

## COMPLEXES OF GRAPH HOMOMORPHISMS

BY

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## ABSTRACT

$\text{Hom}(G, H)$  is a polyhedral complex defined for any two undirected graphs  $G$  and  $H$ . This construction was introduced by Lovász to give lower bounds for chromatic numbers of graphs. In this paper we initiate the study of the topological properties of this class of complexes.

We prove that  $\text{Hom}(K_m, K_n)$  is homotopy equivalent to a wedge of  $(n - m)$ -dimensional spheres, and provide an enumeration formula for the number of the spheres. As a corollary we prove that if for some graph  $G$ , and integers  $m \geq 2$  and  $k \geq -1$ , we have  $\varpi_1^k(\text{Hom}(K_m, G)) \neq 0$ , then  $\chi(G) \geq k + m$ ; here  $\mathbb{Z}_2$ -action is induced by the swapping of two vertices in  $K_m$ , and  $\varpi_1$  is the first Stiefel–Whitney class corresponding to this action.

Furthermore, we prove that a fold in the first argument of  $\text{Hom}(G, H)$  induces a homotopy equivalence. It then follows that  $\text{Hom}(F, K_n)$  is homotopy equivalent to a direct product of  $(n - 2)$ -dimensional spheres, while  $\text{Hom}(\overline{F}, K_n)$  is homotopy equivalent to a wedge of spheres, where  $F$  is an arbitrary forest and  $\overline{F}$  is its complement.

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## 1. Introduction

1.1. DEFINITION OF THE MAIN OBJECT. For any graph  $G$ , we denote the set of its vertices by  $V(G)$ , and the set of its edges by  $E(G)$ ,  $E(G) \subseteq V(G) \times V(G)$ . All the graphs in this paper are undirected, so  $(x, y) \in E(G)$  implies  $(y, x) \in E(G)$ . Unless otherwise specified, our graphs are finite and may contain loops.

*Definition 1.1:* For two graphs  $G$  and  $H$ , a **graph homomorphism** from  $G$  to  $H$  is a map  $\phi: V(G) \rightarrow V(H)$ , such that if  $x, y \in V(G)$  are connected by an edge, then  $\phi(x)$  and  $\phi(y)$  are also connected by an edge.

We denote the set of all homomorphisms from  $G$  to  $H$  by  $\text{Hom}_0(G, H)$ .

*Definition 1.2:*  $\text{Hom}(G, H)$  is a polyhedral complex whose cells are indexed by all functions  $\eta: V(G) \rightarrow 2^{V(H)} \setminus \{\emptyset\}$ , such that if  $(x, y) \in E(G)$ , then  $\eta(x) \times \eta(y) \subseteq E(H)$ .

The closure of a cell  $\eta$  consists of all cells indexed by  $\tilde{\eta}: V(G) \rightarrow 2^{V(H)} \setminus \{\emptyset\}$ , which satisfy  $\tilde{\eta}(v) \subseteq \eta(v)$ , for all  $v \in V(G)$ .

The set of vertices of  $\text{Hom}(G, H)$  is precisely  $\text{Hom}_0(G, H)$ . Since all cells of  $\text{Hom}(G, H)$  are products of simplices, the geometric realization of  $\text{Hom}(G, H)$  is defined in a straightforward fashion.

On the intuitive level, one can think of each  $\eta: V(G) \rightarrow 2^{V(H)} \setminus \{\emptyset\}$ , satisfying the conditions of Definition 1.2, as associating non-empty lists of vertices of  $H$  to vertices of  $G$  with the condition on this collection of lists being that any choice of one vertex from each list will yield a graph homomorphism from  $G$  to  $H$ .

A direct geometric construction of  $\text{Hom}(G, H)$  is as follows. Consider the partially ordered set  $P_{G, H}$  of all  $\eta$  as in Definition 1.2, with the partial order defined by  $\tilde{\eta} \leq \eta$  if and only if  $\tilde{\eta}(v) \subseteq \eta(v)$ , for all  $v \in V(G)$ . Then the order complex  $\Delta(P_{G, H})$  is a barycentric subdivision of  $\text{Hom}(G, H)$ . A cell  $\tau$  of  $\text{Hom}(G, H)$  corresponds to the union of all the simplices of  $\Delta(P_{G, H})$  labeled by the chains with the maximal element  $\tau$ .

In this paper we study properties of the complexes  $\text{Hom}(G, H)$ . More specifically, we compute the homotopy type of  $\text{Hom}(G, H)$  for several families of  $G$  and  $H$  and also derive some information about natural finite group actions on these complexes.

1.2. HISTORIC MOTIVATION. A particularly frequently studied special case of a graph homomorphism is that of a vertex coloring: for a graph  $G$  a vertex coloring of  $G$  with  $n$  colors is simply a graph homomorphism from  $G$  to  $K_n$ .

Here  $K_n$  denotes an unlooped complete graph on  $n$  vertices, that is  $V(K_n) = [n]$ ,  $E(K_n) = \{(x, y) | x, y \in [n], x \neq y\}$ .

Historically, one was especially interested in the question of existence of vertex colorings with a specified number of colors. From this point of view, the minimal possible number of colors in a vertex coloring is of special importance. It is called the *chromatic number* of the graph, and is denoted by  $\chi(G)$ .

The Kneser conjecture was posed in 1955, see [16], and concerned chromatic numbers of a specific family of graphs, later called *Kneser graphs*. For  $n, k \in \mathbb{Z}$ ,  $n \geq 2$ ,  $1 \leq k \leq n/2$ , the Kneser graph  $\Gamma_{k,n}$  is the graph whose vertices are all  $k$ -subsets of  $[n]$ , and edges are all pairs of disjoint  $k$ -subsets; here  $1 \leq k \leq n/2$ .

In 1978 L. Lovász solved the Kneser conjecture by finding geometric obstructions of Borsuk–Ulam type to the existence of graph colorings.

**THEOREM 1.3** (Kneser–Lovász, [16, 22]):  $\chi(\Gamma_{k,n}) = n - 2k + 2$ .

To show the inequality  $\chi(\Gamma_{k,n}) \geq n - 2k + 2$  Lovász associated a simplicial complex  $\mathcal{N}(G)$ , called the *neighborhood complex*, to an arbitrary graph  $G$ , and then used the connectivity information of the topological space  $\mathcal{N}(G)$  to find obstructions to the colorability of  $G$ .

**THEOREM 1.4** (Lovász, [22]): *Let  $G$  be a graph, such that  $\mathcal{N}(G)$  is  $k$ -connected for some  $k \in \mathbb{Z}$ ,  $k \geq -1$ ; then  $\chi(G) \geq k + 3$ .*

The main topological tool which Lovász employed was the Borsuk–Ulam theorem. See also [1] for the extension to hypergraphs, which used the generalization of the Borsuk–Ulam theorem from [6].

We shall define the complex  $\mathcal{N}(G)$  in Section 4, where we shall also see that for any graph  $G$  the complex  $\mathcal{N}(G)$  is homotopy equivalent to  $\text{Hom}(K_2, G)$ . This fact leads one to consider the family of Hom complexes as a natural context in which to look for further obstructions to the existence of graph homomorphisms.

Accordingly, Lovász has made the following conjecture, [23]. Let  $C_m$  be a cycle with  $m$  vertices:  $V(C_m) = \mathbb{Z}_m$ ,  $E(C_m) = \{(x, x+1), (x+1, x) | x \in \mathbb{Z}_m\}$ .

