}e_j)\|_p^{p'},$$

which yields that the set

$$\sigma = \{1 \leqq j \leqq n ; \| P^*(n^{1/p}e_j)\|_p \geqq 1/2^{1/p'+1/2} \cdot C'_p\}$$

has cardinality

$$|\sigma| \geqq n/2^{1+p'/2}(C'_p)^{2p'}.$$

The proof can be now easily completed by constructing an operator $T: l_p^n \to l_p^n$ with $\| Te_i \|_p = 1$; $1 \leqq i \leqq n$, so that its range is contained in $g^2 H_m$, i.e. rank $T = [n^{2/p'}]$. □

While Proposition 3.9 is clearly false for $1 \leqq p < 2$, in view of Proposition 3.12, a weaker version still holds.

PROPOSITION 3.13. *For every* $1 \leqq p < 2$, $M < \infty$ *and* $c > 0$, *there exists a* $d = d(p, M, c) > 0$ *so that, whenever* T *is a linear operator on* l_p^n *of norm* $\| T \| \leqq M$, *for which*

$$\int \left\| \sum_{i=1}^{n} \varepsilon_i Te_i \right\|_p d\varepsilon \geqq cn^{1/p},$$

then there exist a subset η *of* $\{1, 2, \ldots, n\}$ *and a one-to-one mapping* π *from* η *into* $\{1, 2, \ldots, n\}$ *so that*

$$|\eta| \geqq dn \quad \text{and} \quad |e^*_{\pi(i)} Te_i| \geqq d,$$

for all $i \in \eta$.

PROOF. The condition imposed above on the vectors $x_i = Te_i$; $1 \leqq i \leqq n$, implies that

$$\left\| \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} \right\|_p \geqq cn^{1/p}.$$

On the other hand, it is entirely trivial that

$$\left\| \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \right\|_p = \left(\sum_{i=1}^{n} \| x_i \|_p^p \right)^{1/p} \leqq Mn^{1/p}.$$

Therefore, by an interpolation argument as in the proof of Proposition 3.9, we conclude that

$$cn^{1/p} \leqq \left\| \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} \right\|_p \leqq \left\| \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \right\|_p^{p/2} \cdot \left\| \max_{1 \leqq i \leqq n} |x_i| \right\|_p^{1-p/2},$$

which yields that

$$\left\| \max_{1 \leqq i \leqq n} |x_i| \right\|_p \geqq c^{2/(2-p)}/M^{p/(2-p)}.$$

The rest of the proof goes exactly as in that of Proposition 3.9. □

4. Applications to the geometry of Banach spaces

The results proved in the previous section will be used in the present one in order to solve some problems concerning the existence of "well" complemented copes of l_p^n-spaces in L_p which were raised by W. B. Johnson and G. Schechtman [13]. The main part of this section is devoted to a study of those subspaces of L_p whose euclidean distance (i.e., the distance to a Hilbert space of the same dimension) is maximal.

We discuss first the isometric version. It is well-known that the euclidean distance of an l_p^n-space satisfies

$$d_{l_p^n} = d(l_p^n, l_2^n) = n^{|1/p-1/2|},$$

for any n and $1 \leqq p \leqq \infty$. Less trivial is the fact that, for a fixed n and $1 \leqq p \leqq \infty$, l_p^n has the largest euclidean distance among all the subspaces of L_p of dimension n. For $p = 1$ or $p = \infty$, this assertion is part of a considerably more general theorem of F. John (see e.g. [10]) which states that *any* n-dimensional Banach space has euclidean distance $\leqq n^{1/2}$. In the case $1 < p \neq 2$, the fact that the euclidean distance of any n-dimensional subspace X of L_p satisfies

$$d_X = d(X, l_2^n) \leqq n^{|1/p-1/2|},$$

was proved by D. R. Lewis [19].

This maximality property of the euclidean distance of l_p^n-spaces raises the question whether these spaces are the only subspaces of L_p which have a maximal euclidean distance. For $p = 1$, the problem was settled in the positive by T. Figiel and W. B. Johnson [9]. We prove below a similar assertion for $1 \leqq p < 2$. The case $p > 2$ is still open.

THEOREM 4.1. *Fix n and $1 < p < 2$. Then any n-dimensional subspace X of L_p whose euclidean distance is maximal, i.e.*

$$d_X = n^{1/p-1/2},$$

is necessarily isometric to l_p^n.

We first need a result which, in particular, gives a simple alternative proof to the aforementioned theorem of D. R. Lewis in the case $1<p<2$.

PROPOSITION 4.2. *Fix n and $1<p<2$, and let (Ω,Σ,μ) be a probability space. The euclidean distance d_X of an arbitrary n-dimensional subspace X of $L_p(\Omega,\Sigma,\mu)$ satisfies the estimate*

$$d_X \leqq \|T\|\cdot\|T^{-1}\| \leqq \|T^{-1}\| \leqq \|F_X\|_p^{1-p/2} \leqq n^{1/p-1/2},$$

where

$$F_X(\omega)=\sup\{|f(\omega)|;\ f\in X,\ \|f\|_p\leqq 1\}$$

and $T\colon X\to L_2(\Omega,\Sigma,\mu)$ is the linear operator defined by

$$Tf=f\cdot F_X^{p/2-1};\qquad f\in X.$$

PROOF. Observe that, for any $f\in X$ with $\|f\|_p\leqq 1$, we have that

$$\begin{aligned}\|Tf\|_2^2&=\int_\Omega |f(\omega)|^2F_X(\omega)^{p-2}d\mu(\omega)\\&=\int_\Omega |f(\omega)|^p\,|f(\omega)/F_X(\omega)|^{2-p}d\mu(\omega)\\&\leqq\int_\Omega |f(\omega)|^p d\mu(\omega),\end{aligned}$$

i.e.

$$\|T\|\leqq 1.$$

On the other hand, by using Hölder's inequality, it follows that, for any $f\in X$, we also have

$$\|f\|_p^p=\int_\Omega |(Tf)(\omega)|^pF_X(\omega)^{p(1-p/2)}d\mu(\omega)\leqq\|Tf\|_2^p\cdot\|F_X\|_p^{p(1-p/2)},$$

i.e.

$$\|T^{-1}\|\leqq\|F_X\|_p^{1-p/2}.$$

Since, by definition, $d_X\leqq\|T\|\cdot\|T^{-1}\|$ it remains to show that $\|F_X\|_p\leqq n^{1/p}$.

By using a well-known characterization of Hilbert spaces, due to S. Kwapien [18], together with a result of N. Tomczak-Jaegermann [30], we get the more trivial estimate

$$d_X\leqq T_2^{(g)}(X)\cdot C_2^{(g)}(X)\leqq Kn^{1/p-1/2},$$

for some constant $K<\infty$, independent of p and n, where $T_2^{(g)}(X)$ and $C_2^{(g)}(X)$ denote the gaussian type 2, respectively cotype 2, constants of X.

Another argument of an elementary nature which proves the same estimate for the euclidean distance as above can be found in [28].

It follows that there exists a sequence $\{f_i\}_{i=1}^n$ of vectors in X so that

$$\left(\sum_{i=1}^n |a_i|^2\right)^{1/2} \leqq \left\|\sum_{i=1}^n a_i f_i\right\|_p \leqq K n^{1/p-1/2}\cdot\left(\sum_{i=1}^n |a_i|^2\right)^{1/2},$$

for any choice of scalars $\{a_i\}_{i=1}^n$. Since any $f\in X$ with $\|f\|_p\leqq 1$ can be represented as a linear combination $f=\sum_{i=1}^n b_i f_i$, for a suitable choice of $\{b_i\}_{i=1}^n$, we conclude that

$$\begin{aligned}|f(\omega)| &\leqq \sum_{i=1}^n |b_i|\cdot|f_i(\omega)| \\ &\leqq \left(\sum_{i=1}^n |b_i|^2\right)^{1/2}\cdot\left(\sum_{i=1}^n |f_i(\omega)|^2\right)^{1/2} \\ &\leqq \left(\sum_{i=1}^n |f_i(\omega)|^2\right)^{1/2},\end{aligned}$$

for all $\omega\in\Omega$. Hence, also the maximal function F_X satisfies

$$F_X(\omega)\leqq\left(\sum_{i=1}^n |f_i(\omega)|^2\right)^{1/2},$$

for all $\omega\in\Omega$, and thus

$$\|F_X\|_p \leqq \left\|\left(\sum_{i=1}^n |f_i|^2\right)^{1/2}\right\|_p \leqq \sqrt{2}\int\left\|\sum_{i=1}^n \varepsilon_i f_i\right\|_p d\varepsilon \leqq K\sqrt{2}\,n^{1/p}.$$

Fix now an integer k and let $\Omega^{(k)}$ denote the product space $\Omega\times\Omega\times\cdots\times\Omega$ (k times) endowed with the corresponding σ-field $\Sigma^{(k)}$ and product measure $\mu^{(k)}$. Let $X^{(k)}$ be the subspace of $L_p(\Omega^{(k)},\Sigma^{(k)},\mu^{(k)})$ obtained by the k-fold tensorization of X, i.e.

$$X^{(k)} = X\otimes X\otimes\cdots\otimes X$$

with $\dim X^{(k)}=n^k$. The maximal function $F_{X^{(k)}}(\omega_1,\omega_2,\ldots,\omega_k)$ of $X^{(k)}$ clearly satisfies

$$\begin{aligned}&F_{X^{(k)}}(\omega_1,\omega_2,\ldots,\omega_k)\\ &\quad\geqq \sup\{|g_1(\omega_1)|\cdot|g_2(\omega_2)|\cdots|g_k(\omega_k)|;\ g_j\in X,\ \|g_j\|_p\leqq 1,\ 1\leqq j\leqq k\}\\ &\quad\geqq F_X(\omega_1)\cdot F_X(\omega_2)\cdots F_X(\omega_k),\end{aligned}$$

for all $(\omega_1, \omega_2, \dots, \omega_k) \in \Omega^{(k)}$. Thus, by the above estimate for the norm of the maximal function, we get that

$$K\sqrt{2}\, n^{k/p} \geqq \| F_{X^{(k)}} \|_p \geqq \| F_X \|_p^k,$$

i.e.

$$\| F_X \|_p \leqq (K\sqrt{2})^{1/k} \cdot n^{1/p}.$$

By letting $k \to \infty$, we easily conclude that

$$\| F_X \|_p \leqq n^{1/p},$$

thus completing the proof. □

Proof of Theorem 4.1. Suppose now that X is a n-dimensional subspace of $L_p(\Omega, \Sigma, \mu)$ whose euclidean distance satisfies

$$d_X = n^{1/p - 1/2}.$$

Then, by Proposition 4.2, we get that

$$\| F_X \|_p = n^{1/p}$$

and, furthermore, that

$$\| T \| = 1.$$

Hence, by a simple compactness argument, we conclude the existence of a function $g \in X$ with $\| g \|_p = 1$ such that

$$\| Tg \|_2^2 = \int_\Omega | g(\omega) |^2 F_X(\omega)^{p-2} d\mu(\omega) = 1.$$

Since $| g(\omega) | \leqq F_X(\omega)$, for all $\omega \in \Omega$, it follows that

$$1 = \int_\Omega | g(\omega)/F_X(\omega) |^2 \cdot F_X(\omega)^p d\mu(\omega) \leqq \int_\Omega | g(\omega)/F_X(\omega) |^p \cdot F_X(\omega)^p d\mu(\omega) = 1,$$

i.e.

$$| g(\omega) | = F_X(\omega)$$

for a.e. ω in the support A of g.

We shall prove in the sequel that, for any $f \in X$, the restriction $f \cdot \chi_A$ of f to A belongs to the one-dimensional subspace $[g]$ of $L_p(\Omega, \Sigma, \mu)$ generated by g. This would imply that

$$X = ([g] \oplus X_1)_p,$$

where

$$X_1 = \{f \in X; f(\omega) = 0, \text{for a.e. } \omega \in A\}.$$

We also have that $\dim X_1 = n - 1$ and that

$$F_X(\omega) = \sup\{\lambda\,|g(\omega)| + (1-\lambda^p)^{1/p}\,|f(\omega)|;\ 0 \leqq \lambda \leqq 1,\ f \in X_1,\ \|f\|_p \leqq 1\}$$
$$= (|g(\omega)|^p + F_{X_1}(\omega)^p)^{1/p},$$

for a.e. $\omega \in \Omega$, which clearly yields that

$$\|F_{X_1}\|_p = (n-1)^{1/p}.$$

Furthermore, it is easily verified that

$$d_X \leqq (1 + d_{X_1}^{2p/(2-p)})^{1/p}$$

from where it follows that

$$d_{X_1} = (n-1)^{1/p - 1/2}.$$

Consequently, X_1 has the same properties as X if we replace n by $n-1$. Repeating the procedure for n times, we conclude that X is isometric to l_p^n.

In order to prove that $f\chi_A \in [g]$, for any $f \in X$, we fix $f \in X$ and $t > 0$, and note that

$$|g(\omega) + tf(\omega)| \leqq \|g + tf\|_p \cdot F_X(\omega); \qquad \omega \in \Omega.$$

Thus, by restricting this inequality to $\omega \in A$ and taking into account the fact that $\|g\|_p = 1$, we get that

$$\frac{|g(\omega)+tf(\omega)|^p - |g(\omega)|^p}{t} \leqq \left[\left(\int_A \frac{|g(\omega')+tf(\omega')|^p - |g(\omega')|^p}{t}\, d\mu(\omega') + t^{p-1}\int_{\Omega\sim A} |f(\omega')|^p d\mu(\omega')\right)\right] |g(\omega)|^p,$$

for all $\omega \in A$ (use the fact that on this set $F_X = |g|$). Since $p > 1$ we obtain, by letting $t \to 0$, $t > 0$, that

$$p\,|g(\omega)|^{p-1}(\operatorname{sgn} g(\omega))f(\omega) \leqq p\,|g(\omega)|^p \int_A |g(\omega')|^{p-1}(\operatorname{sgn} g(\omega'))f(\omega')d\mu(\omega'),$$

i.e.

$$f(\omega)/g(\omega) \leqq \int_A |g(\omega')|^p (f(\omega')/g(\omega'))d\mu(\omega'),$$

for all $\omega \in A$. However, the same inequality holds when f is replaced by $-f$, i.e.

$$f(\omega)/g(\omega) \geqq \int_A |g(\omega')|^p (f(\omega')/g(\omega'))d\mu(\omega'); \qquad \omega \in A.$$

This means that

$$f\chi_A = g\int_A |g(\omega')|^p (f(\omega')/g(\omega'))d\mu(\omega') \in [g],$$

thus completing the proof. □

We pass now to the study of the isomorphic case when we consider n-dimensional subspaces X of L_p whose euclidean distance satisfies

$$d_X \geqq cn^{|1/p-1/2|}$$

for some constant $c > 0$, independent of n. Of course, one cannot expect to prove in this case that X is well isomorphic to l_p^n but just that X contains a subspace Y of dimension k proportional to n which is well isomorphic to l_p^k.

For $p = 1$, this fact was proved in [13] (see also [4]) while for $1 < p < 2$ it is still an open problem. In the case $p > 2$, the assertion is false: there exist n-dimensional subspaces of L_p; $p > 2$, which contain copies of l_p^m only for $m \leqq Cn^{2/p'}$, for some constant $C < \infty$. Such examples are provided by the so-called random subspaces, on which the L_1- and L_2-norm are equivalent (cf. [9]).

The situation is different if we consider subspaces of L_p, $1 < p < \infty$, of maximal euclidean distance (in the isomorphic sense) which are also well-complemented in L_p. W. B. Johnson and G. Schechtman [13] proved that such a subspace X of L_p of $\dim X = n$ should contain, for each $\varepsilon > 0$, a well-complemented subspace X_ε of $k = \dim X_\varepsilon \geqq n^{1-\varepsilon}$ which is well isomorphic to l_p^k. They also raised the question whether this assertion is true with $n^{1-\varepsilon}$ being replaced by dn, for some $d > 0$. The following result shows that their problem has a positive solution.

