ISRAEL JOURNAL OF MATHEMATICS **152** (2006), 371–380

# DENSITY THEOREMS AND EXTREMAL HYPERGRAPH PROBLEMS

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* Research was partially supported by NSF Grant DMS 0300529.

** Research was partially supported by the Deutsche Forschungsgemeinschaft within the European graduate program 'Combinatorics, Geometry, and Computation' (No. GRK 588/2) and by grant SCHA 1263/1-1.

† Research was supported by a scholarship from CAPES, Brazil.

‡ Research was supported by MEXT Grant-in-Aid for Scientific Research (B) 16340027.

Received September 2, 2004

ABSTRACT

We present alternative proofs of density versions of some combinatorial partition theorems originally obtained by E. Szemerédi, H. Furstenberg and Y. Katznelson. These proofs are based on an extremal hypergraph result which was recently obtained independently by W. T. Gowers and B. Nagle, V. Rödl, M. Schacht, J. Skokan by extending Szemerédi's regularity lemma to hypergraphs.

## 1. Introduction

In 1977, Furstenberg [2] gave an alternative proof of Szemerédi's celebrated theorem [13] regarding the upper density of sets containing no arithmetic progression of fixed length (for other proofs of Szemerédi's theorem see also [7, 15]). Refining the techniques of this proof Furstenberg and Katznelson were later able to derive several other density versions of combinatorial partition theorems. The following, which can be viewed as a density version of Gallai–Witt's theorem, is one of them (see [3]). We denote by $[-N;N]$ the set $\{-N,-N+1,\dots,N\}$.

THEOREM 1: *Let $T$ be a finite subset of $\mathbb{R}^d$, and let $\delta>0$. Then there exists a finite subset $C$ of $\mathbb{R}^d$ such that any subset $Y\subset C$ with $|Y|>\delta|C|$ contains a homothetic copy of $T$, i.e., a set of the form $\mathbf{y}+\lambda T$ for some $\mathbf{y}\in\mathbb{R}^d$ and some $\lambda\in\mathbb{R}\setminus\{0\}$.*

*Furthermore, if $T\subset[-t;t]^d$ for some positive integer $t$, then $C=[-N;N]^d$ has the above property for every sufficiently large $N=N(t,d,\delta)$.*

Note that for $d=1$, Theorem 1 implies Szemerédi's density theorem [13]. For a fixed $d$, the special case of the above result allows us to find a homothetic copy of a full-dimensional cube $[-t;t]^d$ in a dense subset of a sufficiently large cube $[-N;N]^d$. Two other results in a similar vein, also due to Furstenberg and Katznelson [4], address the complementary case when the dimension is allowed to grow.

THEOREM 2: *Let $\mathbb{F}_q$ be the finite field with $q$ elements. Then for every positive integer $d$ and every $\delta>0$, there exists $M_0=M_0(q,d,\delta)$ such that, for $M\geq M_0$, any subset $Y\subset\mathbb{F}_q^M$ with $|Y|>\delta|\mathbb{F}_q^M|=\delta q^M$ contains a $d$-dimensional affine subspace.*

THEOREM 3: *Let $G$ be a finite abelian group, and let $\delta > 0$. Then there exists $M_0 = M_0(G, \delta)$ such that if $M \geq M_0$ and $Y$ is a subset of $G^M$ with $|Y| > \delta |G|^M$, then $Y$ contains a coset of a subgroup of $G^M$ isomorphic to $G$.*

The proofs by Furstenberg and Katznelson of the above theorems rely on ergodic theory and do not yield any bounds on $N$ and $M_0$. The purpose of this note is to present proofs of these theorems based on an extremal hypergraph result (see Theorem 4 below). It is not known, however, whether a similar approach yields an alternative proof of the density version of the theorem of Hales and Jewett [8], which was established by Furstenberg and Katznelson [5].

For a set $V$ and an integer $k \geq 1$, let $\binom{V}{k}$ be the family of all $k$-element subsets of $V$. A subset $\mathcal{H}^{(k)} \subseteq \binom{V}{k}$ is a **$k$-uniform hypergraph** on the vertex set $V$. As usual, we refer to the $k$-element sets in the hypergraph $\mathcal{H}^{(k)}$ as **edges**. We write $K_t^{(k)}$ for a clique of order $t$, namely the complete $k$-uniform hypergraph on $t$ vertices with $\binom{t}{k}$ edges.