**CONJECTURE 1.5** (Lovász): *Let  $G$  be a graph, such that  $\text{Hom}(C_{2r+1}, G)$  is  $k$ -connected for some  $r, k \in \mathbb{Z}$ ,  $r \geq 1$ ,  $k \geq -1$ ; then  $\chi(G) \geq k + 4$ .*

Our proof of Conjecture 1.5 was announced in [4]. The full version of the proof consists of 2 parts: the study of the important general properties of Hom complexes, appearing in this paper, and more detailed and specific spectral sequence calculations, appearing in [5]. However, the general study undertaken in this paper contains more than the results which we later use for our spectral sequence computations.

There was a more general conjecture, also due to Lovász.

CONJECTURE 1.6 (Lovász): *Let  $T, G$  be two graphs; then*

$$(1.1) \quad \chi(G) \geq \chi(T) + \text{conn Hom}(T, G) + 1.$$

Here,  $\text{conn } X$  is the connectivity of the topological space  $X$ , i.e., the maximum  $k$  such that  $X$  is  $k$ -connected. Note that Conjecture 1.5 is the special case of the Conjecture 1.6 corresponding to  $T = C_{2r+1}$ .

Hoory and Linial, [15], gave a counterexample to the Conjecture 1.6. In their counterexample  $G = K_5$ , and  $T$  is a graph with 9 vertices and 22 edges. Furthermore,  $\chi(T) = 5$ , but  $\text{Hom}(T, G)$  is connected, showing that equation (1.1) is false in general.

In this paper we show that Conjecture 1.6 is true for  $T = K_m$  (Lovász himself proved (1.1) for  $T = K_2$ ). More specifically, let  $\mathbb{Z}_2$  act on  $K_m$  for  $m \geq 2$ , by swapping the vertices 1 and 2 and fixing the vertices  $3, \dots, m$ . Since the graph homomorphism flips an edge, it induces a free  $\mathbb{Z}_2$ -action on  $\text{Hom}(K_m, G)$ , for an arbitrary graph  $G$  without loops.

For a CW complex  $X$  on which  $\mathbb{Z}_2$  acts freely let  $\varpi_1(X)$  denote its *first Stiefel–Whitney class*; see [28]. As a corollary of our computations, we prove the following theorem.

THEOREM 1.7: *Let  $G$  be a graph, and let  $m, k \in \mathbb{Z}$ , be such that  $m \geq 2$ ,  $k \geq -1$ . If  $\varpi_1^k(\text{Hom}(K_m, G)) \neq 0$ , then  $\chi(G) \geq k + m$ .*

Note that if a  $\mathbb{Z}_2$ -space  $X$  is  $k$ -connected, then there exists a  $\mathbb{Z}_2$ -map  $S_a^{k+1} \rightarrow X$ . The functoriality of Stiefel–Whitney classes and the fact that  $\varpi_1^{k+1}(S_a^{k+1}) \neq 0$  imply that  $\varpi_1^{k+1}(X) \neq 0$ . Therefore Theorem 1.7 implies Conjecture 1.6 for  $T = K_m$ .

1.3. PLAN OF THE PAPER. In Section 2 we define notations, describe the category of graphs and graph homomorphisms, and give several examples of  $\text{Hom}$  complexes. Furthermore, we list many simple, but fundamental properties of the  $\text{Hom}$  construction.

In Section 3 we describe two results from topological combinatorics which we need for our arguments: a proposition from Discrete Morse theory, and a Quillen-type result.

In Section 4 we see first that in general  $\text{Hom}(K_2, G)$  is homotopy equivalent to the neighborhood complex  $\mathcal{N}(G)$ , implying in particular that  $\text{Hom}(K_2, K_n) \simeq S^{n-2}$ . We observe that in fact  $\text{Hom}(K_2, K_n)$  is a boundary complex of a polytope, on which the natural  $\mathbb{Z}_2$ -action on the first argument induces an antipodal

action. In subsection 4.3 we prove the central result of this section, namely we show that  $\text{Hom}(K_m, K_n)$  is homotopy equivalent to a wedge of  $(n - m)$ -dimensional spheres, and provide an enumeration formula for the number of the spheres. As a corollary we derive Theorem 1.7.

In Section 5 we prove that a fold in the first argument of  $\text{Hom}(G, H)$  induces a homotopy equivalence. As a corollary, we show for an arbitrary forest  $F$  that  $\text{Hom}(F, K_n)$  is homotopy equivalent to a direct product of  $(n - 2)$ -dimensional spheres, while  $\text{Hom}(\overline{F}, K_n)$  is homotopy equivalent to a wedge of spheres. For an arbitrary tree  $T$  with  $\mathbb{Z}_2$ -action we describe the  $\mathbb{Z}_2$ -homotopy type of  $\text{Hom}(T, K_n)$ .

To conclude the introduction, we refer the reader to a recent survey of the previous studies of other complexes related to graph colorings; see [25].

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**2. Basic facts about Hom complexes**

2.1. TERMINOLOGY. ◦ For a graph  $G$  we distinguish between looped and unlooped complements, namely we let  $\overline{G}^l$  be the graph defined by

$$V(\overline{G}^l) = V(G), E(\overline{G}^l) = (V(G) \times V(G)) \setminus E(G),$$

while  $\overline{G}$  is the graph defined by

$$V(\overline{G}) = V(G), E(\overline{G}) = \{(x, y) \in V(G) \times V(G) \mid x \neq y, (x, y) \notin E(G)\}.$$

◦ For a graph  $G$  and  $S \subseteq V(G)$  we denote by  $G[S]$  the graph on the vertex set  $S$  induced by  $G$ , that is  $V(G[S]) = S, E(G[S]) = (S \times S) \cap E(G)$ . For  $S \subseteq V(G)$  we set  $G - S$  to be the graph  $G[V(G) \setminus S]$ . For  $v \in V(G)$  we shall sometimes simply write  $G - v$  instead of  $G - \{v\}$ .

◦ For a graph  $G$  and  $A \subseteq V(G)$ , let

$$N(A) = \{w \in V(G) \mid (v, w) \in E(G), \forall v \in A\}$$

denote the set of all common neighbors of the vertices of  $A$ . In particular,  $N(\emptyset) = V(G)$ , and  $N(v) := N(\{v\})$  is simply the set of all neighbors of  $v$ , with the convention being that  $v$  is its own neighbor if and only if  $(v, v) \in E(G)$ . If needed, we will also specify the graph by writing  $N_G(A)$ .

◦ For two arbitrary graphs  $G$  and  $H$  we let  $G \times H$  denote the *direct product* of  $G$  and  $H$ :

$$\begin{aligned} V(G \times H) &= V(G) \times V(H), E(G \times H) \\ &= \{((x, y), (\tilde{x}, \tilde{y})) \mid x, \tilde{x} \in V(G), y, \tilde{y} \in V(H), (x, \tilde{x}) \in E(G), (y, \tilde{y}) \in E(H)\}. \end{aligned}$$

◦ For two arbitrary graphs  $G$  and  $H$  we let  $G \coprod H$  denote the *disjoint union* of  $G$  and  $H$ .

◦ For  $n \in \mathbb{Z}, n \geq 1$ , we let  $L_n$  denote the graph defined by  $V(L_n) = [n], E(L_n) = \{(x, y) \mid |x - y| = 1\}$ .

◦ Let  $\Omega$  be the graph defined by  $V(\Omega) = [2], E(\Omega) = \{(1, 2), (2, 1), (1, 1)\}$ .

◦ For an arbitrary graph  $G$ , we let  $G^\circ$  denote the *loop completion* of  $G$ , that is  $V(G^\circ) = V(G), E(G^\circ) = E(G) \cup \{(v, v) \mid v \in V(G)\}$ .