Theorem 4.3. *For every $1 < p < \infty$, $M < \infty$ and $c > 0$, there exists a constant $C = C(p, M, c) < \infty$ so that, whenever X is a n-dimensional subspace of L_p for which*

(i) $d_X \geqq cn^{|1/p-1/2|}$

and

(ii) *there is a projection P of norm $\|P\| \leqq M$ from L_p onto X,*
then there exists a subspace Y of X and a linear projection R from L_p onto Y such that

$$k = \dim Y \geqq n/C, \quad d(Y, l_p^k) \leqq C \quad \text{and} \quad \|R\| \leqq C.$$

The deduction of Theorem 4.3 from Corollary 3.2 is basically known and in the easier case $p = 1$ can be found in [13]. We give here all the details for the sake of completeness. The proof requires the following result, an extension of which will be discussed in Section 7.

PROPOSITION 4.4. *For every $1 < p < \infty$ and $K > \infty$, there is a $D = D(p, K)$ such that, whenever $\{g_i\}_{i=1}^n$ and $\{h_i\}_{i=1}^n$ are sequences of functions in L_p, respectively $L_{p'}$, for which*

(i) $\|\sum_{i=1}^n a_i g_i\|_p \leqq K(\sum_{i=1}^n |a_i|^p)^{1/p}$, *for all* $\{a_i\}_{i=1}^n$,

(ii) $\|\sum_{i=1}^n b_i h_i\|_{p'} \leqq K(\sum_{i=1}^n |b_i|^{p'})^{1/p'}$, *for all* $\{b_i\}_{i=1}^n$, *and*

(iii) $\langle g_i, h_i \rangle = 1$, *for all* $1 \leqq i \leqq n$,

then there exist a subset σ of $\{1, 2, \ldots, n\}$ with $|\sigma| \geqq n/D$ and a projection R from L_p onto its subspace $[g_i]_{i \in \sigma}$ so that $\|R\| \leqq D$ and

$$D \left\| \sum_{i \in \sigma} a_i g_i \right\|_p \geqq \left(\sum_{i \in \sigma} |a_i|^p \right)^{1/p},$$

for any choice of $\{a_i\}_{i \in \sigma}$. In other words, the sequence $\{g_i\}_{i \in \sigma}$ is KD-equivalent to the unit vector basis of $l_p^{|\sigma|}$ and its linear span is D-complemented in L_p.

PROOF. Let $\{e_i\}_{i=1}^n$ and $\{e_i^*\}_{i=1}^n$ denote, as usual, the unit vector basis of l_p^n, respectively $l_{p'}^n$, and consider the operator $T: l_p^n \to l_p^n$, defined by

$$Te_i = \sum_{j=1}^n \langle g_i, h_j \rangle e_j\,; \qquad 1 \leqq i \leqq n.$$

It is easily checked that

$$|\langle Tx, y \rangle| \leqq K^2 \|x\|_p \cdot \|y\|_{p'},$$

for any choice of $x \in l_p^n$ and $y \in l_{p'}^n$, i.e. $\|T\| \leqq K^2$. Since, by (iii), $e_i^* Te_i = \langle g_i, h_i \rangle = 1$, for all $1 \leqq i \leqq n$, we can apply Corollary 3.2 and conclude the existence of a $d = d(K)$ and of a subset σ of $\{1, 2, \ldots, n\}$ such that $|\sigma| \geqq dn$, $R_\sigma T R_\sigma$ restricted to $R_\sigma l_p^n$ is invertible and its inverse satisfies

$$\|(R_\sigma T R_\sigma)^{-1}\| < 2.$$

Thus, by (i) and (ii), it follows that, for any $g \in [g_i]_{i \in \sigma}$ of the form $g = \sum_{i \in \sigma} a_i g_i$, we have

$$K\|g\|_p \geqq \sup\left\{\sum_{j\in\sigma} b_j\langle g, h_j\rangle;\ \sum_{j\in\sigma}|b_j|^{p'}\leqq 1\right\}$$

$$=\left(\sum_{j\in\sigma}|\langle g, h_j\rangle|^p\right)^{1/p}$$

$$=\left\|R_\sigma T\left(\sum_{i\in\sigma} a_i e_i\right)\right\|_p$$

$$\geqq\left(\sum_{i\in\sigma}|a_i|^p\right)^{1/p}\Big/2$$

$$\geqq \|g\|_p/2K,$$

i.e. $\{g_i\}_{i\in\sigma}$ is $2K^2$-equivalent to the unit vector basis of $l_p^{|\sigma|}$ and it remains to show that $[g_i]_{i\in\sigma}$ is well complemented in L_p. To this end, define the operator $Q: L_p \to [g_i]_{i\in\sigma}$, by setting

$$Qf=\sum_{j\in\sigma}\langle f, h_j\rangle g_j\,; \qquad f\in L_p.$$

Then, by (i) and linearization, as above, we obtain that

$$\|Qf\|_p \leqq K\left(\sum_{j\in\sigma}|\langle f, h_j\rangle|^p\right)^{1/p}\leqq K^2\|f\|_p\,; \qquad f\in L_p,$$

i.e.

$$\|Q\|\leqq K^2.$$

On the other hand, by using twice the inequalities above, we conclude that, for any element $g=\Sigma_{i\in\sigma} a_i g_i$ in $[g_i]_{i\in\sigma}$,

$$\|Qg\|_p\geqq\left(\sum_{j\in\sigma}|\langle g, h_j\rangle|^p\right)^{1/p}\Big/2K\geqq\|g\|_p/4K^2,$$

which means that Q, restricted to $[g_i]_{i\in\sigma}$, is an invertible operator whose inverse satisfies

$$\|(Q_{|[g_i]_{i\in\sigma}})^{-1}\|\leqq 4K^2.$$

It follows that

$$R=(Q_{|[g_i]_{i\in\sigma}})^{-1}\cdot Q$$

is a linear projection from L_p onto its subspace $[g_i]_{i\in\sigma}$ with norm $\|R\|\leqq 4K^4$. □

PROOF OF THEOREM 4.3. In view of the fact that X is well complemented in L_p, the statement of the theorem is self-dual and, therefore, there is no loss of generality in assuming that $1<p<2$. Suppose also that $L_p = L_p(\Omega, \Sigma, \mu)$, for some probability space (Ω, Σ, μ).

By the result of S. Kwapien [18] which has already been mentioned above and condition (i), there exists a constant C_0, depending only on c, so that

$$n^{1/p-1/2} \leqq C_0 T_2^{(g)}(X) \leqq C_0 T_2(X),$$

where $T_2(X)$ denotes the usual type 2 constant of X (recall that $T_2^{(g)}(X)$ stands for the gaussian type 2 constant of X). But, by N. Tomczak-Jaegermann [30], the type 2 constant of a n-dimensional space can be computed with only n vectors, up to a universal constant. This means that there are a $d = d(c) > 0$ and vectors $\{x_i\}_{i=1}^n$ in X such that

$$\left\| \left(\sum_{i=1}^n |x_i|^2 \right)^{1/2} \right\|_p \geqq d n^{1/p-1/2} \left(\sum_{i=1}^n \|x_i\|_p^2 \right)^{1/2}.$$

Put $y_i = x_i / \|x_i\|_p$; $1 \leqq i \leqq n$, and observe that

$$\begin{aligned} d^p n^{1-p/2} & \left(\sum_{i=1}^n \|x_i\|_p^2 \right)^{p/2} \\ & \leqq \int_\Omega \left(\sum_{i=1}^n |x_i(\omega)|^p \cdot \|x_i\|_p^{2-p} \cdot |y_i(\omega)|^{2-p} \right)^{p/2} d\mu(\omega) \\ & \leqq \int_\Omega \left(\sum_{i=1}^n |x_i(\omega)|^p \cdot \|x_i\|_p^{2-p} \right)^{p/2} \cdot \left(\max_{1 \leqq i \leqq n} |y_i(\omega)| \right)^{p(2-p)/2} d\mu(\omega). \end{aligned}$$

By using Hölder's inequality with $r = 2/p$ and $r' = 2/(2-p)$, we get that

$$d^{2/(2-p)} \cdot n^{1/p} \leqq \left\| \max_{1 \leqq i \leqq n} |y_i| \right\|_p.$$

Let $\{\eta_j\}_{j=1}^n$ be a partition of Ω into mutually disjoint subsets such that

$$\max_{1 \leqq i \leqq n} |y_i(\omega)| = |y_j(\omega)|,$$

for $\omega \in \eta_j$; $1 \leqq j \leqq n$. Then

$$d^{2/(2-p)} \cdot n^{1/p} \leqq \left(\sum_{j=1}^n \| y_j \chi_{\eta_j} \|_p^p \right)^{1/p},$$

and a simple probabilistic argument shows the existence of a subset τ of $\{1, 2, \ldots, n\}$ of cardinality $|\tau| \geqq d_1 n$ such that

$$\int_{\eta_j} |y_j(\omega)|^p d\mu(\omega) \geqq d_1; \qquad j \in \tau,$$

where $d_1 = d^{2p/(2-p)}/2$.

Consider now the functions

$$z_j = |y_j|^{p-1}(\operatorname{sgn} y_j)\chi_{\eta_j} \in L_{p'},$$

and put $u_j = P^*(z_j)$; $j \in \tau$. Then

$$\langle y_j, u_j \rangle = \langle y_j, z_j \rangle = \int_{\eta_j} |y_j(\omega)|^p d\mu(\omega) \geqq d_1,$$

for all $j \in \tau$, and also

$$\left\| \sum_{j\in\tau} b_j u_j \right\|_{p'} \leqq M \left(\sum_{j\in\tau} |b_j|^{p'} \right)^{1/p'},$$

for any choice of $\{b_j\}_{j\in\tau}$.

We would like now to apply Proposition 4.4 to the vectors $\{y_j\}_{j\in\tau}$ and $\{u_j\}_{j\in\tau}$ which, essentially speaking, satisfy the conditions (ii) and (iii) there. The problem is, however, that $\{y_j\}_{j\in\tau}$ need not satisfy (i) and, therefore, should be replaced by a different system of vectors. To this end, take a so that $4a^{1-2/p'} \cdot K_G M = d_1$ and, for each $j \in \tau$, put

$$\delta_j = \left\{ \omega \in \Omega;\ |u_j(\omega)| > a \left(\sum_{i\in\tau} |u_i(\omega)|^2 \right)^{1/2} \right\},$$

$$v_j = u_j \chi_{\delta_j} \quad \text{and} \quad w_j = u_j - v_j.$$

Then, for $j \in \tau$, we have

$$\langle P(y_j \chi_{\delta_j}), u_j \rangle = \langle y_j, v_j \rangle \geqq d_1 - \langle y_j, w_j \rangle.$$

However, by Hölder's inequality,

$$\begin{aligned} \sum_{j\in\tau} |\langle y_j, w_j \rangle| &\leqq \left\| \left(\sum_{j\in\tau} |y_j|^p \right)^{1/p} \right\|_p \cdot \left\| \left(\sum_{j\in\tau} |w_j|^{p'} \right)^{1/p'} \right\|_{p'} \\ &= |\tau|^{1/p} \cdot \left\| \left(\sum_{j\in\tau} |w_j|^{p'} \right)^{1/p'} \right\|_{p'}. \end{aligned}$$

On the other hand,

$$|w_j|^{p'} \leqq |u_j|^2 \cdot |w_j|^{p'-2} \leqq a^{p'-2} |u_j|^2 \left(\sum_{i\in\tau} |u_i|^2 \right)^{p'/2-1}; \qquad j \in \tau,$$

from which it follows that

$$\left(\sum_{j\in\tau}|w_j|^{p'}\right)^{1/p'}\leqq a^{1-2/p'}\left(\sum_{i\in\tau}|u_i|^2\right)^{1/2}.$$

Thus, by Grothendieck's inequality and the choice of a, we get that

$$\begin{aligned}\sum_{j\in\tau}|\langle y_j, w_j\rangle| &\leqq a^{1-2/p'}\cdot|\tau|^{1/p}\cdot\left\|\left(\sum_{i\in\tau}|u_i|^2\right)^{1/2}\right\|_{p'}\\ &\leqq a^{1-2/p'}\cdot K_G M|\tau|\\ &= d_1|\tau|/4.\end{aligned}$$

A simple probabilistic argument shows that there exists a subset τ_1 of τ of cardinality $|\tau_1|\geqq|\tau|/2\geqq d_1 n/2$ for which

$$|\langle y_j, w_j\rangle|\leqq d_1/2;\qquad j\in\tau_1,$$

i.e.

$$\langle P(y_j\chi_{\delta_j}), u_j\rangle\geqq d_1/2;\qquad j\in\tau_1.$$

In order to complete the proof, we now apply Proposition 4.4 to the functions $g_j = P(y_j\chi_{\delta_j})\in X$ and by $h_j = u_j/\langle g_j, u_j\rangle\in L_{p'}$; $j\in\tau_1$. □

Another problem raised in [13] is whether any copy of l_p^n in L_p ; $1<p<\infty$, contains in turn a copy of l_p^k which is well complemented in L_p with k proportional to n. The cases $p=2$ and $p=\infty$ are entirely trivial while the case $p=1$ was solved in [13] and [5]. We solve here the case $1<p\neq 2$, again in the positive.

THEOREM 4.5. *For every $1<p<\infty$ and $M<\infty$, there is a constant $A=A(p,M)<\infty$ such that, whenever $\{f_i\}_{i=1}^n$ is a sequence of functions in L_p which satisfies*

$$M^{-1}\left(\sum_{i=1}^n|a_i|^p\right)^{1/p}\leqq\left\|\sum_{i=1}^n a_i f_i\right\|_p\leqq M\left(\sum_{i=1}^n|a_i|^p\right)^{1/p},$$

for all $\{a_i\}_{i=1}^n$, then there exist a subset σ of $\{1,2,\dots,n\}$ of cardinality $|\sigma|\geqq n/A$ and a projection R from L_p onto $[f_i]_{i\in\sigma}$ with $\|R\|\leqq A$.

PROOF. Since both $\|(\Sigma_{i=1}^n|f_i|^p)^{1/p}\|_p$ and $\|(\Sigma_{i=1}^n|f_i|^2)^{1/2}\|_p$ are between $M^{-1}n^{1/p}$ and $M\cdot n^{1/p}$ we conclude, by a simple interpolation argument, that

$$\left\|\max_{1\leqq i\leqq n}|f_i|\right\|_p\geqq n^{1/p}/M^{(2+p)/|2-p|},$$

provided, of course, that $p \neq 2$. Then, exactly as in the proof of the previous result, we find a $c = c(p, M) > 0$, a subset τ of $\{1, 2, \ldots, n\}$ and mutually disjoint subsets $\{\eta_i\}_{i \in \tau}$ of Ω so that $|\tau| \geqq cn$ and

$$\int_{\eta_i} |f_i(\omega)|^p d\mu(\omega) \geqq c; \qquad i \in \tau.$$

The proof can be completed now by applying Proposition 4.4 to the functions $g_i = f_i / \int_{\eta_i} |f_i(\omega)|^p d\mu(\omega)$ and $h_i = |f_i|^{p-1}(\operatorname{sgn} f_i) \cdot \chi_{\eta_i}$; $i \in \tau$. □

REMARK. Proposition 4.4 can be reformulated as a factorization theorem which improves a recent result of T. Figiel, W. B. Johnson and G. Schechtman [11]. More precisely, it follows from Proposition 4.4 and some of the arguments used to prove Theorem 4.3 that, for every $1 < p < \infty$ and $M < \infty$, there exists a constant $C = C(p, M) < \infty$ such that, whenever $T: l_p^n \to L_p$ is an operator of norm $\leqq M$ satisfying the condition

$$(*) \qquad \left\| \left(\sum_{i=1}^{n} |Te_i|^2 \right)^{1/2} \right\|_p \geqq n^{1/p},$$

then there exists an integer $k \geqq n/C$ and an operator $R: L_p \to l_p^k$ with $\|R\| \leqq C$ such that the identity operator I on l_p^k factors through T as

$$I = RTJ,$$

where J is the formal identity map from l_p^k onto a subspace of l_p^n generated by a certain set of k unit vectors.