THEOREM 4: *Let $t$ and $k$ be fixed integers with $t > k \geq 2$. Suppose that a $k$-uniform hypergraph $\mathcal{H}^{(k)}$ on $n$ vertices contains only $o(n^t)$ copies of $K_t^{(k)}$ as a subhypergraph. Then one can delete $o(n^k)$ edges of $\mathcal{H}^{(k)}$ to make it $K_t^{(k)}$-free.*

We will also use the following immediate corollary of Theorem 4.

COROLLARY 5: *Let $\mathcal{H}^{(k)}$ be a $k$-uniform hypergraph on $n$ vertices. Suppose that for each edge $H$ of $\mathcal{H}^{(k)}$ there exists precisely one clique $K_{k+1}^{(k)}$ in $\mathcal{H}^{(k)}$ which contains $H$. Then $|E(\mathcal{H}^{(k)})| = o(n^k)$ (where $E(\mathcal{H}^{(k)})$ denotes the edge set of the hypergraph $\mathcal{H}^{(k)}$).*

*Proof:* Since every edge of $\mathcal{H}^{(k)}$ sits in precisely one copy of $K_{k+1}^{(k)}$, the number of copies of $K_{k+1}^{(k)}$ in $\mathcal{H}^{(k)}$ is $|E(\mathcal{H}^{(k)})|/(k+1) \leq \binom{n}{k}/(k+1) = o(n^{k+1})$. By Theorem 4 (applied with $t = k+1$), we can delete only $o(n^k)$ edges of $\mathcal{H}^{(k)}$ in order to make it $K_{k+1}^{(k)}$-free. On the other hand, we need to remove at least one edge per clique, that is, at least $|E(\mathcal{H}^{(k)})|/(k+1)$ edges. Therefore, $|E(\mathcal{H}^{(k)})| = o(n^k)$. ∎

Theorem 4 is a consequence of the method independently developed by Gowers [6] and Nagle, Rödl, Schacht, and Skokan [9, 11]. This method is based on an extension of Szemerédi's regularity lemma [14] from graphs to $k$-uniform hypergraphs. The formal proof of Theorem 4 is given in [10] (see also [6] and [9] for the case $t = k+1$). The proof of Theorem 4 is purely combinatorial and combined with the arguments below give the first quantitative proofs of the density theorems, Theorems 1–3.

The proof of Theorem 1 in the case $d = 1$ (i.e., Szemerédi's theorem) based on Corollary 5 was given by Frankl and Rödl [1]. The essential part of the reduction of Theorem 1 to Corollary 5 was already discovered by Solymosi in [12]. We present this proof in Section 2 (see also [6]). Our proof of Theorem 2 and Theorem 3 extends an idea from [1].

## 2. Proof of Theorem 1

In this section we present a proof of Theorem 1. We first prove the special case when the finite configuration $T$ is a subset of the integer lattice (see Lemma 6 below). The proof of Lemma 6 is based on Corollary 5. This reduction was first considered by Solymosi in [12].

LEMMA 6: *For all positive integers $t$, $d$ and every $\delta > 0$, there exists $N_0 = N_0(t, d, \delta)$ such that for $N \geq N_0$ any subset $Y \subset [-N; N]^d$ with $|Y| > \delta(2N+1)^d$ contains a homothetic copy of $[-t; t]^d$.*

*Proof:* Suppose, on the contrary, that there exists $Y \subset [-N; N]^d$ with $|Y| > \delta(2N+1)^d$ which contains no homothetic copy of $[-t; t]^d$. Set $k = (2t+1)^d - 1$ and $W = Y \times [-N; N]^{k-d}$. Then $|W| > \delta(2N+1)^k$. We shall show that $W$ contains no homothetic copy of a simplex $S$ defined below, but this contradicts Corollary 5 as we will see.