◦ For a polyhedral complex  $K$  we let  $\mathcal{P}(K)$  denote its *face poset*, that is a partially ordered set of the faces ordered by inclusion.

◦ For any finite category  $C$  (in particular a finite poset) we denote by  $\Delta(C)$  the realization of the nerve of that category.

◦ For a poset  $P$  we let  $\text{Bd}(P)$  denote the barycentric subdivision of  $P$ , that is the poset of all the chains in the given poset ordered by inclusion. For a polyhedral complex  $K$  we let  $\text{Bd}(K)$  denote the barycentric subdivision of  $K$ . Clearly,  $\text{Bd}(K) = \Delta(\mathcal{P}(K))$ , and  $\mathcal{P}(\Delta(P)) = \text{Bd}(P)$ .

◦ For any finite poset  $P$ , we let  $P^{op}$  denote the finite poset which has the same set of elements as  $P$ , but the opposite partial order. Also, for any finite poset  $P$ , whenever a subset of the elements of  $P$  is considered as a poset, the partial order is taken to be induced from  $P$ .

◦ **Top** is a category having topological spaces as objects, and continuous maps as morphisms.

**2.2. THE CATEGORY GRAPHS.** It is an easy check that a composition of two graph homomorphisms is again a graph homomorphism. We denote a composition of  $\phi \in \text{Hom}_0(G, H)$  and  $\psi \in \text{Hom}_0(H, K)$  by  $\psi \circ \phi \in \text{Hom}_0(G, K)$ .

Since the composition is associative and since for any graph  $G$  we have a unique identity homomorphism in  $\text{Hom}_0(G, G)$ , we can define a category **Graphs** as the one having graphs as objects, and graph homomorphisms as morphisms.

One can check that the direct product of graphs is a categorical *product* in **Graphs**, while the disjoint union of graphs is a categorical *coproduct* in **Graphs**.

Note that with the above notations  $K_1^o$  is a graph consisting of one vertex and one loop; it is the *terminal object* of **Graphs**. The empty graph is the *initial object* of **Graphs**.

2.3. EXAMPLES OF **Hom** COMPLEXES. To start with, we have various trivial cases:

- $\text{Hom}(K_1, H)$  is a simplex with  $|V(H)|$  vertices;

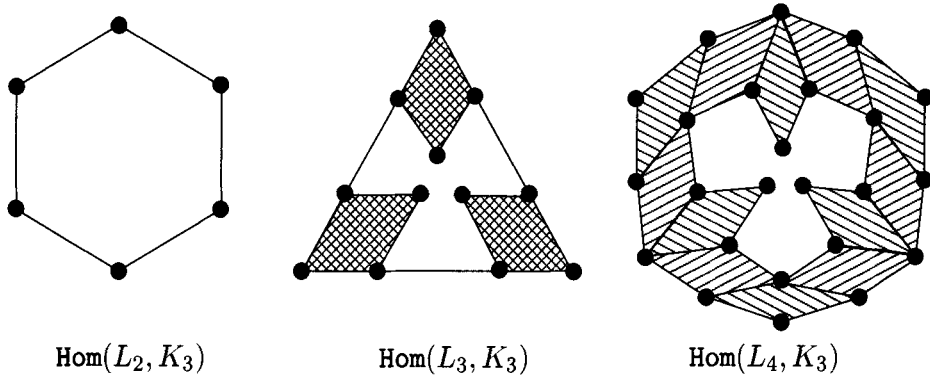


Figure 2.1. Complexes of 3-colorings of strings.

- $\text{Hom}(H, K_1) = \emptyset$ , unless  $E(H) = \emptyset$ , in which case  $\text{Hom}(H, K_1)$  is a point;
- more generally,  $\text{Hom}(G, H) = \emptyset$  if  $\chi(G) > \chi(H)$ ;
- $\text{Hom}(K_1^o, H)$  is a simplex with vertices indexed by the looped vertices of  $H$ ;
- $\text{Hom}(H, K_1^o)$  is a point, as mentioned above;
- $\text{Hom}(G, K_n^o)$  is a direct product of  $|V(G)|$  simplices, each simplex having  $n$  vertices;
- $\text{Hom}(G, K_2) = \emptyset$  if  $G$  is not bipartite; it consists of  $2^c$  points, if  $G$  bipartite and has  $c$  connected components;
- $\text{Hom}(C_{2r+1}, C_{2p+1}) = \emptyset$  if and only if  $r < p$ ;
- $\text{Hom}(C_{2r+1}, C_{2r+1})$  is a disjoint union of  $4r + 2$  points, for  $r \geq 1$ ;
- $\text{Hom}(C_{2r+1}, C_{2r-1})$  is a disjoint union of two cycles, each of length  $4r^2 - 1$ , for  $r \geq 2$ ;
- $\text{Hom}(C_6, K_3)$  consists of 6 isolated points, 6 solid cubes and 18 squares connected as shown on Figure 2.3. The left part of Figure 2.3 is incomplete for the purpose of visualizing; it shows the 6 points, 6 cubes and some of the squares. The right part shows the link of each of the 6 vertices, where two of the cubes touch. The closed star of such a vertex consists of 2 solid cubes and 3 squares.

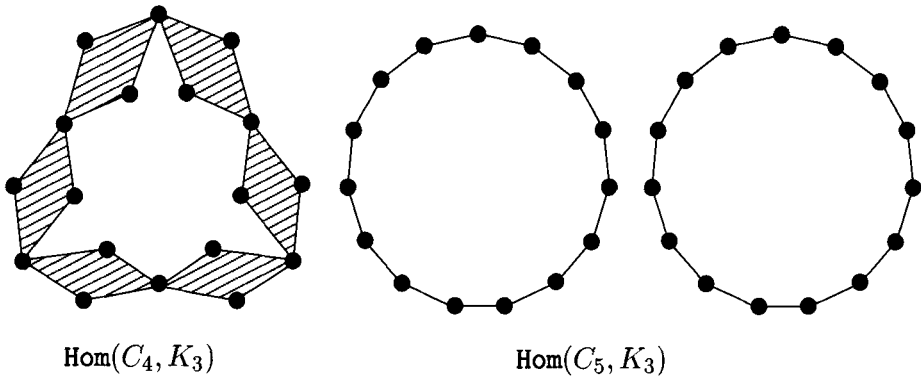


Figure 2.2. Complexes of 3-colorings of 4- and 5-cycles.

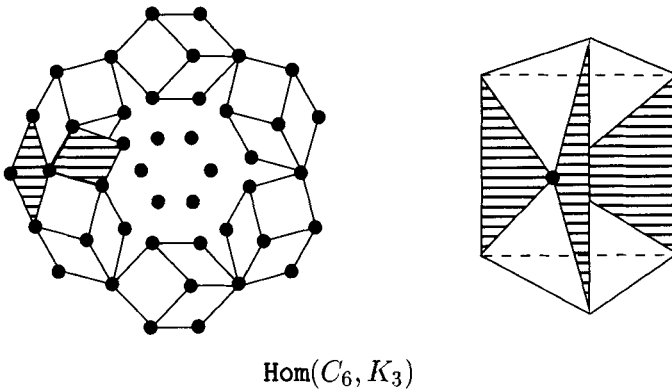


Figure 2.3. Complexes of 3-colorings of a 6-cycle.