This factorization result is an immediate consequence of Proposition 4.4 and the fact that (*) implies that

$$\left\| \max_{1 \leqq i \leqq n} |Te_i| \right\|_p \geqq cn^{1/p},$$

for some constant $c = c(p, M) > 0$, which further yields the existence of a subset σ of $\{1, 2, \ldots, n\}$ with $|\sigma| \geqq n/C$ and of mutually disjoint subsets $\{A_i\}_{i \in \sigma}$ so that $\|\chi_{A_i}(Te_i)\|_p \geqq c$; $i \in \sigma$.

In the aforementioned paper [11], the authors prove a weaker factorization theorem asserting that, under the same assumptions, there are a k, as above, and operators J_ε and R_ε with $\|R_\varepsilon\| < C$; $\varepsilon \in \{-1, +1\}^k$, so that

$$I = \int R_\varepsilon T J_\varepsilon d\varepsilon.$$

5. "Unbounded" operators on l_r^n-spaces: $1 \leqq r \leqq \infty$

The invertibility results discussed in Sections 1 and 3 apply to "bounded" operators on l_p^n-spaces; $1 \leqq p \leqq \infty$, in the sense that at least one of the constants appearing in the various statements (measuring either the cardinality of the set of vectors onto which the operator is restricted or the norm of the inverse) depends on the norm of the given operator.

In the present section, we discuss some unexpected invertibility theorems. The main feature of these results is that, given a $n \times n$ matrix with 1's on the diagonal which acts as a "bounded" operator on some l_p^n-space; $1 \leqq p \leqq \infty$, one can find, for a whole interval of values r, a submatrix of rank proportional to n which is "well" invertible on l_r^n. Furthermore, the constants appearing in the statements depend only on the norm of the matrix as an operator on the original l_p^n-space and not on the value of r under consideration.

We now state our main result.

Theorem 5.1. *For every $1 \leqq p \leqq \infty$ and $M < \infty$, there exists a constant $c = c(p, M) > 0$ such that, whenever T is a linear operator of l_p^n of norm $\|T\|_p \leqq M$ whose matrix relative to the unit vector basis has 1's on the diagonal, then, for any $1 \leqq r \leqq p$ if $2 < p \leqq \infty$, or, for any $1 \leqq r \leqq 2$ if $1 \leqq p \leqq 2$, there is a subset σ of $\{1, 2, \ldots, n\}$ such that $|\sigma| \geqq cn$ and*

$$\left\| \sum_{i \in \sigma} a_i T e_i \right\|_r \geqq c \left(\sum_{i \in \sigma} |a_i|^r \right)^{1/r},$$

for all $\{a_i\}_{i \in \sigma}$.

Moreover, for every $\varepsilon > 0$ (and p and M, as above), there exists a constant $d = d(p, M, \varepsilon) > 0$ such that, for any T as above, one can choose the subset σ of $\{1, 2, \ldots, n\}$ with the property that $R_\sigma T R_\sigma$ restricted to $R_\sigma l_r^n$ is invertible and its inverse satisfies

$$\|(R_\sigma T R_\sigma)^{-1}\|_r < 1 + \varepsilon,$$

for all $2 \leqq r \leqq p$ if $2 < p \leqq \infty$ or $p \leqq r \leqq 2$ if $1 \leqq p \leqq 2$.

The proof of Theorem 5.1 is based on the following generalization of a result of J. Elton [8] (see also A. Pajor [23] for the extension to the complex case).

Theorem 5.2. *For every $M < \infty$ and $0 < \rho < 1$, there is a constant $c = c(M, \rho) > 0$ such that, whenever $\{x_i\}_{i=1}^n$ is a sequence of vectors in an arbitrary Banach space X which satisfies*

(i) $\int \|\sum_{i=1}^n \varepsilon_i x_i\| \, d\varepsilon \geqq n$

and

(ii) $\|\Sigma_{i \in \eta} x_i\| \leqq M \cdot |\eta|^{\rho} n^{1-\rho}$,

for every subset η *of* $\{1, 2, \ldots, n\}$, *then there exists a subset* σ *of* $\{1, 2, \ldots, n\}$ *so that* $|\sigma| \geqq cn$ *and*

$$\left\| \sum_{i \in \sigma} a_i x_i \right\| \geqq c \cdot \sum_{i \in \sigma} |a_i|,$$

for any choice of $\{a_i\}_{i \in \sigma}$.

PROOF. The difference between Theorem 5.2 and the aforementioned result of J. Elton is that, in the statement above, the vectors $\{x_i\}_{i=1}^n$ are supposed to satisfy condition (ii) instead of the weaker assumption of uniform boundedness. In order to overcome this difficulty, we shall replace the vectors $\{x_i\}_{i=1}^n$ by another sequence $\{\tilde{x}_i\}_{i=1}^n$ of uniformly bounded functions on the closed unit ball of the dual X^* of X.

Since the statement of Theorem 5.2 involves n vectors there is no loss of generality in assuming that the underlying space X is n-dimensional and, thus, that the closed unit ball K of X^* is norm compact.

For each $x \in X$ and $f \in K$, define

$$\hat{x}(f) = f(x),$$

and note that $\hat{x}$ is an element in the space $C(K)$ of all the continuous functions on K so that

$$\|\hat{x}\|_\infty = \sup_{f \in K} |\hat{x}(f)| = \|x\|.$$

Take now $A = (4M)^{1/\rho}$ and, for any $x \in X$, define the A-truncation $\tilde{x} \in C(K)$ of $\hat{x}$ in the following way:

$$\hat{x}(f) = \begin{cases} \hat{x}(f) & \text{if } |\hat{x}(f)| \leqq A, \\ A & \text{if } \hat{x}(f) > A, \\ -A & \text{if } \hat{x}(f) < -A, \end{cases} \qquad f \in K.$$

Let us also introduce the notation: for $f \in K$, set

$$\eta^+(f) = \{1 \leqq i \leqq n\,;\, \hat{x}_i(f) > A\}, \quad \eta^-(f) = \{1 \leqq i \leqq n\,;\, \hat{x}_i(f) < -A\}$$

and

$$\eta(f) = \eta^+(f) \cup \eta^-(f).$$

Then, for any choice of $f \in K$ and signs $\varepsilon_i = \pm 1$, we have

$$\sum_{i=1}^{n} \varepsilon_i \hat{x}_i(f) = \sum_{i=1}^{n} \varepsilon_i \tilde{x}_i(f) + \sum_{i \in \eta^+(f)} \varepsilon_i (\hat{x}_i(f) - A) + \sum_{i \in \eta^-(f)} \varepsilon_i (\hat{x}_i(f) + A),$$

which yields that

$$\left| \sum_{i=1}^{n} \varepsilon_i \hat{x}_i(f) \right| \leqq \left\| \sum_{i=1}^{n} \varepsilon_i \tilde{x}_i(t) \right| + \sum_{i \in \eta(f)} |\hat{x}_i(f)|.$$

On the other hand, in view of condition (ii),

$$A\,|\eta^+(f)| \leqq \sum_{i \in \eta^+(f)} \hat{x}_i(f) \leqq \left\| \sum_{i \in \eta^+(f)} x_i \right\| \leqq M\,|\eta^+(f)|^p n^{1-p},$$

i.e.

$$|\eta^+(f)| \leqq (M/A)^{1/(1-p)} n$$

and, similarly,

$$|\eta^-(f)| \leqq (M/A)^{1/(1-p)} n.$$

By using again condition (ii), we get that

$$\begin{aligned} \sum_{i \in \eta(f)} |\hat{x}_i(f)| &\leqq \left\| \sum_{i \in \eta^+(f)} x_i \right\| + \left\| \sum_{i \in \eta^-(f)} x_i \right\| \\ &\leqq M(|\eta^+(f)|^p + |\eta^-(f)|^p) n^{1-p} \\ &\leqq (2M^{1/(1-p)}/A^{p/(1-p)}) n. \end{aligned}$$

However, the choice of A made above ensures that $2M^{1/(1-p)}/A^{p/(1-p)} = \frac{1}{2}$, i.e., that

$$\sum_{i \in \eta(f)} |\hat{x}_i(f)| \leqq n/2.$$

Consequently,

$$\left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\| = \left\| \sum_{i=1}^{n} \varepsilon_i \hat{x}_i \right\|_\infty \leqq \left\| \sum_{i=1}^{n} \varepsilon_i \tilde{x}_i \right\|_\infty + n/2$$

which, by averaging and (i), yields that

$$\int \left\| \sum_{i=1}^{n} \varepsilon_i \tilde{x}_i \right\|_\infty d\varepsilon \geqq n/2.$$

The advantage of working with the functions $\{\tilde{x}_i\}_{i=1}^n$ in $C(K)$ instead of the original vectors $\{x_i\}_{i=1}^n$ consists of the fact that the former are uniformly bounded by A. Therefore, we can apply the main result of J. Elton [8] in the form stated in the Remark on p. 119 by which there exist a constant $d > 0$, depending only on A and, thus, on M and ρ, a subset σ of $\{1, 2, \dots, n\}$ of cardinality $|\sigma| \geqq dn$ and reals u and v with $v - u \geqq d$ such that if we set

$$U_i = \{f \in K; \tilde{x}_i(f) \leqq u\} \quad \text{and} \quad V_i = \{f \in K; \tilde{x}_i(f) \geqq v\}; \qquad 1 \leqq i \leqq n,$$

then the family $(U_i, V_i)_{i \in \sigma}$ is Boolean independent. This means that, whenever σ_1 and σ_2 are two mutually disjoint subsets of σ, then

$$\Big(\bigcap_{i \in \sigma_1} U_i\Big) \cap \Big(\bigcap_{i \in \sigma_2} V_i\Big) \neq \emptyset.$$

In particular, we get that U_i and V_i are non-void, for all $i \in \sigma$, which implies that $u \geqq -A$ and $v \leqq A$. Thus, for each $i \in \sigma$, we have that

$$U_i \subset \hat{U}_i = \{f \in K; \hat{x}_i(f) \leqq u\} \quad \text{and} \quad V_i \subset \hat{V}_i = \{f \in K; \hat{x}_i(f) \geqq v\}.$$

Indeed, if $f \in U_i$, for some $i \in \sigma$, and e.g. $\tilde{x}_i(f) < A$ then $\hat{x}_i(f) \leqq \tilde{x}_i(f) \leqq u$, i.e. $f \in \hat{U}_i$. If, on the other hand, $f \in U_i$ and $\tilde{x}_i(f) = A$ then $u \geqq A$ which implies that $v > A$ and contradiction.

The inclusion above shows that also the family $(\hat{U}_i, \hat{V}_i)_{i \in \sigma}$ is Boolean independent. The proof can be now completed by using a standard argument. Let $\{a_i\}_{i \in \sigma}$ be an arbitrary sequence of reals, put

$$\sigma_1 = \{i \in \sigma; a_i > 0\} \quad \text{and} \quad \sigma_2 = \sigma \sim \sigma_1,$$

and let f_0 be an element in the intersection

$$\Big(\bigcap_{i \in \sigma_2} \hat{U}_i\Big) \cap \Big(\bigcap_{i \in \sigma_1} \hat{V}_i\Big).$$

Then

$$\begin{aligned} \Big\| \sum_{i \in \sigma} a_i x_i \Big\| &\geqq \sum_{i \in \sigma} a_i \hat{x}_i(f_0) \\ &= \sum_{i \in \sigma_1} a_i \hat{x}_i(f_0) + \sum_{i \in \sigma_2} a_i \hat{x}_i(f_0) \\ &\geqq v \sum_{i \in \sigma_1} |a_i| - u \sum_{i \in \sigma_2} |a_i|. \end{aligned}$$

On the other hand, by replacing a_i with $-a_i$, for all $i \in \sigma$, we also get that

$$\Big\| \sum_{i \in \sigma} a_i x_i \Big\| \geqq v \sum_{i \in \sigma_2} |a_i| - u \sum_{i \in \sigma_1} |a_i|.$$

Hence, by addition, we finally obtain that

$$\left\|\sum_{i\in\sigma} a_i x_i\right\| \geqq (v-u)\cdot \sum_{i\in\sigma} |a_i| \geqq d\cdot \sum_{i\in\sigma} |a_i|,$$

and this completes the proof in the real case. The solution in the real case also implies that, whenever $\{x_i\}_{i=1}^n$ are vectors in an arbitrary complex Banach space, then

$$\left\|\sum_{i\in\sigma} a_i x_i\right\| \geqq d' \sum_{i\in\sigma} |a_i|$$

for any real $\{a_i\}$ and some $d'>0$. The proof of the complex case can then be completed by using A. Pajor [23] Theorem 3.16. □

Before presenting the proof of Theorem 5.1, we need one more result which is of interest in itself.

PROPOSITION 5.3. *For every $c>0$, $1<r\leqq 2$ and every sequence $\{x_j\}_{j=1}^n$ of vectors in l_1^n which satisfies*

$$\left\|\sum_{j=1}^n a_j x_j\right\|_1 \geqq c\cdot \sum_{j=1}^n |a_j|,$$

for all $\{a_j\}_{j=1}^n$, there exists a subset τ of $\{1,2,\dots,n\}$ such that $|\tau|\geqq n/2$ and

$$\left\|\sum_{j\in\tau} a_j x_j\right\|_r \geqq 5^{-1/r'}\cdot c\left(\sum_{j\in\tau} |a_j|^r\right)^{1/r},$$

for any choice of $\{a_j\}_{j\in\tau}$.

PROOF. We shall use again an exhaustion argument. Fix c, r and the sequence $\{x_j\}_{j=1}^n$ in l_1^n, and suppose that the assertion of Proposition 5.3 is false. Then we can construct subsets

$$\tau_1 \supset \tau_2 \supset \cdots \supset \tau_m$$

with $|\tau_m|\geqq n/2$ and vectors $y_i=\sum_{j\in\tau_i} b_{i,j}x_j$; $1\leqq i\leqq m$, so that

$$\sum_{j\in\tau_i} |b_{i,j}|^r = 1,\quad \|y_i\|_r < 5^{-1/r'}\cdot c,\qquad 1\leqq i\leqq m,$$

and if we set

$$\tau_{m+1} = \left\{1\leqq j\leqq n;\ \sum_{i=1}^m |b_{i,j}|^r < 1\right\}$$

then $|\tau_{m+1}|<n/2$. This construction yields that $m\geqq n/2$.