Denote the elements of $[-t; t]^d$ by $\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_k$; without loss of generality, we may further assume that $\mathbf{e}_0$ is the origin, and that $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_d$ are the vectors of the standard basis:

$$\mathbf{e}_0 = (0,0,\ldots,0),\ \mathbf{e}_1 = (1,0,\ldots,0),\ \mathbf{e}_2 = (0,1,\ldots,0),\ \ldots,\ \mathbf{e}_d = (0,0,\ldots,1).$$

For $i = 0, \ldots, k-d$, set $(k-d)$-tuples

$$\mathbf{f}_0 = (0,0,\ldots,0),\ \mathbf{f}_1 = (1,0,\ldots,0),\ \ldots,\ \mathbf{f}_{k-d} = (0,0,\ldots,1).$$

Let us define a $k$-dimensional simplex $S \subseteq [-t; t]^k$ with points $\{\mathbf{s}_0, \ldots, \mathbf{s}_k\}$ by

$$\mathbf{s}_i = \begin{cases} (\mathbf{e}_i, \mathbf{f}_0) & \text{if } i = 0, \ldots, d, \\ (\mathbf{e}_i, \mathbf{f}_{i-d}) & \text{if } i = d+1, \ldots, k. \end{cases}$$

Since $Y$ contains no homothetic copy of $[-t; t]^d$, $W$ contains no homothetic copy of $S$. Let $\{F_0, \ldots, F_k\}$ be the facets, i.e., $(k-1)$-dimensional faces, of $S$, and for $i = 0, \ldots, k$ let $V_i$ be the set of all hyperplanes in $\mathbb{R}^k$ which are parallel to $F_i$ and intersect $[-N; N]^k$.

Let us show that $|V_i| = O(N)$ for each $i$. A normal vector $\mathbf{w} = (w_1, \ldots, w_k)$ of a facet (which is an affine span of $k$ vectors among $\mathbf{s}_0, \ldots, \mathbf{s}_k$) is a non-zero solution of the system

$$\mathbf{r}_i \cdot \mathbf{w} = 0, \quad i = 1, \ldots, k-1$$

where each $\mathbf{r}_i$ is a difference of 2 distinct $\mathbf{s}_j$'s and hence is an integer vector whose coordinates have absolute value less than $2t$.

Therefore, we may assume that the $w_j$ are given, up to sign, by determinants of integer matrices whose entries are coordinates of $\mathbf{r}_i$'s. Hence $|w_j|$ does not exceed $(k-1)!(2t)^{k-1}$. Consequently, the hyperplanes of the form

$$\mathbf{w} \cdot \xi = w_1\xi_1 + w_2\xi_2 + \cdots + w_k\xi_k = b,$$

where $b$ is an integer and $|b| \le k!(2t)^{k-1}N$, cover all the points $\xi = (\xi_1, \ldots, \xi_k) \in [-N; N]^k$. We conclude that at most $2k!(2t)^{k-1}N + 1 = O(N)$ hyperplanes parallel to the given facet are needed to cover all the points of $[-N; N]^k$.

Next, we are going to define a $(k+1)$-partite $k$-uniform hypergraph $\mathcal{H}^{(k)}$ with vertex partition $V_0 \cup \cdots \cup V_k$. Let $H$ be a set of $k$ vertices of $\mathcal{H}^{(k)}$ with the property $|H \cap V_i| \le 1$ for all $i$. Then those $k$ hyperplanes corresponding to $H$ determine a point $\mathbf{p} \in \mathbb{R}^k$. We put $H$ in $\mathcal{H}^{(k)}$ if and only if $\mathbf{p} \in W$.

Each $H \in E(\mathcal{H}^{(k)})$ determines a point $\mathbf{p} \in W$. On the other hand, for each $i = 0, \ldots, k$, each point $\mathbf{p} \in W$ determines a vertex $v \in V_i$, which corresponds to a hyperplane parallel to $F_i$ and passing through $\mathbf{p}$. In this way, $\mathbf{p}$ determines the $k+1$ vertices of a clique $K_{k+1}^{(k)}$ in $\mathcal{H}^{(k)}$.

Suppose that $k+1$ hyperplanes determined by a clique $K_{k+1}^{(k)}$ do not meet one point. Then these planes define a simplex homothetic to $S$ in $W$, which is a contradiction. Thus every clique $K_{k+1}^{(k)}$ must determine a point $\mathbf{p} \in W$. This means that for every $H \in E(\mathcal{H}^{(k)})$ there is precisely one clique $K_{k+1}^{(k)}$ which contains $H$. This implies that $|E(\mathcal{H}^{(k)})| = (k+1)|W|$. Finally, we have $|W| = o(N^k)$ by Corollary 5. This contradicts our earlier assumption $|W| > \delta(2N+1)^k$. ∎

We now deduce Theorem 1 from Lemma 6.