- $\text{Hom}(K_n, K_n)$  is a disjoint union of  $n!$  points;
- $\text{Hom}(K_{n-1}, K_n)$  is the Cayley graph of  $\mathcal{S}_n$  with the set of generators consisting of  $n - 1$  transpositions  $\{(a, n) | a = 1, \dots, n - 1\}$ . Indeed, every vertex of  $\text{Hom}(K_{n-1}, K_n)$  is an injection  $\iota: [n - 1] \rightarrow [n]$ , which can be identified with a permutation of  $[n]$  by writing out the values of  $\iota$  and then writing the missing element of  $[n]$  in the last position. An edge is a changing of one arbitrary value of  $\iota$ , say  $\iota(a)$ , to the missing value, which is precisely the same as acting with the transposition  $(a, n)$  on the corresponding permutation.
- $\text{Hom}(K_2, K_4)$  is the full 2-skeleton of the 3-cell depicted on Figure 2.4.



◦  $\text{Hom}(C_7, K_3)$  is homeomorphic to a disjoint union of two Möbius bands. The local structure of each Möbius band is shown on Figure 2.4. The middle cycle which is painted bold has length 21 in each band, and all visible squares on the picture are filled with 2-cells.

◦ It is not difficult to count the number of connected components of  $\text{Hom}(C_t, K_3)$ . Denote this number  $c_t$ ; the general formula is

$$c_t = \begin{cases} \lfloor (t+1)/3 \rfloor, & \text{if } 3 \nmid t, \\ t/3 + 5, & \text{if } 3 \mid t, \end{cases}$$

for  $t \geq 3$ . The crucial fact for deriving this formula for  $c_t$  is to notice that the connected components of  $\text{Hom}(C_t, K_3)$  are indexed with the number of times  $C_t$  can wind around the triangle  $K_3$ , with the sign encoding the direction.

*Remark 2.1:* As the examples above show, for the considered values of  $m$  and  $n$ , the spaces  $\text{Hom}(C_m, C_n)$  are either empty or consist of several connected components, with each component either being a point, or homotopy equivalent to a circle. Recently, this fact has been proved for all values of  $m$  and  $n$ ; see [12].

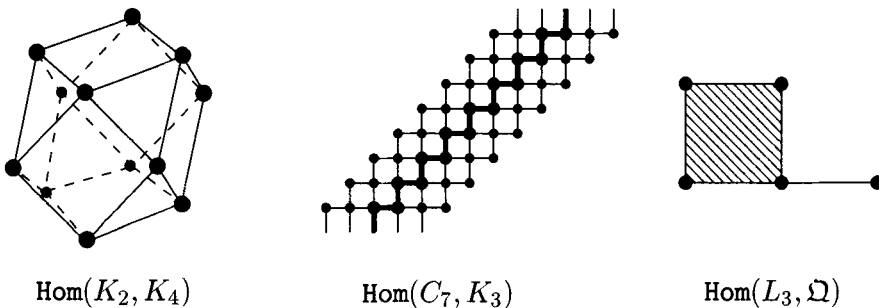


Figure 2.4. Further examples of Hom-complexes.

◦ Let  $V_{n,k}$  denote the Stiefel manifold of all orthonormal  $k$ -frames in  $\mathbb{R}^n$ . Csorba, [10], has made the following conjecture:

CONJECTURE 2.2 (Csorba):  $\text{Hom}(C_5, K_n)$  is homeomorphic to  $V_{n-1,2}$ , for all  $n \geq 1$ .

The cases  $n = 1, 2$  are tautological, as both spaces are empty. The example above verifies the case  $n = 3$ :  $\text{Hom}(C_5, K_3) \cong S^1 \amalg S^1$ . Several cases, including  $n = 4$ , have been recently verified by Csorba and Lutz, [11].

◦ Note that for an arbitrary  $G$ ,  $\text{Hom}(G, \Omega)$  can be interpreted as a cubical cone over the independence complex of  $G$ ; recall that the independence complex of  $G$  is the simplicial complex consisting of all independent sets of  $G$ . When saying *cubical cone* we mean the following construction: given an arbitrary simplicial complex  $\Delta$ , add an extra vertex  $a$ , and for each simplex  $\sigma \in \Delta$  with  $d$  vertices span a  $d$ -dimensional cube  $K_\sigma$  with  $a$  being a vertex of  $K_\sigma$  and  $\sigma$  forming the link of  $a$  in  $K_\sigma$ .

Note that  $\text{Hom}(G, K_3)$  is cubical for any graph  $G$  having no isolated vertices. By a theorem of Gromov, see [9],  $\text{Hom}(G, K_3)$  allows metric with nonpositive curvature if and only if the link of every vertex is a flag complex (which means that each link is the clique complex of its 1-skeleton).

For any  $\varphi \in \text{Hom}_0(G, H)$ , we say that  $\varphi$  has a cubical neighborhood if  $\varphi$  does not belong to any simplex with more than 2 vertices.

**PROPOSITION 2.3:** *If  $\varphi \in \text{Hom}_0(G, H)$  has a cubical neighborhood, then  $\text{lk}_{\text{Hom}(G, H)} \varphi$  is a flag complex.*

*Proof:* Set  $L = \text{lk}_{\text{Hom}(G, H)} \varphi$ , i.e., the simplicial complex whose face poset is  $\text{Hom}(G, H)_{\geq \varphi}$ . For  $v \in V(G)$ , set

$$A_\varphi(v) = \mathbb{N} \left( \bigcup_{w \in \mathbb{N}(v)} \varphi(w) \right).$$

Since  $\varphi$  has a cubical neighborhood, we have  $|A_\varphi(v)| \in \{1, 2\}$ , for any  $v \in V(G)$ . Let  $M(\varphi) \subseteq V(G)$  be the set of all vertices  $v$  with  $|A_\varphi(v)| = 2$ .

Clearly  $L$  has  $M(\varphi)$  as the set of vertices. Furthermore,  $\sigma \subseteq M(\varphi)$ , such that  $|\sigma| \geq 2$ , is a simplex in  $L$ , if and only if, for any two  $a, b \in \sigma$ , and any  $x \in A_\varphi(a)$ ,  $y \in A_\varphi(b)$ , we have  $(x, y) \in E(H)$ . Since this is a local condition depending only on the pair  $(a, b)$ , we conclude that  $L$  is a flag complex. ■

It follows that the cubical complex  $\text{Hom}(G, K_3)$  always allows a metric with nonpositive curvature. Moreover, for any  $\varphi \in \text{Hom}_0(G, K_3)$ , the proof of Proposition 2.3 yields that  $\text{lk}_{\text{Hom}(G, K_3)} \varphi$  is the independence complex of  $G[M(\varphi)]$ .

2.4. GENERAL PROPERTIES OF  $\text{Hom}$  COMPLEXES.

- (1) For any two graphs  $G$  and  $H$ ,  $\text{Hom}(G, H)$  is a regular CW complex.
- (2) Cells of  $\text{Hom}(G, H)$  are direct products of simplices. More specifically, each  $\eta$  as in Definition 1.2 is a product of  $|V(G)|$  simplices, having dimensions  $|\eta(x)| - 1$ , for  $x \in V(G)$ . Thus  $\dim \eta = \sum_{x \in V(G)} |\eta(x)| - |V(G)|$ .
- (3) For any three graphs  $G, H$ , and  $K$ , we have

$$\text{Hom}(G \amalg H, K) = \text{Hom}(G, K) \times \text{Hom}(H, K),$$

and, if  $G$  is connected, and  $G \neq K_1$ , then also

$$\text{Hom}(G, H \amalg K) = \text{Hom}(G, H) \amalg \text{Hom}(G, K),$$

where the equality denotes isomorphism of polyhedral complexes.