Let now $\{\varphi_i\}_{i=1}^m$ be a sequence of r-stable independent random variables over a probability space (Ω, Σ, μ) which are normalized in $L_1(\Omega, \Sigma, \mu)$. Since the norm in L_1-spaces is additive on the positive cone we get that

$$\begin{aligned}
5^{-1/r'} c m^{1/r} &> \left(\sum_{i=1}^{m} \| y_i \|_r^r \right)^{1/r} \\
&= \left\| \left(\sum_{i=1}^{m} | y_i |^r \right)^{1/r} \right\|_r \\
&\geqq n^{-1/r'} \left\| \left(\sum_{i=1}^{m} | y_i |^r \right)^{1/r} \right\|_1 \\
&= n^{-1/r'} \left\| \int_\Omega \left| \sum_{i=1}^{m} \varphi_i(\omega) y_i \right| d\mu(\omega) \right\|_1 \\
&= n^{-1/r'} \int_\Omega \left\| \sum_{j=1}^{n} \left(\sum_{i=1}^{m} \varphi_i(\omega) b_{i,j} \right) x_j \right\|_1 d\mu(\omega) \\
&\geqq c n^{-1/r'} \sum_{j=1}^{n} \int_\Omega \left| \sum_{i=1}^{m} \varphi_i(\omega) b_{i,j} \right| d\mu(\omega) \\
&= c n^{-1/r'} \sum_{j=1}^{n} \left(\sum_{i=1}^{m} | b_{i,j} |^r \right)^{1/r}
\end{aligned}$$

However, the above construction yields that

$$\sum_{i=1}^{m} | b_{i,j} |^r \leqq 2,$$

for all $1 \leqq j \leqq n$, which implies that

$$2^{1/r'} \cdot 5^{-1/r'} m^{1/r} > n^{-1/r'} \sum_{j=1}^{n} \sum_{i=1}^{m} | b_{i,j} |^r = n^{-1/r'} m.$$

This contradicts, as easily checked, the fact that $m \geqq n/2$. □

Proof of Theorem 5.1. Fix $1 \leqq p \leqq \infty$ and $M < \infty$, and let T be a linear operator on l_p^n of norm $\| T \|_p \leqq M$ whose matrix has 1's on the diagonal. Put $x_i = \sqrt{2}\, T e_i$; $1 \leqq i \leqq n$, and note that

$$\int \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\|_1 d\varepsilon = \left\| \int \left| \sum_{i=1}^{n} \varepsilon_i x_i \right| d\varepsilon \right\|_1 \geqq \left\| \left(\sum_{i=1}^{n} | x_i |^2 \right)^{1/2} \right\|_1 \Big/ \sqrt{2} \geqq n,$$

since, for each $1 \leqq j \leqq n$,

$$e_j^* \left(\sum_{i=1}^{n} | x_i |^2 \right)^{1/2} \geqq e_j^*(| x_j |) = \sqrt{2}.$$

Consequently, condition (i) of Theorem 5.2 holds in $X = l_1^n$. Moreover, if η is an arbitrary subset of $\{1,2,\dots,n\}$ then

$$\left\|\sum_{i\in\eta} x_i\right\|_1 \leqq \left\|\sum_{i\in\eta} x_i\right\|_p \cdot n^{1-1/p} \leqq M\sqrt{2}\,|\eta|^{1/p}\cdot n^{1-1/p},$$

i.e., also condition (ii) of Theorem 5.2 holds with $\rho = 1/p$ and M replaced by $M\sqrt{2}$. Thus, by Theorem 5.2 with the notation $c_1 = c(M\sqrt{2}, 1/p)$, one can find a subset σ_1 of $\{1,2,\dots,n\}$ so that $|\sigma_1| \geqq c_1 n$ and

$$\left\|\sum_{i\in\sigma_1} a_i x_i\right\|_1 \geqq c_1 \sum_{i\in\sigma_1} |a_i|,$$

for any choice of $\{a_i\}_{i\in\sigma_1}$. This already completes the proof in the case $r = 1$. In the case $1 < r \leqq 2$, we complete the proof by using Proposition 5.3. Finally, we consider the cases when $2 < r \leqq p$ or when $1 \leqq p < 2$ and $p < r \leqq 2$. Put $S = T - I$ and apply Theorem 3.1. It follows that there exist a constant $c_2 = c_2(p, M, \varepsilon) > 0$ and a subset σ_2 of $\{1,2,\dots,n\}$ such that $|\sigma_2| \geqq c_2 n$ and

$$\|R_{\sigma_2} S R_{\sigma_2}\|_p < \varepsilon/4K_G.$$

Hence, by Corollary 3.4, there exists a subset σ of σ_2 of cardinality $|\sigma| \geqq |\sigma_2| \geqq c_2 n/2$ so that $\|R_\sigma S R_\sigma\|_r < \varepsilon$, for all r between p and 2, including 2. Consequently, for each such r, $R_\sigma T R_\sigma$ restricted to $R_\sigma l_r^n$ is invertible and its inverse satisfies

$$\|(R_\sigma T R_\sigma)^{-1}\|_r < 1 + \varepsilon.$$

In particular, we also get that

$$\left(\sum_{i\in\sigma} |a_i|^r\right)^{1/r} < (1+\varepsilon)\left\|\sum_{i\in\sigma} a_i T e_i\right\|_r,$$

for all $\{a_i\}_{i\in\sigma}$ and r, as above. □

Remark. As we have pointed out in the introduction, if $p = \infty$ then σ can be chosen so that $\|(R_\sigma T R_\sigma)^{-1}\|_r < 1 + \varepsilon$, for *all* $1 \leqq r \leqq \infty$. The same is true, of course, if $r = 1$, by duality.

We present now two examples which show that the range of restricted invertibility given by Theorem 5.1 in both cases: rectangular and square, is best possible.

Example 5.4. For each $p \geqq 2$, there exists a sequence $\{E_{p,n}\}_{n=1}^\infty$ of linear operators on l_p^n such that

(i) $\sup_n \|E_{p,n}\|_p / n^{1/p'} < \infty$,

(ii) the entries of the matrix associated to $E_{p,n}$ have absolute value equal to 1, for all n,

(iii) for any $r > p$ and any subset σ_n of $\{1, 2, \dots, n\}$ with $\sup_n n/|\sigma_n| < \infty$, there exists a vector $x_n \in [e_i]_{i \in \sigma_n}$ so that $\lim_{n\to\infty} \|x_n\|_r = 1$ but $\lim_{n\to\infty} \|(I + n^{-1/p'}E_{p,n})x_n\|_r = 0$.

Before presenting the construction, let us point out some of the features of the operators $A_{p,n} = I + n^{-1/p'}E_{p,n}$; $n = 1, 2, \dots$.

(1) $\sup_n \|A_{p,n}\|_p < \infty$ and the entries of the matrix associated to $A_{p,n}$, which are on the diagonal, tend uniformly to 1, as $n \to \infty$. Furthermore, by (iii), the restriction of $A_{p,n}$ to any set of unit vectors of cardinality proportional to n is not invertible in l_r^n, for $r > p$, i.e. Theorem 5.1 is false for $2 \leqq p < r$.

(2) The adjoint $A_{p,n}^*$ of $A_{p,n}$ has the property that $R_\sigma A_{p,n}^* R_\sigma$ restricted to $R_\sigma l_r^n$ is not "well" invertible in l_r^n, for any $1 \leqq r < p'$ and any $\sigma \subset \{1, 2, \dots, n\}$ of cardinality proportional to n. Indeed, otherwise $R_\sigma A_{p,n} R_\sigma$ restricted to $R_\sigma l_{r'}^n$ would be "well" invertible in $l_{r'}^n$ with $r' > p$, contrary to (iii). This means that, in the range $1 \leqq r < p \leqq 2$, Theorem 5.1 cannot be improved so as to yield restricted square invertibility.

(3) Corollary 3.4 is false for $2 \leqq p < r$ since if $R_\eta A_{p,n} R_\eta$ were a "well" bounded operator on l_r^n, for some subset η of $\{1, 2, \dots, n\}$ of cardinality proportional to n, then, by Corollary 3.2, it would also be "well" invertible in l_r^n when further restricted to a subset σ of η of "large" cardinality. This again contradicts (iii).

In order to describe our construction, fix n and let $E_n(\omega)$ be a $n \times n$ matrix whose entries $(\varepsilon_{i,j}(\omega))_{i,j=1}^n$ are symmetric independent random variables on some probability space (Ω, Σ, μ), each of which taking only the values $+1$ and -1. Let $(g_{i,j}(\omega'))_{i,j=1}^n$ be a matrix of symmetric independent Gaussian random variables over an independent copy (Ω', Σ', μ') of (Ω, Σ, μ). Fix now $p \geqq 2$ and note that the norm $\|E_n(\omega)\|_p$ of $E_n(\omega)$, when considered as an operator on l_p^n, satisfies

$$\begin{aligned}
&\int_\Omega \|E_n(\omega)\|_p d\mu(\omega) \\
&= \sqrt{\pi/2} \int_\Omega \left\| \sum_{i,j=1}^n \varepsilon_{i,j}(\omega) \left(\int_{\Omega'} |g_{i,j}(\omega')| \, d\mu'(\omega') \right) e_i \otimes e_j \right\|_p d\mu(\omega) \\
&\leqq \sqrt{\pi/2} \int_\Omega \int_{\Omega'} \left\| \sum_{i,j=1}^n \varepsilon_{i,j}(\omega) |g_{i,j}(\omega')| \, e_i \otimes e_j \right\|_p d\mu'(\omega') d\mu(\omega) \\
&= \sqrt{\pi/2} \int_{\Omega'} \left\| \sum_{i,j=1}^n g_{i,j}(\omega') e_i \otimes e_j \right\|_p d\mu'(\omega').
\end{aligned}$$

Hence, by Chevet's inequality [7] (see also [12]), we get that

$$\int_\Omega \|E_n(\omega)\|_p d\mu(\omega) \leqq \pi n^{1/p'}.$$

Define

$$A_{p,n}(\omega) = I + n^{-1/p'} E_n(\omega); \qquad \omega \in \Omega,$$

and observe that

$$\int_\Omega \|A_{p,n}(\omega)\|_p d\mu(\omega) \leqq 5.$$

(Note that the same argument involving the use of Chevet's inequality actually yields that

$$\int_\Omega \|A_{p,n}(\omega)\|_q d\mu(\omega) \leqq 5,$$

for any $p' \leqq q \leqq p$.)

In order to prove condition (iii), we need the following lemma.

LEMMA 5.5. *There exists a constant $D < \infty$ such that, for any n and any matrix $(\varepsilon_{i,j}(\omega))_{i,j=1}^n$ of symmetric independent random variables on a probability space (Ω, Σ, μ) which take only the values $+1$ and -1, we have*

$$J_n = \int_\Omega \max\left\{ \sum_{i=1}^n \max_{\substack{1 \leqq k \leqq n \\ k \neq i}} \left| \sum_{j \in \sigma} \varepsilon_{j,i}(\omega) \varepsilon_{j,k}(\omega) \right| ; \sigma \subset \{1, 2, \ldots, n\} \right\} d\mu(\omega)$$

$$\leqq D n^{3/2} (\log n)^{1/2}.$$

PROOF. Let $\mathcal{F}$ be the family of all the maps φ which take the set $\{1, 2, \ldots, n\}$ into itself in such a manner that $\varphi(i) \neq i$, for all $1 \leqq i \leqq n$. Let $\mathcal{U}$ be the family of all the triplets of the form $u = (\sigma, \varphi, \{\theta_i\}_{i=1}^n)$, which range over all $\sigma \subset \{1, 2, \ldots, n\}$, $\varphi \in \mathcal{F}$ and $\theta_i = \pm 1$; $1 \leqq i \leqq n$, and observe that $|\mathcal{U}| \leqq 2^n \cdot n^n \cdot 2^n = (4n)^n$. Put $m = [\log |\mathcal{U}|] + 1$ and note that

$$J_n \leqq \int_\Omega \max_{\mathcal{U}} \left| \sum_{i=1}^n \sum_{j \in \sigma} \theta_i \varepsilon_{j,i}(\omega) \varepsilon_{j,\varphi(i)}(\omega) \right| d\mu(\omega)$$

$$\leqq \left(\sum_{\mathcal{U}} \int_\Omega \left| \sum_{i=1}^n \sum_{j \in \sigma} \theta_i \varepsilon_{j,i}(\omega) \varepsilon_{j,\varphi(i)}(\omega) \right|^m d\mu(\omega) \right)^{1/m}$$

$$\leqq |\mathcal{U}|^{1/m} \max_{\mathcal{U}} \left(\int_\Omega \left| \sum_{i=1}^n \sum_{j \in \sigma} \theta_i \varepsilon_{j,i}(\omega) \varepsilon_{j,\varphi(i)}(\omega) \right|^m d\mu(\omega) \right)^{1/m}$$

$$\leqq e \max_{u \in \mathcal{U}} \tilde{J}_n(u),$$

where, for any fixed $u = (\sigma, \varphi, \{\theta_i\}_{i=1}^n) \in \mathcal{U}$,

$$\tilde{J}_n(u) = \left(\int_\Omega \left| \sum_{i=1}^n \sum_{j \in \sigma} \theta_i \varepsilon_{j,i}(\omega) \varepsilon_{j,\varphi(i)}(\omega) \right|^m d\mu(\omega) \right)^{1/m}.$$

Fix now $u = (\sigma, \varphi, \{\theta_i\}_{i=1}^n) \in \mathcal{U}$ and verify that, for this particular choice of φ, there exists a particular $\{\eta_k\}_{k=1}^l$ of $\{1, 2, \ldots, n\}$ into l mutually disjoint subsets so that $l \leqq d \log n$, for some constant $d < \infty$ independent of n or φ, and

(i) $|\eta_k| \leqq n/2^k$,

(ii) $\varphi(\eta_k) \cap \eta_k = \varnothing$,

for all $1 \leqq k \leqq l$. Thus

$$\tilde{J}_n(u) \leqq \sum_{k=1}^l \left(\int_\Omega \left| \sum_{i \in \eta_k} \sum_{j \in \sigma} \theta_i \varepsilon_{j,i}(\omega) \varepsilon_{j,\varphi(i)}(\omega) \right|^m d\mu(\omega) \right)^{1/m}.$$

In general, for a fixed $j \in \sigma$, $\varepsilon_{j,h}(\omega)$ and $\varepsilon_{j,\varphi(i)}(\omega)$ need not be independent since $\varphi(i)$ might coincide with h. However, for each fixed $1 \leqq k \leqq l$, we conclude, by (ii), that the families $(\varepsilon_{j,k}(\omega))_{j \in \sigma, i \in \eta_k}$ and $(\varepsilon_{j,\varphi(i)}(\omega))_{j \in \sigma, i \in \eta_k}$ are independent. Therefore, by Khintchine's inequality in $L_m(\Omega, \Sigma, \mu)$, we have that

$$\tilde{J}_n(u) \leqq B_m \cdot \sum_{k=1}^l (|\eta_k| \cdot |\sigma|)^{1/2} \leqq B_m n / (\sqrt{2} - 1).$$

This completes the proof, in view of the fact that Khintchine's constant B_m is, as well known, $\leqq \sqrt{m}$. □

We return now to Example 5.4. By Lemma 5.5 and the estimate in mean for the norm of $A_{p,n}(\omega)$, there exists a point $\omega_n \in \Omega$ such that

$$\| A_{p,n}(\omega_n) \|_p \leqq 10$$

and

$$\sum_{i=1}^n \max_{\substack{1 \leqq k \leqq n \\ k \neq i}} \left| \sum_{j \in \sigma} \varepsilon_{j,i}(\omega_n) \varepsilon_{j,k}(\omega_n) \right| \leqq 2Dn^{3/2} (\log n)^{1/2},$$

for any choice of $\sigma \subset \{1, 2, \ldots, n\}$.