*Proof of Theorem 1:* Let $\delta > 0$ be given. Let $T$ be a finite subset of $\mathbb{R}^d$. Let $r = r(T)$ be the $\mathbb{Q}$-dimension of $T$, i.e., the largest number of linearly independent vectors of $T$ over $\mathbb{Q}$. Choose $r$ such vectors $\omega_1, \ldots, \omega_r \in \mathbb{R}^d$ so that $T \subset \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_r$. We define the map $\psi\colon \mathbb{Z}^r \to \mathbb{R}^d$

$$(a_1, \ldots, a_r) \mapsto a_1\omega_1 + \cdots + a_r\omega_r.$$

Since $\omega_1, \ldots, \omega_r$ are linearly independent over $\mathbb{Q}$, the map $\psi$ is injective. Now choose a positive integer $t$ large enough so that $\psi^{-1}(T) \subset [-t;t]^r$ and define $N = N_0(t, r, \delta)$ by Lemma 6. Let $C = \psi([-N;N]^r)$; if $Y \subset C$ and $|Y| = |\psi^{-1}(Y)| > \delta|C| = \delta(2N+1)^r$, then $\psi^{-1}(Y)$ contains a homothetic copy of $[-t;t]^r$, say $\mathbf{y}' + \lambda[-t;t]^r$ for some $\mathbf{y}' \in [-N;N]^r$ and some $\lambda > 0$. Thus $Y \supset \psi(\mathbf{y}' + \lambda[-t;t]^r) = \psi(\mathbf{y}') + \lambda\psi([-t;t]^r) \supset \psi(\mathbf{y}') + \lambda T$, as required. ∎

## 3. Proof of Theorem 2 and Theorem 3

As we shall show at the end of this section, the following lemma, Lemma 7, is more general than Theorem 2 and Theorem 3. Its proof elaborates on a construction first considered by Frankl and Rödl [1, Proposition 2.3].

LEMMA 7: *Let $A$ be a finite, commutative ring with $q$ elements. Then for every $\delta > 0$, there exists $M_0 = M_0(q, \delta)$ such that, for $M \geq M_0$, any subset $Y \subset A^M$ with $|Y| > \delta|A^M| = \delta q^M$ contains a coset of an isomorphic copy (as an $A$-module) of $A$.*

*In other words, there exist $\mathbf{r}$, $\mathbf{u} \in A^M$ such that $\mathbf{r} + \varphi(A) \subseteq Y$, where $\varphi: A \hookrightarrow A^M$, $\varphi(\alpha) = \alpha\mathbf{u}$ for $\alpha \in A$, is an injection.*

*Remark 8:* We later only use Lemma 7 for a commutative ring $A$. We remark that commutativity of the ring $A$ is not used in the proof below. In fact, the proof below works verbatim for an arbitrary finite non-commutative ring and left modules, as well.

*Proof of Lemma 7:* Let $q = |A|$ and let $\alpha_0 = 0$, $\alpha_1 = 1$, $\alpha_2, \ldots, \alpha_{q-1}$ be the elements of the ring $A$. Let $V = A^m$ and suppose $Y$ is a subset of $V$ which does not contain a coset of an isomorphic copy of $A$. We shall define a $q$-partite, $q$-uniform hypergraph $\mathcal{H}^{(q)}$ whose vertex partition classes $V_0, \ldots, V_{q-1}$ are disjoint copies of $V$. For $\mathbf{v}_2, \ldots, \mathbf{v}_{q-1} \in V$ and $\mathbf{y} \in Y$, let

$$H(\mathbf{v}_2, \ldots, \mathbf{v}_{q-1}, \mathbf{y}) = (\mathbf{h}_0, \ldots, \mathbf{h}_{q-1}) \in \prod_{i=0}^{q-1} V_i,$$

where

$$\mathbf{h}_i = \begin{cases} \mathbf{y} + \sum_{j=2}^{q-1} \alpha_j \mathbf{v}_j & \text{if } i = 0, \\ \mathbf{y} + \sum_{j=2}^{q-1} (\alpha_j - 1)\mathbf{v}_j & \text{if } i = 1, \\ \mathbf{v}_i & \text{if } i = 2, 3, \ldots, q-1. \end{cases}$$

Set

$$E(\mathcal{H}^{(q)}) = \{H(\mathbf{v}_2, \ldots, \mathbf{v}_{q-1}, \mathbf{y}) \colon \mathbf{v}_2, \ldots, \mathbf{v}_{q-1} \in V \text{ and } \mathbf{y} \in Y\}. \tag{1}$$