The first formula is obvious. To see the second one, note that for  $\eta: V(G) \rightarrow 2^{V(H) \cup V(K)} \setminus \{\emptyset\}$ , and  $x, y \in V(G)$ , such that  $(x, y) \in E(G)$ , if  $\eta(x) \cap V(H) \neq \emptyset$ , then  $\eta(y) \subseteq V(H)$ , which under assumptions on  $G$  implies that  $\bigcup_{x \in V(G)} \eta(x) \subseteq V(H)$ .

(4)  $\text{Hom}(H, -)$  is a covariant, while  $\text{Hom}(-, H)$  is a contravariant functor from **Graphs** to **Top**.

If  $\phi \in \text{Hom}_0(G, G')$ , then we shall denote the cellular maps induced by composition as  $\phi^H: \text{Hom}(H, G) \rightarrow \text{Hom}(H, G')$  and  $\phi_H: \text{Hom}(G', H) \rightarrow \text{Hom}(G, H)$ .

(5) The map induced by composition

$$\text{Hom}(G, H) \times \text{Hom}(H, K) \longrightarrow \text{Hom}(G, K)$$

is a topological map.

(6) Obviously, it is difficult to decide in general whether  $\text{Hom}(G, K_n)$  is non-empty, let alone  $k$ -connected. It is certainly non-empty if the valency of each vertex is at most  $n - 1$ . The following fact is true in general.

**PROPOSITION 2.4:** *Let  $G$  be any graph. If the maximal valency of  $G$  is equal to  $d$ , then  $\text{Hom}(G, K_n)$  is connected, for all  $n \geq d + 2$ .*

*Proof:* Assume  $\text{Hom}(G, K_n)$  is not connected. Choose  $\phi, \psi \in \text{Hom}_0(G, K_n)$ , such that  $\psi$  and  $\phi$  belong to different connected components, and  $\phi(v) = \psi(v)$  for the maximal possible number of vertices. Pick  $u$ , such that  $\phi(u) \neq \psi(u)$ . If  $\psi(u)$  cannot be changed to  $\phi(u)$ , that is, if  $\tau: V(G) \rightarrow V(H)$ , defined by  $\tau(x) = \psi(x)$  for  $x \neq u$ ,  $\tau(u) = \phi(u)$ , is not a graph homomorphism, then there exists a vertex  $w$ , such that  $(u, w) \in E(G)$ , and  $\psi(w) = \phi(u) \neq \phi(w)$ .

Since the valency of  $w$  is at most  $n - 2$ , we can change  $\psi(w)$  to something else, without changing the number of vertices on which  $\psi$  and  $\phi$  coincide. Once this is done for each such neighbor of  $u$ , we can change  $\psi(u)$  to  $\phi(u)$ , thereby increasing the number of vertices on which  $\psi$  and  $\phi$  coincide, hence obtaining a contradiction to the choice of  $\psi$  and  $\phi$ . ■

This result motivates the following conjecture.

CONJECTURE 2.5: *Let  $G$  be any graph. If the maximal valency of  $G$  is equal to  $d$ , then  $\text{Hom}(G, K_n)$  is  $k$ -connected, for all integers  $k \geq -1$ ,  $n \geq d + k + 2$ .*

Proposition 2.4 corresponds to the case  $k = 0$ ; the case  $k = -1$  is obviously true since the graph of maximal valency  $d$  can be colored with  $d + 1$  colors.

### 3. Tools from topological combinatorics

3.1. DISCRETE MORSE THEORY. For a poset  $P$  with the covering relation  $\succ$ , we define a *partial matching* on  $P$  to be a set  $S \subseteq P$ , and an injective map  $\mu: S \rightarrow P \setminus S$ , such that  $\mu(x) \succ x$ , for all  $x \in S$ . The elements of  $P \setminus (S \cup \mu(S))$  are called *critical*.

The next proposition is a special case, which will be sufficient for our purposes, of a more general result proved by R. Forman; see [14].

PROPOSITION 3.1: *Let  $\Delta$  be a regular CW complex and  $\Delta'$  a subcomplex of  $\Delta$ . Then the following are equivalent:*

- (a) *there is a sequence of collapses leading from  $\Delta$  to  $\Delta'$ ;*
- (b) *there is a partial matching  $\mu$  on  $\mathcal{P}(\Delta)$  with the set of critical cells being  $\mathcal{P}(\Delta')$ , such that there is no sequence  $x_1, \dots, x_t \in \mathcal{P}(\Delta) \setminus \mathcal{P}(\Delta')$ ,  $t \geq 2$ , such that  $\mu(x_1) \succ x_2, \mu(x_2) \succ x_3, \dots, \mu(x_t) \succ x_1$  (such matching is called *acyclic*).*

*Proof:* See [18, Proposition 5.4]. ■

Proposition 3.1 is a part of the Discrete Morse theory; [3, 14, 17, 18] are just some of the references where it has been studied and used.

3.2. A QUILLEN-TYPE RESULT. In this subsection we prove a Quillen-type result which, given a poset map  $\phi$  satisfying certain conditions, provides us with some topological information about the induced simplicial map  $\Delta(\phi)$ .

PROPOSITION 3.2: *Let  $\phi: P \rightarrow Q$  be a map of finite posets. Consider a list of possible conditions on  $\phi$ .*

Condition (A). *For every  $q \in Q$ ,  $\Delta(\phi^{-1}(q))$  is contractible.*

Condition (B). *For every  $p \in P$  and  $q \in Q$  with  $p \in \phi^{-1}(Q_{\geq q})$  the poset  $\phi^{-1}(q) \cap P_{\leq p}$  has a maximal element. In this case we denote this maximal element by  $\max(p, q)$ .*

Condition (B<sup>op</sup>). *Let  $\phi^{op}: P^{op} \rightarrow Q^{op}$  be the poset map induced by  $\phi$ . We require that  $\phi^{op}$  satisfies Condition B. In this case we denote the minimal element of  $\phi^{-1}(q) \cap P_{\geq p}$  by  $\min(p, q)$ .*

Then

- (1) If  $\phi$  satisfies (A) and either (B) or  $(B^{op})$ , then  $\Delta(\phi)$  is a homotopy equivalence.
- (2) If  $\phi$  satisfies (B) and  $(B^{op})$ , and  $Q$  is connected, then for any  $q, q' \in Q$  we have  $\Delta(\phi^{-1}(q)) \simeq \Delta(\phi^{-1}(q'))$ . Furthermore, we have a fibration homotopy long exact sequence:

$$(3.1) \quad \cdots \longrightarrow \pi_i(\Delta(\phi^{-1}(q))) \longrightarrow \pi_i(\Delta(P)) \longrightarrow \pi_i(\Delta(Q)) \longrightarrow \cdots$$

*Proof:* Consider the poset map  $\text{Bd}\phi: \text{Bd}P \rightarrow \text{Bd}Q$ , which maps  $\rho \in \text{Bd}P$ ,  $\rho = (\alpha_1 > \cdots > \alpha_t)$  to  $\{\phi(\alpha_1), \dots, \phi(\alpha_t)\}$ . Since  $\phi$  is order-preserving, the last set is totally ordered, and thus can be interpreted as a chain in  $Q$ .