Now, for each n, fix $\sigma_n \subset \{1, 2, \ldots, n\}$ so that

$$K = \sup_n n / |\sigma_n| < \infty$$

and choose an integer $1 \leqq i_n \leqq n$ which ensures that

$$\left| \sum_{j \in \sigma_n} \varepsilon_{j,i_n}(\omega_n) \varepsilon_{j,k}(\omega_n) \right| \leqq 2Dn^{1/2} \cdot (\log n)^{1/2},$$

for all $1 \leqq k \leqq n$, $k \neq i_n$. Let $r > p$ and consider the vectors

$$x_n = e_{i_n} - \frac{n^{1/p'}}{|\sigma_n| - 1} \sum_{\substack{j \in \sigma_n \\ j \neq i_n}} \varepsilon_{j,i_n}(\omega_n) e_j \,; \qquad n = 1, 2, \ldots,$$

whose norm satisfies

$$1 \leqq \| x_n \|_r \leqq 1 + n^{1/p'} / (|\sigma_n| - 1)^{1/r'} \to 1,$$

as $n \to \infty$. On the other hand, we have that

$$\| A_{p,n}(\omega_n) x_n \|_r \leqq | e^*_{i_n} A_{p,n}(\omega_n) x_n | + n^{1/r} \cdot \max_{\substack{1 \leqq k \leqq n \\ k \neq i_n}} | e^*_k A_{p,n}(\omega_n) x_n |,$$

and it is easily verified that

$$e^*_{i_n} A_{p,m}(\omega_n) x_n = \varepsilon_{i_n,i_n}(\omega_n) / n^{1/p'}$$

and, for $1 \leqq k \leqq n$, $k \neq i_n$,

$$\begin{aligned} | e^*_k A_{p,n}(\omega_n) x_n | &\leqq n^{1/p'} / (|\sigma_n| - 1) + n^{-1/p'} | e^*_k E_n(\omega_n) x_n | \\ &\leqq n^{1/p'} / (|\sigma_n| - 1) + n^{-1/p'} + \left| \sum_{\substack{j \in \sigma_n \\ j \neq i_n}} \varepsilon_{j,i_n}(\omega_n) \varepsilon_{j,k}(\omega_n) \right| \Big/ (|\sigma_n| - 1) \\ &\leqq n^{1/p'} / (|\sigma_n| - 1) + n^{-1/p'} + (2Dn^{1/2} (\log n)^{1/2} + 1) / (|\sigma_n| - 1). \end{aligned}$$

Thus, for n sufficiently large, we obtain that

$$\begin{aligned} \| A_{p,n}(\omega_n) x_n \|_r &\leqq n^{1/r + 1/p'} / (|\sigma_n| - 1) + (3Dn^{1/r + 1/2} (\log n)^{1/2}) / (|\sigma_n| - 1) \\ &\leqq K(2/n^{1/p - 1/r} + 6D(\log n)^{1/2} / n^{1/r' - 1/2}) \\ &\to 0, \end{aligned}$$

as $n \to \infty$. This completes the argument. □

EXAMPLE 5.6. For each $p > 2$, there exists a sequence $\{G_{p,n}\}_{n=1}^{\infty}$ of linear operators on l_p^n so that

(i) $\sup_n \| G_{p,n} \|_p < \infty$,

(ii) the entries of the matrix associated to $G_{p,n}$ tend to 0, as $n \to \infty$,

(iii) for any $1 \leqq r < 2$ and any subset σ_n of $\{1, 2, \ldots, n\}$ with $\sup_n n / |\sigma_n| < \infty$, we have

$$\lim_{n \to \infty} \| R_{\sigma_n} G_{p,n} R_{\sigma_n} \|_r = \infty.$$

We can draw the following conclusions from the existence of the above sequence $\{G_{p,n}\}_{n=1}^{\infty}$:

(1) The operators $B_{p,n} = I + G_{p,n}$; $n = 1, 2, \ldots$ have diagonal tending to 1, as $n \to \infty$, and $\sup_n \|B_{p,n}\|_p < \infty$. However, the operator $R_{\sigma_n} B_{p,n} R_{\sigma_n}$ is not well invertible, for any choice of $1 \leqq r < 2$ and σ_n of cardinality proportional to n, i.e., Theorem 5.1 cannot be improved so as to yield restricted square invertibility for $1 \leqq r < 2$ (where we have only restricted rectangular invertibility, by Theorem 5.1). Indeed, if

$$\sup_n \|(R_{\sigma_n} B_{p,n} R_{\sigma_n})^{-1}\|_r < \infty,$$

for some $1 \leqq r < 2$ and $\{\sigma_n\}_{n=1}^{\infty}$ satisfying $\sup_n n/|\sigma_n| < \infty$, then, by Corollary 3.2 applied to the operators $(R_{\sigma_n} B_{p,n} R_{\sigma_n})^{-1}$; $n = 1, 2, \ldots$, one could find subsets τ_n of σ_n with $\sup_n |\sigma_n|/|\tau_n| < \infty$ so that

$$\sup_n \|R_{\tau_n} B_{p,n} R_{\tau_n}\|_r < \infty,$$

contrary to (iii).

(2) Corollary 3.4 is false, for $p > 2$ and $1 \leqq r < 2$, and also for $1 \leqq p < 2$ and $r > 2$.

We pass now to the construction. Fix an integer n and $p > 2$, take $k = [n^{1-2/p}]$ and suppose, for the sake of simplicity, that k divides n. Put $m = n/k$ and let $\{\eta_i\}_{i=1}^{m}$ be a partition of $\{1, 2, \ldots, n\}$ into mutually disjoint subsets, each of which has cardinality equal to k. Let $(\varepsilon_{i,j}(\omega))_{i=1,j=1}^{m\ \ n}$ be a matrix of symmetric indepenent random variables on a probability space (Ω, Σ, μ), each of which takes only the values $+1$ and -1, and define

$$G_{p,n}(\omega) = n^{-1/p'} \sum_{i=1}^{m} \sum_{j=1}^{n} \varepsilon_{i,j}(\omega) \left(\left(\sum_{h \in \eta_i} e_h \right) \otimes e_j \right).$$

Then, by passing to independent Gaussian random variables, as in Example 5.4, and, by using Chevet's inequality, we get that

$$\int_{\Omega} \|G_{p,n}(\omega)\|_p d\mu(\omega) \leqq \sqrt{\pi/2}\, k^{1/p'} n^{-1/p'} (m^{1/2-1/p} n^{1/p} + m^{1/p'}),$$

which, in view of the condition imposed on k, implies that

$$\int_{\Omega} \|G_{p,n}(\omega)\|_p d\mu(\omega) \leqq 3.$$

Fix $K<\infty$ and choose a point $\omega_n \in \Omega$ that satisfies $\| G_{p,n}(\omega_n)\|_p \leqq 3$, and let $\sigma_n \subset \{1,2,\dots,n\}$ be so that

$$\sup_n n/|\sigma_n| \leqq K.$$

Then one can find an integer $1 \leqq i_n \leqq m$ with the property that

$$|\sigma_n \cap \eta_{i_n}| \geqq k/K.$$

Fix now $1 \leqq r < 2$ and define the vector

$$x_{r,n} = \Big(\sum_{h\in\sigma_n\cap\eta_{i_n}} e_h\Big) \Big/ |\sigma_n \cap \eta_{i_n}|^{1/r} \in [e_i]_{i\in\sigma_n} \subseteq l_r^n$$

and note that $\|x_{r,n}\|_r = 1$, for all n. On the other hand,

$$\begin{aligned}\| R_{\sigma_n} G_{p,n}(\omega_n) x_{r,n}\|_r &= n^{-1/p'} |\sigma_n \cap \eta_{i_n}|^{1/r'} \cdot \Big\| \sum_{j\in\sigma_n} \varepsilon_{i_n,j}(\omega_n) e_j \Big\|_r \\ &= n^{-1/p'} |\sigma_n \cap \eta_{i_n}|^{1/r'} \cdot |\sigma_n|^{1/r} \\ &\geqq n^{(1-2/r')/p}/K \to \infty,\end{aligned}$$

as $n\to\infty$. This proves (iii). □

So far, we have studied in this section only the restricted invertibility of matrices with 1's on the diagonal. Since there are also interesting applications in which the corresponding matrices do not satisfy this assumption we present now a variant of Theorem 5.1 that applies in a more general setting.

THEOREM 5.7. *For every $p \geqq 1$ and $M<\infty$, there exists a constant $b = b(p,M) > 0$ such that, whenever T is a linear operator on l_p^n for which $\|T\|_p \leqq M$ and*

$$\int \Big\| \sum_{i=1}^n \varepsilon_i T e_i \Big\|_p d\varepsilon \geqq n^{1/p},$$

then, for every $1 \leqq r \leqq 2$, there exists a subset τ of $\{1,2,\dots,n\}$ such that $|\tau| \geqq bn$ and

$$\Big\| \sum_{i\in\tau} a_i T e_i \Big\|_r \geqq b \Big(\sum_{i\in\tau} |a_i|^r \Big)^{1/r},$$

for all $\{a_i\}_{i\in\tau}$.

PROOF. Note first that, by Proposition 3.13, the case $1 \leqq p < 2$ reduces immediately to that of matrices with 1's on the diagonal, which is already covered by Theorem 5.1. Suppose, therefore, that $p>2$, let B_p denote Khintchine's constant in L_p and put

$$x_1 = \sqrt{2}\, B_p^p M^{p-1} T e_i\,; \qquad 1 \leqq i \leqq n.$$

Let $s = (\sum_{i=1}^n |x_i|^2)^{1/2}$ be the square functions of the vectors $\{x_i\}_{i=1}^n$ and assume that $s = \sum_{i=1}^n s_i e_i$. Then, for each $1 \leqq i \leqq n$,

$$\begin{aligned} s_i &= \left(\sum_{j=1}^n | e_i^*(x_j)|^2 \right)^{1/2} \\ &= \sup\left\{ \sum_{j=1}^n c_j e_i^*(x_j);\ \sum_{j=1}^n |c_j|^2 \leqq 1 \right\} \\ &\leqq \sqrt{2}\, B_p^p M^p. \end{aligned}$$

Thus, by our hypothesis,

$$\begin{aligned} \sqrt{2}\, B_p^p M^{p-1} n^{1/p} &\leqq \int \left\| \sum_{i=1}^n \varepsilon_i x_i \right\|_p d\varepsilon \\ &\leqq B_p \| s \|_p \\ &= B_p \left(\sum_{i=1}^n | s_i |^p \right)^{1/p} \\ &\leqq B_p (\sqrt{2}\, B_p^p M^p)^{(p-1)/p} \cdot \left(\sum_{i=1}^n | s_i | \right)^{1/p} \end{aligned}$$

from which one easily deduces that

$$\int \left\| \sum_{i=1}^n \varepsilon_i x_i \right\|_1 d\varepsilon \geqq \| s \|_1 / \sqrt{2} \geqq n,$$

i.e., condition (i) of Theorem 5.2 holds for the vectors $\{x_i\}_{i=1}^n$ in l_1^n. As in the proof of Theorem 5.1, we check immediately that also condition (ii) holds with M replaced by $\sqrt{2}\, B_p^p M^p$ and $\rho = 1/p$. Thus, by using Theorem 5.2, we complete the proof in the case $r=1$. Then, by Proposition 5.3, we complete also the case $1 < r \leqq 2$. □

We conclude this section with an application of Theorem 5.7 to the Λ_2-sets problem, whether there exists a constant $K < \infty$ such that, for every integer n, $\varepsilon > 0$ and every set of n characters, there is a subset of cardinality $\geqq n^{1-\varepsilon}$ onto whose linear span the L_1- and L_2-norms are K-equivalent (recall that W. Rudin [25] gave a positive answer for $\varepsilon = \frac{1}{2}$). We present here a partial result, a variant of which was observed before by V. D. Milman and G. Pisier.

Fix n and let $\{w_i\}_{i=1}^{2^n}$ be the sequence of the Walsh elements in $l_\infty^{2^n}$, i.e.

$$w_1 = \sum_{i=1}^{2^n} e_i, \qquad w_2 = \sum_{i=1}^{2^{n-1}} e_i - \sum_{i=2^{n-1}+1}^{2^n} e_i, \dots, \text{ etc.}$$

The operator T on l_2^n, defined by

$$Te_i = w_i/\sqrt{2^n}; \qquad 1 \leqq i \leqq 2^n,$$

is clearly an isometry and, moreover,

$$\int \left\| \sum_{i=1}^{2^n} \varepsilon_i Te_i \right\|_2 d\varepsilon \geqq \left\| \left(\sum_{i=1}^{2^n} | Te_i |^2 \right)^{1/2} \right\|_2 \Big/ \sqrt{2} = \sqrt{2^{n-1}},$$

i.e., $\sqrt{2}\, T$ satisfies both conditions of Theorem 5.7. It follows that there exist a constant $b > 0$ and a subset σ_n of $\{1, 2, \dots, n\}$ so that $|\sigma_n| \geqq b \cdot 2^n$ and

$$\left\| \sum_{i \in \sigma_n} a_i w_i / 2^n \right\|_1 \geqq b \cdot \sum_{i \in \sigma_n} | a_i | / \sqrt{2^n}.$$

This statement can be interpreted better in the setting of function spaces. Let $\{W_i\}_{i=1}^\infty$ denote the sequence of the usual Walsh functions on $[0, 1]$. Then the inequality above implies that, for any $n, c > 0$ and $\eta \subset \sigma_n$ with $|\eta| \geqq c \cdot 2^n$, we have

$$\left\| \sum_{i \in \eta} W_i \right\|_2 \geqq \left\| \sum_{i \in \eta} W_i \right\|_1 \geqq b\, |\eta| / \sqrt{2^n} \geqq b \cdot c \sqrt{2^n} = b \cdot c \left\| \sum_{i \in \eta} W_i \right\|_2,$$

i.e., on "large" sums of elements from $\{W_i\}_{i \in \sigma_n}$, the L_1- and L_2-norms are $(bc)^{-1}$-equivalent. Since sets of 2^n characters cannot contain Λ_2-sets of cardinality proportional to 2^n, one cannot expect to prove that the L_1- and L_2-norms are equivalent on $\{W_i\}_{i \in \sigma_n}$.

More generally, it can be derived from Theorem 5.7 that, given a finite set Λ of characters on a compact abelian group G, there exists a subset Λ_0 of Λ such that $|\Lambda_0| \geqq c\, |\Lambda|$, for some universal constant $c > 0$, and

$$\left\| \sum_{\gamma \in \Lambda_0} a_\gamma \gamma \right\|_{L_1(G)} \geqq c \sum_{\gamma \in \Lambda_0} | a_\gamma | / | \Lambda_0 |^{1/2},$$

for any choice of $\{a_\gamma\}_{\gamma \in \Lambda_0}$.

6. Operators on spaces with an unconditional basis

In Section 3 we proved an invertibility theorem for operators acting on l_p^n-spaces whose corresponding matrix has 1's on the diagonal. In this section we present an extension of this result to the case of operators on spaces with an unconditional basis. The method used here is completely different and, in some sense, simpler than that used in Section 3. However, the rank of the "well" invertible submatrix that we obtain by the present method is not necessarily proportional to the rank n of the original matrix but only of order of magnitude $n^{1-\varepsilon}$, with ε as small as we like. We have not checked whether one can find well invertible submatrices of rank proportional to n. There is another minor restriction, namely, that the underlying spaces have non-trivial cotype or, weaker, than the unconditional basis under consideration satisfy a non-trivial lower estimate.

Before stating the main result, we recall that the unconditional constant of a basis $\{e_i\}_{i=1}^n$ is the smallest constant K so that

$$\left\| \sum_{i=1}^n a_i \varepsilon_i e_i \right\| \leqq K \cdot \left\| \sum_{i=1}^n a_i e_i \right\|,$$

for any choice of scalars $\{a_i\}_{i=1}^n$ and signs $\{\varepsilon_i\}_{i=1}^n$. Such a basis is also called K-unconditional.