(Since $\mathcal{H}^{(q)}$ is $q$-partite and $q$-uniform we may view its edges as ordered $q$-tuples as defined above.) Clearly, $\mathcal{H}^{(q)}$ has $q|V| = q^{m+1}$ vertices and $q^{m(q-2)}|Y|$ edges. Consequently, Lemma 7 follows from Claim 9 below. ∎

CLAIM 9: *Let $\mathcal{H}^{(q)}$ be the hypergraph defined in (1); then*

$$|E(\mathcal{H}^{(q)})| = o(q^{m(q-1)}), \quad \text{as } m \to \infty.$$

*Proof:* First we verify that

$$|H_1 \cap H_2| \leq q-2 \quad \text{for any distinct edges } H_1, H_2 \in E(\mathcal{H}^{(q)}). \tag{2}$$

For that let $H_1 = H(\mathbf{v}_2, \ldots, \mathbf{v}_{q-1}, \mathbf{y}_1)$ and $H_2 = H(\mathbf{w}_2, \ldots, \mathbf{w}_{q-1}, \mathbf{y}_2)$ be a pair of distinct edges in $\mathcal{H}^{(q)}$. It is easy to see that if they intersect in $q-1$ points, we must have $\mathbf{v}_i = \mathbf{w}_i$, $2 \leq i \leq q-1$, and $\mathbf{y}_1 = \mathbf{y}_2$, which implies $H_1 = H_2$.

Let $\mathcal{F}(q)$ be the $q$-uniform hypergraph on $2q$ vertices $a_0, \ldots, a_{q-1}, b_0, \ldots, b_{q-1}$ and $q$ edges $F_i = \{a_0, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_{q-1}\}$ for $i = 0, \ldots, q-1$. We shall show that $\mathcal{H}^{(q)}$ contains only "few" copies of $\mathcal{F}(q)$ (see (14)).

Suppose that $\mathbf{a}_0, \ldots, \mathbf{a}_{q-1}, \mathbf{b}_0, \ldots, \mathbf{b}_{q-1}$ are the vertices of some $\mathcal{F}(q)$ in $\mathcal{H}^{(q)}$, with $\mathbf{a}_i, \mathbf{b}_i \in V_i$.

We first consider $F_0 = \{\mathbf{b}_0, \mathbf{a}_1, \ldots, \mathbf{a}_{q-1}\}$. There are $\mathbf{v}_2, \ldots, \mathbf{v}_{q-1} \in V$ and $\mathbf{y}' \in Y$ such that $F_0 = \{\mathbf{b}_0, \mathbf{a}_1, \ldots, \mathbf{a}_{q-1}\} = H(\mathbf{v}_2, \ldots, \mathbf{v}_{q-1}, \mathbf{y}')$ and consequently

$$\mathbf{b}_0 = \mathbf{y}' + \sum_{j=2}^{q-1} \alpha_j \mathbf{v}_j, \tag{3}$$

$$\mathbf{a}_1 = \mathbf{y}' + \sum_{j=2}^{q-1} (\alpha_j - 1) \mathbf{v}_j, \tag{4}$$

$$\mathbf{a}_i = \mathbf{v}_i \quad \text{for } i = 2, \ldots, q-1. \tag{5}$$

Next, we consider $F_1 = \{\mathbf{a}_0, \mathbf{b}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{q-1}\}$. Since $F_0 \cap F_1 = \{\mathbf{a}_2, \ldots, \mathbf{a}_{q-1}\}$ by (5) we have $F_1 = H(\mathbf{v}_2, \ldots, \mathbf{v}_{q-1}, \mathbf{y}'')$ for some $\mathbf{y}'' \in Y$ such that

$$\mathbf{a}_0 = \mathbf{y}'' + \sum_{j=2}^{q-1} \alpha_j \mathbf{v}_j, \tag{6}$$

$$\mathbf{b}_1 = \mathbf{y}'' + \sum_{j=2}^{q-1} (\alpha_j - 1) \mathbf{v}_j. \tag{7}$$