We set  $\phi^{-1}(\gamma) := \bigcup_{i=1}^t \phi^{-1}(\alpha_i)$  and view it as a subposet of  $P$ . Note that

$$(3.2) \quad (\text{Bd}\phi)^{-1}(\text{Bd}Q_{\leq \gamma}) = \text{Bd}(\phi^{-1}(\gamma)).$$

First we show (1). Because of the symmetry, we restrict our consideration to the case when  $\phi$  satisfies conditions (A) and  $(B^{op})$ . By Quillen’s theorem A, see [26, p. 85], it is enough to show that  $\Delta((\text{Bd}\phi)^{-1}(\text{Bd}Q_{\leq \gamma}))$  is contractible for any  $\gamma \in \text{Bd}Q$ . By (3.2), it is enough to show that  $\Delta(\phi^{-1}(\gamma))$  is contractible for any  $\gamma \in \text{Bd}Q$ . We use induction on the length of the chain  $\gamma = (\alpha_1 > \cdots > \alpha_t)$ . When  $t = 1$ , this is precisely condition (A), so we assume that  $t \geq 2$ .

Define  $\xi: \phi^{-1}(\gamma) \rightarrow \phi^{-1}(\alpha_1)$ , by  $\xi(p) = \min(p, \alpha_1)$ , for  $p \in \phi^{-1}(\gamma)$ . This is well-defined since  $\phi(p) \leq \alpha_1$ . Note that

- (1)  $\xi^2 = \xi$ , since  $\xi|_{\phi^{-1}(\alpha_1)} = \text{id}$ ;
- (2)  $\xi(p) \geq p$ , by the definition of  $\min(p, \alpha_1)$ ;
- (3)  $\xi$  is order-preserving. Indeed, take  $p, p' \in \phi^{-1}(\gamma)$ , such that  $p > p'$ . Then, on one hand  $\xi(p) \geq p > p'$ , on the other hand  $\phi(\xi(p)) = \alpha_1$ , hence, by the definition of  $\min(p', \alpha_1)$ , we have  $\xi(p) \geq \xi(p')$ .

This means that  $\xi$  is a closure map, hence  $\Delta(\xi)$  is homotopy equivalence; see [7, Corollary 10.12]. It follows by induction that  $\Delta(\phi^{-1}(\gamma))$  is contractible for any  $\gamma \in \text{Bd}Q$ .

Next we prove (2). Let  $\gamma, \tilde{\gamma} \in \text{Bd}Q$ , such that  $\gamma > \tilde{\gamma}$ . We want to show that the inclusion map  $i: \phi^{-1}(\tilde{\gamma}) \hookrightarrow \phi^{-1}(\gamma)$  induces a homotopy equivalence of the order complexes. Set  $\gamma' = \gamma \cap Q_{\geq \min \tilde{\gamma}}$ . Then  $\min \tilde{\gamma} = \min \gamma'$ ,  $\max \gamma = \max \gamma'$ , and  $\gamma \geq \gamma'$ .

Consider the sequence of inclusion maps  $\phi^{-1}(\max \gamma) \xrightarrow{i_1} \phi^{-1}(\gamma') \xrightarrow{i_2} \phi^{-1}(\gamma)$ , and let  $\xi: \phi^{-1}(\gamma) \rightarrow \phi^{-1}(\max \gamma)$  be the map defined above. By the argument for part (1) we know that pairs  $(i_1, \xi)$  and  $(i_2 \circ i_1, \xi)$  induce homotopy equivalences

of the order complexes. It follows that the pair  $(i_2, \xi)$  also induces a homotopy equivalence, since

$$\Delta(i_2) \circ \Delta(\xi) = \Delta(i_2) \circ \Delta(i_1 \circ \xi) = \Delta(i_2 \circ i_1) \circ \Delta(\xi) \simeq \text{id}$$

and

$$\Delta(\xi) \circ \Delta(i_2) = \Delta(i_1 \circ \xi) \circ \Delta(i_2) = \Delta(i_1) \circ \Delta(\xi \circ i_2) = \Delta(i_1) \circ \Delta(\xi) \simeq \text{id}.$$

By a symmetric argument the inclusion map  $j_2: \phi^{-1}(\tilde{\gamma}) \hookrightarrow \phi^{-1}(\gamma')$  induces a homotopy equivalence as well. Composing, we get that  $\Delta(i): \Delta(\phi^{-1}(\tilde{\gamma})) \hookrightarrow \Delta(\phi^{-1}(\gamma))$  is a homotopy equivalence.

In the special case  $\gamma = (q > q')$  we get that

$$\Delta(\phi^{-1}(q)) \simeq \Delta(\phi^{-1}(\gamma)) \simeq \Delta(\phi^{-1}(q')).$$

Hence, since  $Q$  is connected as a poset, we get  $\Delta(\phi^{-1}(q)) \simeq \Delta(\phi^{-1}(q'))$  for any  $q, q' \in Q$ .

Finally, the existence of the fibration homotopy long exact sequence (3.1) follows from (3.2) and Quillen’s Theorem B; see [26, p. 89]. ■

*Remark 3.3:* We shall not use Proposition 3.2 (2) in this paper. We have proved it here as a result which is interesting on its own right and might be useful for other computations. A more general version of Proposition 3.2 (1) was proved in [2]; see also [27].

### 4. Complexes of homomorphisms from complete graphs

4.1. THE NEIGHBORHOOD COMPLEX AND  $\text{Hom}(K_2, G)$ . We are now ready to define the neighborhood complex  $\mathcal{N}(G)$  and show that it is homotopy equivalent to  $\text{Hom}(K_2, G)$ . The natural advantage to working with the polyhedral complex  $\text{Hom}(K_2, G)$  instead of the simplicial complex  $\mathcal{N}(G)$  is that  $\text{Hom}(K_2, G)$  possesses a natural free cellular  $\mathbb{Z}_2$ -action induced from the swapping  $\mathbb{Z}_2$ -action on  $K_2$ .

*Definition 4.1:* For an arbitrary graph  $G$  the simplicial complex  $\mathcal{N}(G)$  is defined as follows: its vertices are all non-isolated vertices of  $G$ , and its simplices all the subsets of  $V(G)$  which have a common neighbor.

In other words, the maximal simplices of  $\mathcal{N}(G)$  are  $N(v)$ , for  $v \in V(G)$ .

PROPOSITION 4.2:  $\text{Hom}(K_2, G)$  is homotopy equivalent to  $\mathcal{N}(G)$ .

*Proof:* Let  $P = \mathcal{P}(\text{Hom}(K_2, G))$  and  $Q = \mathcal{P}(\mathcal{N}(G))$ . Consider  $\phi: P \rightarrow Q$  mapping the element  $\eta: \{1, 2\} \rightarrow 2^{V(G)} \setminus \emptyset$  to  $\eta(1) \subseteq V(G)$ . Clearly, the vertices in  $\eta(1)$  have all the vertices in  $\eta(2)$  as their neighbors, hence, since  $\eta(2) \neq \emptyset$ ,  $\phi$  is well-defined. Let us show that  $\phi$  induces homotopy equivalence  $\Delta(\phi): \Delta(P) \rightarrow \Delta(Q)$ .

First, let  $A \in Q$ . We see that  $\phi^{-1}(A)$  is the set of all pairs  $(A, B)$ ,  $A, B \subseteq V(G)$ , such that for all  $x \in A$ , and  $y \in B$ , we have  $(x, y) \in E(G)$ . Clearly,  $\phi^{-1}(A)$  has a maximal element  $(A, \mathcal{N}(A))$ , so  $\Delta(\phi^{-1}(A))$  is a cone, hence contractible.

Second, let us check the Condition (B) of Proposition 3.2. Let  $A \in Q$  and  $(C, D) \in P$ , such that  $\phi(C, D) = C \supseteq A$ . Clearly  $\mathcal{N}(A) \supseteq \mathcal{N}(C) \supseteq D \neq \emptyset$ . Then  $\phi^{-1}(A) \cap P_{\leq(C, D)} = \{(A, B) | B \subseteq D, B \neq \emptyset\}$ . This poset has a maximal element  $(A, D)$ , since  $D \subseteq \mathcal{N}(A)$ . In the notations of Proposition 3.2 we have  $(A, D) = \max((C, D), A)$ .