THEOREM 6.1. *For every $K \geqq 1$, $M \geqq 1$, $1 < r < \infty$, $c_r > 0$ and $1 > \varepsilon > 0$, there exists a constant $C = C(K, M, r, c_r, \varepsilon) < \infty$ such that, whenever $n \geqq C$, X is a Banach space with a normalized K-unconditional basis $\{e_i\}_{i=1}^n$ which satisfies a lower r-estimate with constant c_r, i.e.*

$$\left\| \sum_{i=1}^n a_i e_i \right\| \geqq c_r \left(\sum_{i=1}^n |a_i|^r \right)^{1/r},$$

for all $\{a_i\}_{i=1}^n$, and $T: X \to X$ is a linear operator of norm $\|T\| \leqq M$ whose matrix relative to $\{e_i\}_{i=1}^n$ has 1's on the diagonal, then there exists a subset σ of $\{1, 2, \dots, n\}$ of cardinality $|\sigma| > n^{1-\varepsilon}$ for which $R_\sigma T R_\sigma$ restricted to $R_\sigma X$ is invertible and its inverse satisfies

$$\|(R_\sigma T R_\sigma)^{-1}\| \leqq D.$$

(R_σ denotes, as before, the restriction operator defined by $R_\sigma(\Sigma_{i=1}^n a_i e_i) = \Sigma_{i \in \sigma} a_i e_i$, for all $\{a_i\}_{i=1}^n$.)

The proof of Theorem 6.1 requires a preliminary lemma which is essentially known.

LEMMA 6.2. *Fix m and let $p(x)=\sum_{i=0}^{m} b_i x^i$ be a polynomial of degree m which satisfies $|p(x)|\leqq 1$, for $0\leqq x\leqq 1$. Then*

$$\max_{0\leqq i\leqq m} |b_i|\leqq (m+3)^{3m}.$$

PROOF. We shall proceed by induction. For $m=1$, the assertion is trivial. Suppose now that it is true for some m and consider a polynomial $p(x)=\sum_{i=0}^{m+1} b_i x^i$ of degree $m+1$ which satisfies $|p(x)|\leqq 1$, for $0\leqq x\leqq 1$. By integration, we get that

$$\left|\sum_{i=0}^{m+1} b_i x^i/(i+1)\right|\leqq 1,$$

for all $1\leqq x\leqq 1$, from which it easily follows that

$$\left|\sum_{i=0}^{m} b_i x^i((m+2)/(i+1)-1)\right|\leqq m+3,$$

again, for all $0\leqq x\leqq 1$. Hence, by the induction hypothesis,

$$|b_i|/(m+1)(m+3)\leqq |b_i|((m+2)/(i+1)-1)/(m+3)\leqq (m+3)^{3m},$$

i.e.

$$|b_i|\leqq (m+3)^{3m+2}; \qquad 0\leqq i\leqq m,$$

and also

$$|b_{m+1}|\leqq 1+\sum_{i=0}^{m}|b_i|\leqq 1+(m+1)(m+3)^{3m+2}<(m+4)^{3(m+1)}. \qquad \square$$

PROOF OF THEOREM 6.1. Fix the constants K, M, r, c_r and ε, and let $\{e_i\}_{i=1}^{n}$ and T satisfy the conditions of the statement. Note that there is no loss of generality in assuming that $K=1$, i.e., that $\{e_i\}_{i=1}^{n}$ is 1-unconditional.

Let $(a_{i,j})_{i,j=1}^{n}$ be the matrix of T relative to the basis $\{e_i\}_{i=1}^{n}$, i.e.

$$Te_i=\sum_{j=1}^{n} a_{i,j}e_j; \qquad 1\leqq i\leqq n.$$

By our hypothesis, $a_{i,i}=1$, for all $1\leqq i\leqq n$. Put $S=T-I$ and let $(b_{i,j})_{i,j=1}^{n}$ be the matrix associated to S, i.e., $b_{i,i}=0$ and $b_{i,j}=a_{i,j}$, for all $1\leqq i,\ j\leqq n,\ i\neq j$.

The assumption that $\{e_i\}_{i=1}^{n}$ satisfies a non-trivial lower estimate is needed in order to select a submatrix of $(b_{i,j})_{i,j=1}^{n}$ of rank propositional to n which has "small" entries. More precisely, since

$$M^r \geqq \| Te_i \|^r \geqq c_r^r \sum_{j=1}^{n} | a_{i,j} |^r,$$

i.e.

$$(M/c_r)^r \geqq \sum_{j=1}^{n} | a_{i,j} |^r,$$

for all $1 \leqq i \leqq n$, it follows from [13] or [4] that, with $\tau = \varepsilon^2/16$, there exists a subset η of $\{1, 2, \ldots, n\}$ of cardinality

$$| \eta | \geqq n^{1-2\tau}/16$$

so that

$$\sum_{j \in \eta} | b_{i,j} |^r = \sum_{\substack{j \in \eta \\ j \neq i}} | a_{i,j} |^r \leqq (M/c_r)^r / n^\tau,$$

for every $i \in \eta$. In particular, we conclude that

$$| b_{i,j} | \leqq M/c_r \cdot n^{\tau/r},$$

for all $i, j \in \eta$.

Take $\delta = 1/n^{\vee\tau}$ and let $\{\xi_i\}_{i \in \eta}$ be a sequence of independent random variables of mean δ over a probability space (Ω, Σ, μ) taking only the values 0 and 1. For $\omega \in \Omega$, put

$$\eta(\omega) = \{ i \in \eta ;\ \xi_i(\omega) = 1 \},$$

$$S(\omega) = R_{\eta(\omega)} S R_{\eta(\omega)},$$

and, for l being a fixed integer so that $l > 3r/2\tau^2$, let $(b_{i,j}^{(l)}(\omega))_{i,j=1}^{n}$ be the matrix associated to the l-power $S(\omega)^l$ of $S(\omega)$.

We introduce now the following notation. For fixed integers $i, j \in \eta$, put

$$\Gamma_{i,j} = \{ (i, i_1, i_2, \ldots, i_{l-1}, j);\ i_h \in \eta, 1 \leqq h < l \}$$

and, for $\gamma = (i, i_1, i_2, \ldots, i_{l-1}, j) \in \Gamma_{i,j}$, denote

$$s_\gamma = b_{i,i_1} \cdot b_{i_1,i_2} \cdots b_{i_{l-1},j} \quad \text{and} \quad \varphi_\gamma = \xi_i \cdot \xi_{i_1} \cdot \xi_{i_2} \cdot \xi_{i_{l-1}} \cdot \xi_j .$$

Then

$$b_{i,j}^{(l)}(\omega) = \sum_{\gamma \in \Gamma_{i,j}} s_\gamma \varphi_\gamma(\omega),$$

for all $\omega \in \Omega$, and thus, by integration,

$$\begin{aligned}(M+1)^{2l} &\geqq \int_{\Omega} \|S(\omega)^l\|^2 d\mu(\omega) \\ &\geqq \int_{\Omega} |b_{i,j}^{(l)}(\omega)|^2 d\mu(\omega) \\ &= \int_{\Omega} \sum_{\gamma,\gamma'\in\Gamma_{i,j}} s_\gamma s_{\gamma'} \varphi_\gamma(\omega)\varphi_{\gamma'}(\omega) d\mu(\omega) \\ &= \sum_{h=1}^{2l+2} b_h \delta^h,\end{aligned}$$

for a suitable sequence $\{b_h\}_{h=1}^{2l+2}$ of reals. Since

$$0 \leqq \sum_{h=1}^{2l+2} b_h x^h \leqq (M+1)^{2l},$$

for all $0 \leqq x \leqq 1$ (and not only for the particular choice of $x = \delta$ made above), we get, by using Lemma 6.2, that

$$\max_{1\leqq h \leqq 2l+2} |b_h| \leqq (M+1)^{2l} \cdot (2l+5)^{6(l+1)}.$$

On the other hand, observe that, for each $1 \leqq h \leqq 2l+2$, b_h is the sum of all the products $s_\gamma \cdot s_{\gamma'}$ for which the union $\gamma \cup \gamma'$ contains exactly h distinct integers. This sum has at most $(2l)^{2l} \cdot n^h$ summands of the form $s_\gamma \cdot s_{\gamma'}$ each of which is bounded by $(M/c_r \cdot n^{\tau/r})^{2l}$. Thus

$$|b_h| \leqq (2l \cdot M/c_r)^{2l} \cdot n^{h-2l\tau/r}; \qquad 1 \leqq h \leqq 2l+2.$$

We are now able to evaluate the expression $\Sigma_{h=1}^{2l+2} b_h \delta^h$. We choose an integer m so that

$$1/\tau \leqq m < 2/\tau$$

and use the first estimate for b_h with $h > m$ and the second for b_h with $1 \leqq h \leqq m$. It follows that

$$\begin{aligned}\sum_{h=1}^{2l+2} b_h\delta^h &\leqq \sum_{h=1}^{m} |b_n|\delta^h + \sum_{h=m+1}^{2l+2} |b_h|\delta^h \\ &\leqq m \cdot \max_{1\leqq h\leqq m} |b_h| + \delta^{m+1}\cdot(2l+2) \max_{m<h\leqq 2l+2} |b_h| \\ &\leqq m(2l\cdot M/c_r)^{2l}\cdot n^{m-2l\cdot\tau/r} + \delta^{m+1}\cdot(M+1)^{2l}\cdot(2l+5)^{6l+7}.\end{aligned}$$

In view of our concrete choice of l, m and δ, we obtain the existence of a constant C_1, independent of n, so that

$$\int_\Omega |b_{i,j}^{(l)}(\omega)|^2 d\mu(\omega) = \sum_{h=1}^{2l+2} b_h \delta^h \leqq C_1(1/n^{1/\tau} + 1/n^{1/\vee\tau}) \leqq 2C_1/n^{1/\vee\tau},$$

for all $i, j \in \eta$. Hence

$$\begin{aligned}\int_\Omega \|S(\omega)^l\| d\mu(\omega) &\leqq \int_\Omega \sum_{i,j\in\eta} |b_{i,j}^{(l)}(\omega)| d\mu(\omega) \\ &\leqq n^2 \cdot \max_{i,j\in\eta} \int_\Omega |b_{i,j}^{(l)}(\omega)| d\mu(\omega) \\ &\leqq n^2 \cdot \max_{i,j\in\eta} \left(\int_\Omega |b_{i,j}^{(l)}(\omega)|^2 d\mu(\omega)\right)^{1/2} \\ &\leqq (2C_1)^{1/2} \cdot n^{2-1/2\vee\tau}.\end{aligned}$$

Since $\tau < 1/16$ it follows easily that there exists a constant C_2, independent of n, so that, for $n \geqq C_2$, we have

$$\int_\Omega \|S(\omega)^l\| d\mu \leqq \tfrac{1}{2}.$$

Hence, one can find a point ω_0 in the set

$$D = \left\{\omega \in \Omega; \left|\sum_{i\in\eta} \xi_i(\omega) - \delta|\eta|\right| \geqq \delta|\eta|/2\right\}$$

such that

$$\|S(\omega_0)^l\| \leqq \tfrac{1}{2}.$$

Put

$$C = 2l \cdot \max\{\|S(\omega_0)\|, \|S(\omega_0)^2\|, \dots, \|S(\omega_0)^{l-1}\|, 16^{2/\varepsilon}\}$$

and note that the inverse of $R_{\xi(\omega_0)}TR_{\eta(\omega_0)}$ restricted to $R_{\xi(\omega_0)}X$ satisfies

$$\begin{aligned}\|(R_{\eta(\omega_0)}TR_{\eta(\omega_0)})^{-1}\| &\leqq \sum_{k=0}^\infty \|S(\omega_0)^k\| \\ &= \sum_{u=0}^\infty \sum_{h=0}^{l-1} \|S(\omega_0)^{lu+h}\| \\ &\leqq \sum_{u=0}^\infty \sum_{h=0}^{l-1} \|S(\omega_0)^l\|^u \cdot \|S(\omega_0)^h\| \\ &\leqq C.\end{aligned}$$

Furthermore, since $\omega_0 \in D$, it follows that

$$|\eta(\omega_0)| = \sum_{i\in\eta} \xi_i(\omega_0) \geqq \delta|\eta|/2 \geqq n^{1-2\tau-\sqrt{\tau}}/32 > n^{1-3\sqrt{\tau}}/16.$$

Hence, in view of the fact that $\tau = \varepsilon^2/16$, we conclude that, for n sufficiently large, we have

$$|\eta(\omega_0)| > n^{1-\varepsilon}.$$

This, of course, completes the proof. □

Theorem 6.1 can be used in order to prove the following result which, in some sense, improves Theorem 1.1 from [6].

Corollary 6.3. *For every $K \geqq 1$, $M \geqq 1$, $1 < r < \infty$, $c_r > 0$ and $\varepsilon > 0$, there exists a constant $D = D(K, M, r, c_r, \varepsilon) < \infty$ such that, whenever $X = X_1 \oplus X_2$ is a Banach space with a normalized K-unconditional basis $\{e_i\}_{i=1}^n$ which satisfies a lower r-estimate with constant c_r and the projections P_1 and P_2 onto X_1, respectively X_2, associated with the above direct sum, have norms $\leqq M$, then there exists a subset σ of $\{1,2,\dots,n\}$ of cardinality $|\sigma| > n^{1-\varepsilon}$ with the property that at least for one of the factors, say X_1, the following holds*:

(i) *$\{P_1e_i\}_{i\in\sigma}$ is D-equivalent to $\{e_i\}_{i\in\sigma}$,*

(ii) *there exists a linear projection Q of norm $\|Q\| \leqq D$ from X onto $[P_1e_i]_{i\in\sigma}$.*

Proof. Let $\{e_i^*\}_{i=1}^n$ be, as usual, the biorthogonal functionals associated to $\{e_i\}_{i=1}^n$ and notice that, at least for one of the factors, say X_1, one can find a subset η of $\{1,2,\dots,n\}$ of cardinality $|\eta| \geqq n/2$ so that

$$e_i^* P_1 e_i \geqq 1/2; \qquad i \in \eta.$$

Consider now the linear operator T: $[e_i]_{i\in\eta} \to X_1$, which is defined by

$$Te_i = P_1e_i / e_i^* P_1 e_i\,; \qquad i \in \eta.$$

This operator clearly has norm $\leqq 2KM$. Therefore, $R_\eta T$ is an operator of norm $\leqq 2K^2M$ on $[e_i]_{i\in\eta}$ and its matrix relative to $\{e_i\}_{i\in\eta}$ has 1's on the diagonal. Hence, by Theorem 6.1, there exist a $C < \infty$ and a subset σ of η of cardinality $|\sigma| > (n/2)^{1-\varepsilon}$ such that $\|(R_\sigma TR_\sigma)^{-1}\| \leqq C$. This already implies that $\{P_1e_i\}_{i\in\sigma}$ is $2CK^2M$-equivalent to $\{e_i\}_{i\in\sigma}$. Furthermore, it is is easily verified that

$$Q = T \cdot (R_\sigma TR_\sigma)^{-1} \cdot R_\sigma$$

is a projection of norm $\leqq 2CK^2M$ from X onto its subspace $[P_1e_i]_{i\in\sigma}$. □

7. Non-operator type results

So far, we proved invertibility results for "large" submatrices of matrices which map the unit vectors $\{e_i\}_{i=1}^n$ into vectors $\{x_i\}_{i=1}^n$ of norm one or about one. The boundedness of the matrix is equivalent to the existence of a corresponding upper estimate for the vectors $\{x_i\}_{i=1}^n$ while the assertion of restricted invertibility can be interpreted as the existence of a lower estimate which holds for a subset $\{x_i\}_{i\in\sigma}$ of cardinality $|\sigma|$ proportional to n.

The purpose of this section is to present a quite general situation in which lower estimates hold without assuming the existence of suitable upper estimates. In some sense, the main result is an extension of Proposition 4.4 to the present setting, i.e., without assuming condition (i) there.