Similarly, for $2 \leq i \leq q-1$, we infer that $F_i = \{\mathbf{a}_0, \ldots, \mathbf{a}_{i-1}, \mathbf{b}_i, \mathbf{a}_{i+1}, \ldots, \mathbf{a}_{q-1}\} = H(\mathbf{v}_2, \ldots, \mathbf{v}_{i-1}, \mathbf{w}_i, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_{q-1}, \mathbf{y}_i)$ for some $\mathbf{w}_i \in V$ and $\mathbf{y}_i \in Y$ such that

$$\mathbf{a}_0 = \mathbf{y}_i + \alpha_i(\mathbf{w}_i - \mathbf{v}_i) + \sum_{j=2}^{q-1} \alpha_j \mathbf{v}_j, \tag{8}$$

$$\mathbf{a}_1 = \mathbf{y}_i + (\alpha_i - 1)(\mathbf{w}_i - \mathbf{v}_i) + \sum_{j=2}^{q-1} (\alpha_j - 1)\mathbf{v}_j, \tag{9}$$

$$\mathbf{b}_i = \mathbf{w}_i. \tag{10}$$

From (6) and (8) we infer that for $2 \leq i \leq q-1$ we have

$$\mathbf{y}'' = \mathbf{y}_i + \alpha_i(\mathbf{w}_i - \mathbf{v}_i) \iff \mathbf{y}'' - \mathbf{y}_i = \alpha_i(\mathbf{w}_i - \mathbf{v}_i). \tag{11}$$

Moreover, comparing (4) and (9) yields

$$\mathbf{y}' = \mathbf{y}_i + (\alpha_i - 1)(\mathbf{w}_i - \mathbf{v}_i) \iff \mathbf{y}' - \mathbf{y}_i = (\alpha_i - 1)(\mathbf{w}_i - \mathbf{v}_i) \tag{12}$$

for $2 \leq i \leq q-1$. Equations (11) and (12) for $2 \leq i \leq q-1$ give

$$\alpha_i(\mathbf{y}' - \mathbf{y}_i) = (\alpha_i - 1)(\mathbf{y}'' - \mathbf{y}_i) \iff \mathbf{y}_i = \alpha_i(\mathbf{y}' - \mathbf{y}'') + \mathbf{y}''. \tag{13}$$

Note that the last equation also holds for $i = 0, 1$ with $\mathbf{y}_0 = \mathbf{y}''$ and $\mathbf{y}_1 = \mathbf{y}'$, since $\alpha_0 = 0$ and $\alpha_1 = 1$. We also observe that due to (3), (6), and $\mathbf{a}_0 \neq \mathbf{b}_0$ we have $\mathbf{y}' \neq \mathbf{y}''$.

Now let

$$A_{\text{irreg}} = \{a \in A\colon ab = 0 \text{ for some } b \in A, b \neq 0\}$$

be the set of zero-divisors in $A$ and set $s = |A_{\text{irreg}}|$. Since $1 \notin A_{\text{irreg}}$, $s < q$. If $\mathbf{u} = \mathbf{y}' - \mathbf{y}'' \notin A^m_{\text{irreg}}$, then $\varphi\colon A \hookrightarrow A^M$ given by $\varphi(\alpha) = \alpha\mathbf{u}$ is an injective $A$-module homomorphism, and hence (13) implies $\mathbf{y}'' + \varphi(A) \subseteq Y$, which contradicts our assumption on $Y$. Hence if $\mathcal{H}^{(q)}$ contains some $\mathcal{F}(q) = \{F_0, \ldots, F_{q-1}\}$, there exist $\mathbf{v}_2, \ldots, \mathbf{v}_{q-1} \in V$ and $\mathbf{y}', \mathbf{y}'' \in Y$ with $\mathbf{y}' - \mathbf{y}'' \in A^m_{\text{irreg}}$ such that (3)–(10) hold. Conversely, given such quantities, at most one $\mathcal{F}(q)$ is determined: from (4) and (6), we find $\mathbf{a}_0$ and $\mathbf{a}_1$; subtracting (9) from (8), we obtain $\mathbf{a}_0 - \mathbf{a}_1 = \mathbf{w}_i - \mathbf{v}_i + \sum_{2 \leq j \leq q-1} \mathbf{v}_j$, whence the $\mathbf{w}_i$'s are determined. Finally, (8) determines the $\mathbf{y}_i$'s. Hence the number $\#\{\mathcal{F}(q) \subset \mathcal{H}^{(q)}\}$ of copies of $\mathcal{F}(q)$ in $\mathcal{H}^{(q)}$ is bounded by the number of tuples $(\mathbf{v}_2, \ldots, \mathbf{v}_{q-1}, \mathbf{y}', \mathbf{y}'')$ satisfying the above conditions, and therefore

$$\#\{\mathcal{F}(q) \subset \mathcal{H}^{(q)}\} \leq q^{m(q-2)} \times |Y| \times s^m = o(q^{mq}), \tag{14}$$

where the last assertion used $|Y| \leq |V| = |A^m| = q^m$ and $s < q$.