Since Conditions (A) and (B) are satisfied,  $\Delta(\phi)$  is a homotopy equivalence by Proposition 3.2. This shows that  $\text{Bd}(\text{Hom}(K_2, G)) \simeq \text{Bd}(\mathcal{N}(G))$ , hence the result. ■

As Proposition 4.2 shows, the original complexes  $\mathcal{N}(G)$  correspond to  $K_2$ -type obstructions to colorability. The Lovász' idea behind his Conjecture 1.5 was that the next natural class of obstructions should come from the maps from odd cycles  $C_{2r+1}$  to our graph.

4.2.  $\text{Hom}(K_2, K_n)$  AS A BOUNDARY COMPLEX OF A POLYTOPE. Let  $M_n$  denote the Minkowski sum

$$[-1/2, 1/2]^n + [(-1/2, -1/2, \dots, -1/2), (1/2, 1/2, \dots, 1/2)],$$

where  $[-1/2, 1/2]^n$  denotes the cube in  $\mathbb{R}^n$  with vertices all points whose coordinates have the absolute value  $1/2$ .  $M_n$  is a zonotope in  $\mathbb{R}^n$ . Its dual,  $M_n^*$ , is the polytope associated to the hyperplane arrangement  $\mathcal{A} = \{\mathcal{A}_1, \dots, \mathcal{A}_{n+1}\}$  defined by

$$\mathcal{A}_i = \begin{cases} (x_i = 0), & \text{for } 1 \leq i \leq n; \\ (\sum_{j=1}^n x_j = 0), & \text{for } i = n + 1. \end{cases}$$

In the proof of the next proposition we identify each cell  $\eta: V(K_2) \rightarrow 2^{V(K_n)} \setminus \{\emptyset\}$  with the ordered pair  $(A, B)$  of non-empty subsets of  $[n]$ , by taking  $A = \eta(1)$  and  $B = \eta(2)$ .

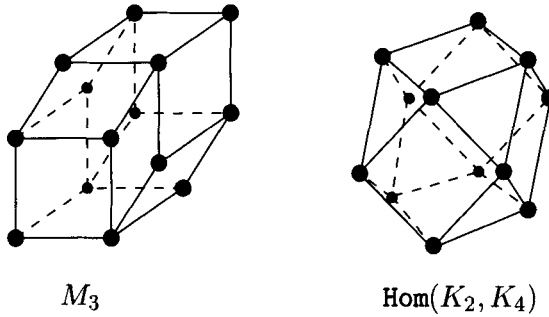


Figure 4.1. Complex of 4-colorings of an edge and its dual.

PROPOSITION 4.3:  $\text{Hom}(K_2, K_{n+1})$  is isomorphic as a cell complex to the boundary complex of  $M_n^*$ . The  $\mathbb{Z}_2$ -action on  $\text{Hom}(K_2, K_{n+1})$ , induced by the flip action of  $\mathbb{Z}_2$  on  $K_2$ , corresponds under this isomorphism to the central symmetry.

*Proof:* Set  $P = \mathcal{P}(\text{Hom}(K_2, K_{n+1}))^{op}$ . We shall see that  $P$  is isomorphic to the face poset of  $M_n$ , which we denote by  $Q = \mathcal{F}(M_n)$ . We shall denote the future isomorphism by  $\rho$ .

First, note that faces of the cube  $[-1/2, 1/2]^n$  are encoded by  $n$ -tuples of  $1/2$ ,  $-1/2$ , and  $*$ , where  $*$  denotes the coordinate where the value can be chosen arbitrarily from the interval  $[-1/2, 1/2]$ . For an arbitrary  $n$ -tuple  $x$ , we let  $\text{supp}(x) \subseteq [n]$  denote the set of the indices of coordinates which are either non-zero, or are denoted with a  $*$ . Additionally, for an arbitrary number  $k$ , we let  $\text{supp}(x, k) \subseteq [n]$  denote the set of the indices of the coordinates which are equal to  $k$  (in particular, they cannot be denoted with a  $*$ ).

Vertices of  $M_n$  are labeled by all  $n$ -tuples of  $1$ ,  $-1$ , and  $0$ , such that  $1$  and  $-1$  are not present simultaneously, and not all the coordinates are equal to  $0$ , that is  $v$  is a vertex of  $M_n$  if and only if  $v \in \{0, 1\}^n$ , or  $v \in \{0, -1\}^n$ , and  $v \neq (0, \dots, 0)$ . These vertices correspond to atoms in  $P$  as follows:

$$v \xleftrightarrow{\rho} \begin{cases} (\text{supp}(v), [n + 1] \setminus \text{supp}(v)), & \text{if } v \in \{0, 1\}^n; \\ ([n + 1] \setminus \text{supp}(v), \text{supp}(v)), & \text{if } v \in \{0, -1\}^n. \end{cases}$$

Clearly, restricted to atoms,  $\rho$  is a bijection.

Those faces of  $M_n$  which are contained in the closed star of  $(1, \dots, 1)$  can be indexed by  $f \in \{0, 1, *\}^n$ , where  $|\text{supp}(f, 1)| \geq 1$ . Symmetrically, those faces of  $M_n$  which are contained in the closed star of  $(-1, \dots, -1)$  can be indexed by



$f \in \{0, -1, *\}^n$ , where  $|\text{supp}(f, -1)| \geq 1$ . For these faces  $\rho$  can be defined as follows:

$$f \xleftarrow{\rho} \begin{cases} (\text{supp}(f, 1), \text{supp}(f, 0) \cup \{n + 1\}), & \text{if } f \in \overline{\text{St}(1, \dots, 1)}; \\ (\text{supp}(f, 0) \cup \{n + 1\}, \text{supp}(f, -1)), & \text{if } f \in \overline{\text{St}(-1, \dots, -1)}. \end{cases}$$

Finally, we consider the faces of  $M_n$  which are not in

$$\overline{\text{St}(1, \dots, 1)} \cup \overline{\text{St}(-1, \dots, -1)}.$$

Each such face is a convex hull of the union of two faces,  $f \cup \tilde{f}$ , such that  $f \in \overline{\text{St}(1, \dots, 1)}$ ,  $\tilde{f} \in \overline{\text{St}(-1, \dots, -1)}$ , with the condition that  $\text{supp}(f, 0) = \text{supp}(\tilde{f}, -1)$ ,  $\text{supp}(f, 1) = \text{supp}(\tilde{f}, 0)$ . The element of  $P$  associated to such a face under  $\rho$  is  $(\text{supp}(f, 1), \text{supp}(f, 0)) = (\text{supp}(\tilde{f}, 0), \text{supp}(\tilde{f}, -1))$ .

It is an easy exercise to check that  $\rho$  defines a poset isomorphism between  $P$  and  $Q$ , which in turn induces the required cell complex isomorphism.

Finally, a brief scanning through the definition of  $\rho$  in different cases reveals that  $\rho$  is equivariant with respect to the described  $\mathbb{Z}_2$ -actions on both sides. Hence the last part of the proposition follows. ■

The cellular map  $\phi$ , defined in Proposition 4.2, is in this case going from the boundary of an  $(n - 2)$ -dimensional polytope  $M_n^*$  to the boundary of an  $(n - 2)$ -dimensional simplex. It would be interesting to see whether it has interesting additional properties in the context of zonotopes and also to find out what other graphs  $G$  provide a connection to polytopes.