THEOREM 7.1. *For every* $1<p\leqq 2$, $K<\infty$ *and* $c>0$, *there exists a constant* $d=d(p,K,c)>0$ *such that, whenever* $\{g_i\}_{i=1}^n$ *and* $\{h_i\}_{i=1}^n$ *are normalized sequences in* L_p, *respectively* $L_{p'}$, *for which*

(1) $\|\Sigma_{i=1}^n b_i h_i\|_{p'}\leqq K(\Sigma_{i=1}^n |b_i|^{p'})^{1/p'}$, *for any choice of* $\{b_i\}_{i=1}^n$,
and

(2) $|\langle g_i, h_i\rangle|\geqq c$, *for all* $1\leqq i\leqq n$,
then there exists a subset σ *of* $\{1,2,\dots,n\}$ *so that* $|\sigma|\geqq dn$ *and*

$$\Big\|\sum_{i\in\sigma} a_i g_i\Big\|_p \geqq d\Big(\sum_{i\in\sigma}|a_i|^p\Big)^{1/p},$$

for all $\{a_i\}_{i\in\sigma}$.

The first step in the proof is to pass from the function space framework to a sequence space one. A connection between these two settings is given by the following very simple lemma.

LEMMA 7.2. *Fix* $1\leqq p\leqq\infty$ *and let* $\{g_i\}_{i=1}^n$ *and* $\{h_i\}_{i=1}^n$ *be normalized sequences of functions in* L_p, *respectively* $L_{p'}$, *which satisfy the conditions* (1) *and* (2) *of Theorem* 7.1.

Let $\{e_j\}_{j=1}^n$ *denote the unit vectors in* l_p^n *and, for* $1\leqq i\leqq n$, *put*

$$x_i=\sum_{j=1}^n \langle g_i, h_j\rangle \frac{\overline{\langle g_i, h_i\rangle}}{|\langle g_i, h_i\rangle|}\, e_j.$$

Then, for any choice of $\{a_i\}_{i=1}^n$, *we have*

$$\Big\|\sum_{i=1}^n a_i x_i\Big\|_p \leqq K\Big\|\sum_{i=1}^n a_i \frac{\overline{\langle g_i, h_i\rangle}}{|\langle g_i, h_i\rangle|}\, g_i\Big\|_p.$$

PROOF. By linearization we get, for any choice of $\{a_i\}_{i=1}^n$, that

$$\left\|\sum_{i=1}^n a_ix_i\right\|_p=\left(\sum_{j=1}^n\left|\sum_{i=1}^n a_i\langle g_i,h_j\rangle\frac{\overline{\langle g_i,h_i\rangle}}{|\langle g_i,h_i\rangle|}\right|^p\right)^{1/p}$$

$$=\sup\left\{\left|\sum_{j=1}^n b_j\sum_{i=1}^n a_i\langle g_i,h_j\rangle\frac{\overline{\langle g_i,h_i\rangle}}{|\langle g_i,h_i\rangle|}\right|;\sum_{j=1}^n|b_j|^{p'}\leqq 1\right\}$$

$$=\sup\left\{\left|\left\langle\sum_{i=1}^n a_i\frac{\overline{\langle g_i,h_i\rangle}}{|\langle g_i,h_i\rangle|}g_i,\sum_{j=1}^n b_jh_j\right\rangle\right|;\sum_{j=1}^n|b_j|^{p'}\leqq 1\right\}$$

$$\leqq K\left\|\sum_{i=1}^n a_i\frac{\overline{\langle g_i,h_i\rangle}}{|\langle g_i,h_i\rangle|}g_i\right\|_p.$$

□

We need also the following lemma.

LEMMA 7.3. *For every $1<r<p\leqq 2$, there is an $\alpha=\alpha(p,r)>0$ such that, whenever $\{x_i\}_{i=1}^k$ is a sequence of vectors in l_1^k, for which*

$$\left\|\sum_{i=1}^k a_ix_i\right\|_1\geqq c\sum_{i=1}^k|a_i|-\alpha ck^{1/r'}\cdot\left(\sum_{i=1}^k|a_i|^r\right)^{1/r},$$

for some $c>0$, k and all $\{a_i\}_{i=1}^k$, then there exists a subset τ of $\{1,2,\dots,k\}$ of cardinality $|\tau|\geqq k/2$ so that

$$\left\|\sum_{i\in\tau}a_ix_i\right\|_p\geqq\frac{c}{4}\cdot\left(\sum_{i\in\tau}|a_i|^p\right)^{1/p},$$

for any choice of $\{a_i\}_{i\in\tau}$.

PROOF. We use again an exhaustion argument. Suppose that the assertion is false. Then one can construct subsets $\tau_1\supset\tau_2\supset\cdots\supset\tau_l$ of $\{1,2,\dots,k\}$ with $|\tau_l|\geqq k/2$ and vectors $\{y_i\}_{i=1}^l$ such that

$$y_i=\sum_{j\in\tau_i}b_{i,j}x_j,\quad \|y_i\|_p<c/4\quad\text{and}\quad\sum_{j\in\tau_i}|b_{i,j}|^p=1;\qquad 1\leqq i\leqq l.$$

For those $j\notin\tau_i$; $1\leqq i\leqq l$, we put $b_{i,j}=0$. The procedure is stopped after, say, m steps when the set

$$\tau_{m+1}=\left\{j\in\tau_m;\sum_{i=1}^m|b_{i,j}|^p<1\right\}$$

has cardinality $<k/2$. An easy computation shows that

$$k \leqq 2m.$$

Let now $\{\psi_i\}_{i=1}^m$ be a sequence of independent p-stable random variables over a probability space (Ω, Σ, μ) which have norm one in $L_1(\Omega, \Sigma, \mu)$. Then, by our assumption (with α to be determined later),

$$\begin{aligned}
&cm^{1/p} \cdot k^{1/p'}/4 \\
&> k^{1/p'} \cdot \left\| \left(\sum_{i=1}^m |y_i|^p\right)^{1/p} \right\|_p \\
&\geqq \left\| \left(\sum_{i=1}^m |y_i|^p\right)^{1/p} \right\|_1 \\
&= \left\| \int_\Omega \left| \sum_{i=1}^m \psi_i(\omega) y_i \right| d\mu(\omega) \right\|_1 \\
&= \int_\Omega \left\| \left(\sum_{j=1}^k \left(\sum_{i=1}^m \psi_i(\omega) b_{i,j} \right) x_j \right\|_1 d\mu(\omega) \\
&\geqq c \cdot \sum_{j=1}^k \int_\Omega \left| \sum_{i=1}^m \psi_i(\omega) b_{i,j} \right| d\mu(\omega) - \alpha \cdot c \cdot k^{1/r'} \cdot \int_\Omega \left(\sum_{j=1}^k \left| \sum_{i=1}^m \psi_i(\omega) b_{i,j} \right|^r \right)^{1/r} d\mu(\omega) \\
&\geqq c \cdot \sum_{j=1}^k \left(\sum_{i=1}^m |b_{i,j}|^p \right)^{1/p} - \alpha \cdot c \cdot k^{1/r'} \|\psi_1\|_r \left(\sum_{j=1}^k \left(\sum_{i=1}^m |b_{i,j}|^p \right)^{r/p} \right)^{1/r}.
\end{aligned}$$

However, as readily verified, we have

$$\sum_{i=1}^m |b_{i,j}|^p \leqq 2,$$

for all $1 \leqq j \leqq k$. Thus

$$\begin{aligned}
m^{1/p} \cdot k^{1/p'}/4 &> \sum_{j=1}^k \sum_{i=1}^m |b_{i,j}|^p / 2^{1/p'} - 2^{1/p} \cdot \alpha \cdot k \|\psi_1\|_r \\
&\geqq m/2^{1/p'} - 2^{2/p} \cdot \alpha \cdot k^{1/p'} \cdot m^{1/p} \cdot \|\psi_1\|_r,
\end{aligned}$$

i.e.

$$1 > 2^{2/p'} - 2^{2+2/p} \cdot \alpha \cdot \|\psi_1\|_r,$$

which is contradictory if α if chosen small enough. □

Proof of Theorem 7.1. Fix $1 < p \leqq 2$, $K < \infty$ and $c > 0$, and consider sequences $\{g_i\}_{i=1}^n$ and $\{h_i\}_{i=1}^n$ which satisfy the conditions (1) and (2). Choose $1 < r < p$ and let $\alpha = \alpha(p, r)$ be given by Lemma 7.3.

By (1), the matrix $\{\langle g_i, h_j\rangle\}_{i,j=1}^n$ has the property that

$$\left(\sum_{j=1}^n |\langle g_i, h_j\rangle|^p\right)^{1/p} = \sup\left\{\sum_{j=1}^n b_j\langle g_i, h_j\rangle;\ \sum_{j=1}^n |b_j|^{p'} \leqq 1\right\} \leqq K,$$

for all $1 \leqq i \leqq n$. Thus, by [13], one can find a $d_1 = d_1(p, r, K, c) > 0$ and a subset σ_1 of $\{1, 2, \dots, n\}$ such that $|\sigma_1| \geqq d_1 n$ and

$$\left(\sum_{\substack{j\in\sigma_1\\ j\neq i}} |\langle g_i, h_j\rangle|^p\right)^{1/p} \leqq \alpha \cdot c/12B_{r'},$$

where $B_{r'}$ denotes the constant in Khintchine's inequality in $L_{r'}$.

Let $\{e_i\}_{i\in\sigma_1}$ and $\{e_i^*\}_{i\in\sigma_1}$ stand for the unit vectors in $l_1^{|\sigma_1|}$, respectively $l_\infty^{|\sigma_1|}$, put

$$x_i = \sum_{j\in\sigma_1} \left\langle g_i, h_j\right\rangle \frac{\overline{\langle g_i, h_i\rangle}}{|\langle g_i, h_i\rangle|}\, e_j\,; \qquad i \in \sigma_1,$$

and, for each tuple of signs $\varepsilon = (\varepsilon_j)_{j\in\sigma_1} \in \{-1, +1\}^{|\sigma_1|}$, consider the vector

$$u(\varepsilon) = \sum_{j\in\sigma_1} \varepsilon_j e_j^*.$$

Then, for each $i \in \sigma_1$, we have

$$\langle x_i, u(\varepsilon)\rangle = \sum_{j\in\sigma_1} \varepsilon_j\langle g_i, h_j\rangle \frac{\overline{\langle g_i, h_i\rangle}}{|\langle g_i, h_i\rangle|} = \varepsilon_i|\langle g_i, h_i\rangle| + v_i(\varepsilon),$$

where

$$v_i(\varepsilon) = \sum_{\substack{j\in\sigma_1\\ j\neq i}} \varepsilon_j\langle g_i, h_j\rangle \frac{\overline{\langle g_i, h_i\rangle}}{|\langle g_i, h_i\rangle|}.$$

By Khintchine's inequality in $L_{r'}$, we get that

$$\begin{aligned}\int\left(\sum_{i\in\sigma_1} |v_i(\varepsilon)|^{r'}\right)^{1/r'} d\varepsilon &\leqq \left(\sum_{i\in\sigma_1}\int\left|\sum_{\substack{j\in\sigma_1\\ j\neq i}} \varepsilon_j\langle g_i, h_j\rangle\right|^{r'} d\varepsilon\right)^{1/r'}\\
&\leqq B_{r'}\cdot\left(\sum_{i\in\sigma_1}\left(\sum_{\substack{j\in\sigma_1\\ j\neq i}} |\langle g_i, h_j\rangle|^2\right)^{r'/2}\right)^{1/r'}\\
&\leqq B_{r'}\cdot\left(\sum_{i\in\sigma_1}\left(\sum_{\substack{j\in\sigma_1\\ j\neq i}} |\langle g_i, h_j\rangle|^p\right)^{r'/p}\right)^{1/r'}\\
&\leqq \alpha\cdot c\cdot|\sigma_1|^{1/r'}/12.\end{aligned}$$

Consider now the set

$$\mathscr{E}=\left\{(\varepsilon_i)_{i\in\sigma_1}\in\{-1,+1\}^{|\sigma_1|};\ \left(\sum_{i\in\sigma_1}|v_i(\varepsilon)|^{r'}\right)^{1/r'}<\alpha\cdot c\cdot|\sigma_1|^{1/r'}/4\right\}$$

and observe that

$$|\mathscr{E}|\geqq 3\cdot 2^{|\sigma_1|}/4.$$

Therefore, by using [27] or [29], we conclude the existence of a subset σ_2 of σ_1 of cardinality $k=|\sigma_2|\geqq|\sigma_1|/2$ so that, for each tuple $(\varepsilon_i)_{i\in\sigma_2}$, there exists an extension $(\varepsilon_i)_{i\in\sigma_1}\in\mathscr{E}$.

Fix scalars $\{a_j\}_{j\in\sigma_2}$, write $a_j=b_j+ic_j$ with b_j and c_j reals, for all $j\in\sigma_2$, and choose signs $(\theta'_j)_{j\in\sigma_2}$ and $(\theta''_j)_{j\in\sigma_2}$ so that $b_j\theta'_j=|b_j|$ and $c_j\theta'_j=|c_j|$; $j\in\sigma_2$. By the above choice of σ_2, one can find in $\mathscr{E}$ extensions $\varepsilon'=(\varepsilon'_j)_{j\in\sigma_1}$ and $\varepsilon''=(\varepsilon''_j)_{j\in\sigma_1}$ of $(\theta'_j)_{j\in\sigma_2}$, respectively $(\theta''_j)_{j\in\sigma_2}$. It follows that

$$\begin{aligned}
&2\left\|\sum_{j\in\sigma_2}a_jx_j\right\|_1\\
&\quad\geqq\left|\left\langle\sum_{j\in\sigma_2}a_jx_j,u(\varepsilon')-iu(\varepsilon'')\right\rangle\right|\\
&\quad=\left|\sum_{j\in\sigma_2}a_j\langle x_j,u(\varepsilon')-iu(\varepsilon'')\rangle\right|\\
&\quad\geqq\left|\sum_{j\in\sigma_2}(b_j+ic_j)(\varepsilon'_j-i\varepsilon''_j)|\langle g_j,h_j\rangle|\right|-\left|\sum_{j\in\sigma_2}a_j(v_j(\varepsilon')-iv_j(\varepsilon'')\right|\\
&\quad\geqq c\cdot\sum_{j\in\sigma_2}|a_j|-\left(\sum_{j\in\sigma_2}|a_j|^r\right)^{1/r}\left[\left(\sum_{j\in\sigma_2}|v_j(\varepsilon')|^{r'}\right)^{1/r'}+\left(\sum_{j\in\sigma_2}|v_j(\varepsilon'')|^{r'}\right)^{1/r'}\right]\\
&\quad\geqq c\cdot\sum_{j\in\sigma_2}|a_j|-\alpha\cdot c\,|\sigma_1|^{1/r'}\cdot\left(\sum_{j\in\sigma_2}|a_j|^r\right)^{1/r}\Big/2\\
&\quad\geqq c\sum_{i\in\sigma_2}|a_i|-\alpha\cdot c\cdot k^{1/r'}\cdot\left(\sum_{i\in\sigma_2}|a_i|^r\right)^{1/r},
\end{aligned}$$

i.e., the conditions of Lemma 7.3 are satisfied. Consequently, there is a subset τ of σ_2 of cardinality

$$|\tau|\geqq k/2\geqq d_1n/4$$

so that

$$\left\|\sum_{i\in\tau}a_ix_i\right\|_p\geqq\frac{c}{4}\cdot\left(\sum_{i\in\tau}|a_i|^p\right)^{1/p},$$

for all $\{a_i\}_{i\in\tau}$. The proof can be now completed by using Lemma 7.2. □

The following immediate consequence of Theorem 7.1 describes the most common situation when this result is used in applications.

COROLLARY 7.4. *For every* $1 < p \leqq 2$ *and* $c > 0$, *there exists a* $d = d(p,c) > 0$ *so that, whenever* $\{f_i\}_{i=1}^n$ *is a normalized sequence in* L_p *for which one can find mutually disjoint sets* $\{A_i\}_{i=1}^n$ *with the property that*

$$\int_{A_i} |f_i|^p d\mu \geqq c,$$

for all $1 \leqq i \leqq n$, *then there exists a subset* σ *of* $\{1, 2, \ldots, n\}$ *such that* $|\sigma| \geqq dn$ *and*

$$\left\| \sum_{i \in \sigma} a_i f_i \right\|_p \geqq d \cdot \left(\sum_{i \in \sigma} |a_i|^p \right)^{1/p},$$

for any choice of scalars $\{a_i\}_{i \in \sigma}$.