Let $\mathcal{H}^{(q-1)}$ be the $(q-1)$-th shadow of $\mathcal{H}^{(q)}$

$$\mathcal{H}^{(q-1)} = \{H' \colon |H'| = q-1 \text{ and } H' \subset H \text{ for some } H \in E(\mathcal{H}^{(q)})\}.$$

Due to (2), for any set $Q$ of $q$ vertices spanning a clique $K_q^{(q-1)}$ in $\mathcal{H}^{(q-1)}$ the following holds: either $Q$ is an edge in $\mathcal{H}^{(q)}$ or $Q \subset V(\mathcal{F}(q))$ for some copy of $\mathcal{F}(q)$ in $\mathcal{H}^{(q)}$. Therefore, due to the definition of $\mathcal{H}^{(q)}$ in (1) and (14), the number of cliques $K_q^{(q-1)}$ in $\mathcal{H}^{(q-1)}$ is bounded by $|E(\mathcal{H}^{(q)})| + o(q^{mq}) = |Y|q^{m(q-2)} + o(q^{mq})$. Since $|Y| \leq q^m$ and $|V(\mathcal{H}^{(q-1)})| = |V(\mathcal{H}^{(q)})| = q^{m+1}$, we infer that the number of copies of $K_q^{(q-1)}$ in $\mathcal{H}^{(q-1)}$ is $o(q^{mq})$ which is $o(q^{(m+1)q}) = o(|V(\mathcal{H}^{(q-1)})|^q)$.

Hence, Theorem 4 (applied to $\mathcal{H}^{(q-1)}$ with $n = |V(\mathcal{H}^{(q-1)})| = q^{m+1}$, $t = q$, and $k = q-1$) yields that it suffices to delete at most $o(|V(\mathcal{H}^{(q-1)})|^{q-1}) = o(q^{(m+1)(q-1)}) = o(q^{m(q-1)})$ edges from $\mathcal{H}^{(q-1)}$ to make it clique free. But due to (2) each deleted edge destroys at most one copy of $K_q^{(q-1)}$ in $\mathcal{H}^{(q-1)}$ originating from an edge of $\mathcal{H}^{(q)}$ and, therefore, $|E(\mathcal{H}^{(q)})| = o(q^{m(q-1)})$ as claimed. ∎

In the rest of this paper we derive Theorem 2 and Theorem 3 from Lemma 7.

*Proof of Theorem 2:* Consider the ring $A = \mathbb{F}_q \oplus \cdots \oplus \mathbb{F}_q = \bigoplus_{i=1}^d \mathbb{F}_q$. Then $A^m \cong \mathbb{F}_q^{md}$ as an $\mathbb{F}_q$-vector space, and a submodule of $A^m$ isomorphic to $A$ is a $d$-dimensional subspace of $\mathbb{F}_q^{md}$. Therefore, Lemma 7 implies Theorem 2 for every sufficiently large $M \equiv 0 \pmod{d}$.

In general, if $M = md+r$, $0 \leq r < d$, $\mathbb{F}_q^M$ is the disjoint union of $|\mathbb{F}_q^M|/|\mathbb{F}_q^{md}| = q^r$ copies of $\mathbb{F}_q^{md}$; therefore one of these translates, say $V$, intersects $Y$ in more than $\delta q^M/q^r = \delta q^{md}$ elements and hence $Y \cap V$ (thus $Y$) will contain a $d$-dimensional subspace. ∎

Finally, we close this paper with the proof of Theorem 3.

*Proof of Theorem 3:* Since $G$ is abelian, we may write

$$G \cong \mathbb{Z}/p_1^{e_1} \times \cdots \times \mathbb{Z}/p_r^{e_r}$$

where $p_i$ are (not necessarily distinct) primes and $e_i$ are positive integers. Using this isomorphism, we want to view $G$ as the additive group of the ring $A = \mathbb{Z}/p_1^{e_1} \times \cdots \times \mathbb{Z}/p_r^{e_r}$.

Then $A^m$ and $G^m$ are isomorphic abelian groups. Similarly, a submodule $A$ of $A^m$ is isomorphic to $G$, also as an abelian group. The theorem then follows from Lemma 7. ∎

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