PROOF. Take $g_i = f_i$, $\hat{h}_i = |f_i|^{p-1}(\operatorname{sgn} f_i)\chi_{A_i}$ and $h_i = \hat{h}_i / \|\hat{h}_i\|_{p'}$; $1 \leqq i \leqq n$, and apply Theorem 7.1. □

8. Remarks on some estimates

In Sections 1 and 3, it was proved that, for every $1 < p < \infty$, there is a function $\delta_p(\varepsilon)$ such that, whenever S is an operator on l_p^n of norm $\|S\|_p \leqq 1$ whose matrix relative to the unit vector basis of l_p^n has 0's on the diagonal, then, for some subset σ of $\{1, 2, \ldots, n\}$ of cardinality $|\sigma| \geqq \delta_p(\varepsilon)n$, the inequality

$$\|R_\sigma S R_\sigma\|_p < \varepsilon$$

holds. As usual, R_σ denotes the restriction operator. Clearly, from the definition of $\delta_p(\varepsilon)$ it follows that

$$\delta_p(\varepsilon \cdot \varepsilon') \geqq \delta_p(\varepsilon) \cdot \delta_p(\varepsilon')$$

and, therefore, also that

$$\delta_p(\varepsilon) > \varepsilon^k,$$

for some $k = k(p)$ and all $0 < \varepsilon < \frac{1}{2}$. This implies that all the estimates obtained in Section 4 related to finite-dimensional L_p-problems are of a polynomial nature.

Notice that the method used in Section 3 to obtain $\delta_p(\varepsilon)$ does not yield directly a function satisfying the above condition. We shall show in what follows how to proceed more effectively. We present in detail only the case $p = 2$. The case of a general p, which is similar, is left to the reader.

Since the considerations below involve different norms for a linear operator $W: \mathbf{R}^m \to \mathbf{R}^n$, we shall denote by $\| W \|_{p\to q}$ its norm when W is considered as an operator from l_p^m into l_q^n. Instead of $\| W \|_{p\to p}$ we shall continue to use the notation $\| W \|_p$.

We also recall that the matrix $(b_{i,j})_{i=1,j=1}^{m \ \ n}$, corresponding to the operator W, is defined by

$$We_i = \sum_{j=1}^{n} b_{i,j} e_j \,; \qquad 1 \leqq i \leqq m.$$

Proposition 8.1. *There exists a constant $C < \infty$ such that, whenever $0 < \delta < 1$, n is an integer, $\{\xi_j\}_{j=1}^n$ is a sequence of independent random variables of mean δ over some probability space (Ω, Σ, μ) taking only the values 0 and 1, $m = [\delta n]$ and $T: l_2^m \to l_2^n$ is a linear operator of norm $\| T \|_2 \leqq 1$ whose matrix relătive to the unit vector bases is denoted by $(a_{i,j})_{i=1,j=1}^{m \ \ n}$, then*

$$\int_\Omega \Big\| \sum_{i=1}^{m} \sum_{j=1}^{n} \xi_j(\omega) a_{i,j} \otimes e_j \Big\|_{2\to 1} d\mu(\omega) \leqq C\delta^{1/8} m^{1/2}.$$

The proof of Proposition 8.1 requires the following lemma.

Lemma 8.2. *For every linear operator $T: l_2^m \to l_1^n$ and every $\varepsilon > 0$, there exists a subset η of $\{1, 2, \ldots, n\}$ so that*

(i) $|\eta| \leqq K_G (\| T \|_{2\to 1}/\varepsilon)^2$

and

(ii) $\| R_{\eta^c} T \|_2 < \varepsilon$.

Proof. As we have already seen before, it follows from Grothendieck's inequality and Pietsch's factorization theorem that there exist non-negative reals $\{\lambda_j\}_{j=1}^n$ such that

(1) $(\sum_{j=1}^n \lambda_j^2)^{1/2} \leqq K_G \| T \|_{2\to 1}$

and

(2) $\sum_{j=1}^n (\sum_{i=1}^m a_{i,j} b_i)^2/\lambda_j^2 \leqq \sum_{i=1}^m |b_i|^2$,

for any choice of $\{b_i\}_{i=1}^m$. Then, in order to complete the proof, it suffices to take

$$\eta = \{1 \leqq j \leqq n \,;\, \lambda_j \geqq \varepsilon\}. \qquad \square$$

Proof of Proposition 8.1. Fix $0 < \delta < 1$ and an integer n, take $m = [\delta n]$ and let $T: l_2^m \to l_2^n$ be a linear operator of norm $\leqq 1$. Let $\{\xi_j\}_{j=1}^n$ be a sequence of independent random variables of mean δ over a probability space (Ω, Σ, μ) which take only the values 0 and 1. Choose now two independent copies (Ω', Σ', μ') and $(\Omega'', \Sigma'', \mu'')$ of (Ω, Σ, μ) and let $\{\xi'_j\}_{j=1}^n$ and $\{\xi''_j\}_{j=1}^n$ be two sequences of independent random variables of mean $\sqrt{\delta}$ over (Ω', Σ', μ'), respectively $(\Omega'', \Sigma'', \mu'')$, which again take only the values 0 and 1.

Fix $\omega' \in \Omega'$ and apply Lemma 8.2 with $\varepsilon = \delta^{1/8}$ to the operator $T_{\omega'}: l_2^m \to l_2^n$ which is determined by the matrix $(\xi'_j(\omega')a_{i,j})_{i=1,j=1}^{m\ \ n}$, where $(a_{i,j})_{i=1,j=1}^{m\ \ n}$ is the matrix corresponding to the original operator T. It follows that there exists a subset $\eta(\omega')$ of $\{1, 2, \ldots, n\}$ such that

(i) $|\eta(\omega')| \leqq K_G \delta^{-1/4} \| T_{\omega'} \|_{2\to 1}^2$

and

(ii) $\| R_{\eta(\omega')^c} T_{\omega'} \|_2 < \delta^{1/8}$.

For $\omega' \in \Omega'$ and $\omega'' \in \Omega''$, we shall set

$$\tau'(\omega') = \{1 \leqq j \leqq n\, ; \xi'_j(\omega') = 1\}; \qquad \tau''(\omega'') = \{1 \leqq j \leqq n\, ; \xi''_j(\omega'') = 1\}$$

and

$$\tau(\omega', \omega'') = \tau'(\omega'') \cap \tau''(\omega'').$$

Then we get that

$$\begin{aligned}
I &= \int_\Omega \left\| \sum_{i=1}^m \sum_{j=1}^m \xi_j(\omega) a_{i,j} e_i \otimes e_j \right\|_{2\to 1} d\mu(\omega) \\
&= \int_{\Omega''} \int_{\Omega'} \left\| \sum_{i=1}^m \sum_{j=1}^n \xi'_j(\omega')\xi''_j(\omega'') a_{i,j} e_i \otimes e_j \right\|_{2\to 1} d\mu'(\omega') d\mu''(\omega'') \\
&= \int_{\Omega''} \int_{\Omega'} \| R_{\tau''(\omega'')} T_{\omega'} \|_{2\to 1} d\mu'(\omega') d\mu''(\omega'') \\
&\leqq \int_{\Omega''} \int_{\Omega'} (\| R_{\eta(\omega')\cap\tau''(\omega'')} T_{\omega'} \|_{2\to 1} + \| R_{\eta(\omega')^c \cap \tau''(\omega'')} T_{\omega'} \|_{2\to 1}) d\mu'(\omega') d\mu''(\omega'') \\
&\leqq \int_{\Omega''} \int_{\Omega''} (|\eta(\omega') \cap \tau''(\omega'')|^{1/2} \| T_{\omega'} \|_2 + |\tau(\omega', \omega'')|^{1/2} \| R_{\eta(\omega')^c} T_{\omega'} \|_2) d\mu'(\omega') d\mu''(\omega'') \\
&\leqq \int_{\Omega''} \int_{\Omega'} \left(\left[\sum_{j\in\eta(\omega')} \xi''_j(\omega'') \right]^{1/2} + \delta^{1/8} |\tau(\omega', \omega'')|^{1/2} \right) d\mu'(\omega') d\mu''(\omega'') \\
&\leqq \delta^{1/4} \int_{\Omega'} |\eta(\omega')|^{1/2} d\mu'(\omega') + \delta^{1/8} \left(\int_{\Omega''} \int_{\Omega'} \sum_{j=1}^n \xi'_j(\omega')\xi''_j(\omega'') d\mu'(\omega') d\mu''(\omega'') \right) \\
&\leqq K_G \delta^{1/8} \int_{\Omega'} \| T_{\omega'} \|_{2\to 1} d\mu'(\omega') + \delta^{5/8} n^{1/2}.
\end{aligned}$$

However, by the estimate for $I(\tau)$ obtained in the proof of Proposition 1.10 together with Proposition 1.8, we conclude the existence of a constant $A < \infty$ such that

$$\int_{\Omega'} \| T_{\omega'} \|_{2\to 1} d\mu'(\omega') \leqq 8 \max \left\{ \left\| \sum_{j=1}^{n} c_j \xi'_j \right\|_m ; c = \sum_{j=1}^{n} c_j e_j \in l_2^n, \sum_{j=1}^{n} | c_j |^2 \leqq 1 \right\}$$

$$\leqq 8 A m^{1/2}$$

$$\leqq 8 A (\delta n)^{1/2}.$$

This, of course, completes the proof. ■

The proof of Proposition 1.10 can now be modified by using Proposition 8.1 in order to evaluate there the expression $I(\tau)$. The outcome of this modification is that the expression $(\log(1/\delta))^{-1/2}$, appearing in the statement of Proposition 1.10, is replaced by $\delta^{1/8}$. Consequently, the function $\delta_2(\varepsilon)$, which was defined in the introduction of this section, satisfies the following inequality:

COROLLARY 8.3. *There exists a constant* $c > 0$ *such that*

$$\delta_2(\varepsilon) \geqq c \varepsilon^8,$$

for all $0 < \varepsilon < 1$.

REMARK. In a similar manner, one can show that, for each $1 \leqq p \leqq \infty$, there are constants d and $\rho > 0$ so that

$$\delta_p(\varepsilon) \geqq d \varepsilon^\rho; \qquad 0 < \varepsilon < 1.$$

Proposition 1 can be also used to improve an estimate obtained by B. S. Kashin [15] for the upper triangular projection A^+ of a $n \times n$ matrix A. Before stating our result, let us introduce some additional notation. If π is a permutation of the integers $\{1, 2, \ldots, n\}$, i.e., if π is an element of the symmetric group $\Lambda = \mathrm{Sym}(n)$, endowed with the normalized invariant measure λ, and $A = (a_{i,j})_{i,j=1}^n$ is a matrix acting as a linear operator on $\mathbf{R}^n$, then we denote by A_π the operator corresponding to the matrix $(a_{i,\pi(j)})_{i,j=1}^n$.

THEOREM 8.4. *For every* $1 \leqq q < 2$, *there exists a constant* $C_q < \infty$ *such that, whenever* A *is a linear operator on* l_2^n, *then*

$$\int_\Lambda \| (A_\pi)^+ \|_{2\to q} d\lambda(\pi) \leqq C_q n^{1/q - 1/2} \cdot \| A \|_2.$$

PROOF. Fix $1 \leqq q < 2$ and an integer n, and assume, for sake of simplicity, that $n = 2^l$. Next, by proceeding as in [15] and writing the upper-triangular projection of A as an element in the projective tensor algebra $l_\infty^n \hat{\otimes} l_\infty^n$, we get that

$$A^{+}=\sum_{j=1}^{l}\sum_{k=1}^{2^{h-1}} R_{\tau_{h,2k-1}} A R_{\tau_{h,2k}},$$

where

$$\tau_{h,k}=\{j;(k-1)n2^{-h}\leqq j<kn2^{-h}\},$$

for all $1\leqq k\leqq 2^{h}$, $1\leqq h\leqq l$. Since, for each $1\leqq h\leqq l$, we clearly have

$$\left\|\sum_{k=1}^{2^{h-1}} R_{\tau_{h,2k-1}} A R_{\tau_{h,2k}}\right\|_2\leqq\|A\|_2,$$

it follows, by interpolation with ϑ satisfying $1/q=\vartheta/1+(1-\vartheta)/2$ (i.e. $\vartheta=2/q-1$), that

$$\begin{aligned}\|(A_\pi)^{+}\|_{2\to q}&\leqq\sum_{h=1}^{l}\left\|\sum_{k=1}^{2^{h-1}} R_{\tau_{h,2k-1}} A_\pi R_{\tau_{h,2k}}\right\|_{2\to q}\\&\leqq\left(\sum_{h=1}^{l}\left\|\sum_{k=1}^{2^{h-1}} R_{\tau_{h,2k-1}} A_\pi R_{\tau_{h,2k}}\right\|_{2\to 1}^{\vartheta}\right)\cdot\|A\|_2^{1-\vartheta}\\&\leqq\left[\sum_{h=1}^{l}\left(\sum_{k=1}^{2^{h-1}}\|R_{\tau_{h,2k-1}} A_\pi R_{\tau_{h,2k}}\|_{2\to 1}^{2}\right)^{\vartheta/2}\right]\cdot\|A\|_2^{1-\vartheta},\end{aligned}$$

for any choice of $\pi\in\Lambda$. Hence, by averaging over $\pi\in\Lambda$, we get that

$$\begin{aligned}\int_\Lambda\|(A_\pi)^{+}\|_{2\to q}d\lambda(\pi)&\leqq\left[\sum_{h=1}^{l}\left(\sum_{k=1}^{2^{h-1}}\int_\Lambda\|R_{\tau_{h,2k-1}} A_\pi R_{\tau_{h,2k}}\|_{2\to 1}^{2}d\lambda(\pi)\right)^{\vartheta/2}\right]\cdot\|A\|_2^{1-\vartheta}\\&\leqq\left[\sum_{h=1}^{l}\left(\sum_{k=1}^{2^{h-1}}(2^{-h}n)^{1/2}\|A\|_2\int_\Lambda\|R_{\tau_{h,2k-1}} A_\pi R_{\tau_{h,2k}}\|_{2\to 1}^{2}d\lambda(\pi)\right)^{\vartheta/2}\right]\cdot\|A\|_2^{1-\vartheta}.\end{aligned}$$

On the other hand, by Proposition 8.1 applied to τ satisfying $|\tau|=2^{-h}n$ and $\delta=2^{-h}$; $1\leqq h\leqq l$, we have that

$$\begin{aligned}&\int_\Lambda\|R_\tau A_\pi R_\tau\|_{2\to 1}d\lambda(\pi)\\&=\int_\Lambda\left\|\sum_{i\in\tau}\sum_{j\in\pi^{-1}(\tau)}\alpha_{i,j}e_i\otimes e_j\right\|_{2\to 1}d\lambda(\pi)\\&=\text{Average}\left\{\left\|\sum_{i\in\tau}\sum_{j\in\sigma}a_{i,j}e_i\otimes e_j\right\|_{2\to 1};\sigma\subset\{1,2,\dots,n\},|\sigma|=|\tau|\right\}\\&\leqq C2^{-h/8}(2^{-h}n)^{1/2}\|A\|_2.\end{aligned}$$

The proof can now be completed by using the above fact with $\tau=\tau_{h,2k-1}$; $1\leqq h\leqq l, 1\leqq k\leqq 2^{h-1}$. ■

Added in proof. K. Ball (private communication) has recently found a nice and simple argument to prove that the assertion of Theorem 1.2 directly implies that of Theorem 1.6.

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