*Remark 4.4:*  $\text{Hom}(K_m, K_n)$  can be viewed as a deleted product of simplices; see [24]. In this context it is well-known, probably due to van Kampen, that for  $m = 2$  it is a boundary of a polytope.

4.3. THE HOMOTOPY TYPE OF  $\text{Hom}(K_m, K_n)$ . We can still get fairly detailed information about the topology of the spaces of homomorphisms between complete graphs in general.

PROPOSITION 4.5:  $\text{Hom}(K_m, K_n)$  is homotopy equivalent to a wedge of  $(n - m)$ -dimensional spheres.

*Proof:* We use induction on  $m$  and on  $n - m$ . The base is provided by the cases  $\text{Hom}(K_1, K_n)$ , which is a simplex with  $n$  vertices, hence contractible, and  $\text{Hom}(K_n, K_n)$ , which consists of  $n!$  points, that is a wedge of  $n! - 1$  spheres of dimension 0. We assume now that  $m \geq 2$  and  $n \geq m + 1$ .

For  $i \in [m]$ , let  $A_i$  be the subcomplex of  $\text{Hom}(K_m, K_n)$  defined by

$$A_i = \{\eta : [m] \rightarrow 2^{[n]} \setminus \{\emptyset\} \mid n \notin \eta(j), \text{ for } j \in [m], j \neq i\}.$$

Since any two vertices of  $K_m$  are connected by an edge,  $n$  cannot be in  $\eta(i_1) \cap \eta(i_2)$ , for  $i_1 \neq i_2$ . This implies that  $\bigcup_{i=1}^m A_i = \text{Hom}(K_m, K_n)$ .

Clearly, for any  $i \neq j, i, j \in [m]$ , we have

$$A_i \cap A_j = \{\eta : [m] \rightarrow 2^{[n]} \setminus \{\emptyset\} \mid n \notin \eta(k), \text{ for all } k \in [m]\},$$

so  $A_i \cap A_j$  is isomorphic to  $\text{Hom}(K_m, K_{n-1})$ , hence, by induction, it is  $(n - m - 2)$ -connected.

We shall now see that each  $A_i$  is  $(n - m - 1)$ -connected. Since all  $A_i$ 's are isomorphic to each other, it is enough to consider  $A_1$ . Let us describe a partial matching on  $\mathcal{P}(A_1)$ . For  $\eta \in \mathcal{P}(A_1)$ , such that  $n \notin \eta(1)$ , we set  $\tilde{\eta}(\eta) := \tilde{\eta}$ , defined by

$$\tilde{\eta}(i) = \begin{cases} \eta(1) \cup \{n\}, & \text{for } i = 1; \\ \eta(i), & \text{for } i = 2, 3, \dots, m. \end{cases}$$

Obviously, this is an acyclic matching and the critical cells form a subcomplex  $\tilde{A} \subseteq A$  defined by:  $\eta \in \tilde{A}$  if and only if  $\eta(1) = \{n\}$ . Thus  $\tilde{A} = \text{Hom}(K_{m-1}, K_{n-1})$ . Since, by Proposition 3.1,  $\tilde{A}$  is homotopy equivalent to  $A_1$ , and  $\tilde{A}$  is  $(n - m - 1)$ -connected by the induction assumption, we conclude that  $A_i$  is  $(n - m - 1)$ -connected for any  $i$ .

It follows from [7, Theorem 10.6(ii)] that  $\text{Hom}(K_m, K_n)$  is  $(n - m - 1)$ -connected. Since the dimension of  $\text{Hom}(K_m, K_n)$  is  $n - m$ , it follows from [7, (9.19)] that  $\text{Hom}(K_m, K_n)$  is homotopy equivalent to a wedge of spheres. ■

One can use the construction in the proof of Proposition 4.5 to count the number of the spheres in the wedge. Let us say that  $\text{Hom}(K_m, K_n)$  is homotopy equivalent to a wedge of  $f(m, n)$  spheres. Let  $S(-, -)$  denote the Stirling numbers of the second kind, and  $SF_k(x) = \sum_{n \geq k} S(n, k)x^n$  denote the generating function for these numbers. It is well-known that

$$SF_k(x) = x^k / (1 - x)(1 - 2x) \cdots (1 - kx).$$

For  $m \geq 1$ , let  $F_m(x) = \sum_{n \geq 1} f(m, n)x^n$  be the generating function for the number of the spheres. Clearly,  $F_1(x) = 0$ , and  $F_2(x) = x^2 / (1 - x)$ .

**PROPOSITION 4.6:** *The numbers  $f(m, n)$  satisfy the following recurrence relation:*

$$(4.1) \quad f(m, n) = mf(m - 1, n - 1) + (m - 1)f(m, n - 1)$$

for  $n > m \geq 2$ , with the boundary values  $f(n, n) = n! - 1$ ,  $f(1, n) = 0$  for  $n \geq 1$ , and  $f(m, n) = 0$  for  $m > n$ .

The generating function  $F_m(x)$  is given by the equation

$$(4.2) \quad F_m(x) = (m! \cdot x \cdot SF_{m-1}(x) - x^m)/(1 + x).$$

As a consequence, the following non-recursive formulae are valid:

$$(4.3) \quad f(m, n) = (-1)^{m+n+1} + m!(-1)^n \sum_{k=m}^n (-1)^k S(k-1, m-1),$$

and

$$(4.4) \quad f(m, n) = \sum_{k=1}^{m-1} (-1)^{m+k+1} \binom{m}{k+1} k^n,$$

for  $n \geq m \geq 1$ .

*Proof:* Let  $\chi(m, n)$  denote the non-reduced Euler characteristics of the complexes  $\text{Hom}(K_m, K_n)$ , and, for  $i = 1, \dots, m$ , let  $A_i$  be as in the proof of Proposition 4.5. Since  $\text{Hom}(K_m, K_n) = \bigcup_{i=1}^m A_i$ ,  $A_i \cap A_j = \text{Hom}(K_m, K_{n-1})$ , for all  $i \neq j$ , and  $A_i \simeq \text{Hom}(K_{m-1}, K_{n-1})$ , for  $i \in [m]$ , by simple inclusion-exclusion counting we conclude that

$$(4.5) \quad \chi(m, n) = m\chi(m-1, n-1) - (m-1)\chi(m, n-1),$$

for  $n > m \geq 2$ , additionally  $\chi(n, n) = n!$ ,  $\chi(1, n) = 1$ , for  $n \geq 1$ . Since  $\chi(m, n) = 1 + (-1)^{m-n}f(m, n)$ , a simple computation shows the validity of the relation (4.1).

For  $m \geq 1$ , let  $G_m(x) = \sum_{n \geq 1} \chi(m, n)x^n$ . Multiplying each side of equation (4.5) by  $x^n$  and summing over all  $n$  yields

$$G_m(x) = m \cdot x \cdot G_{m-1}(x) - (m-1) \cdot x \cdot G_m(x),$$

implying

$$G_m(x) = \frac{mx}{1 + (m-1)x} G_{m-1}(x),$$

for  $m \geq 1$ , and hence, since  $G_0(x) = 1/(1-x)$ , we get

$$(4.6) \quad G_m(x) = \frac{m! \cdot x^m}{(1-x)(1+x)(1+2x)\dots(1+(m-1)x)} \\ = m! \cdot x \cdot (-1)^{m-1} \cdot SF_{m-1}(-x)/(1-x),$$

for  $m \geq 0$ . By multiplying the identity  $f(m, n) = (-1)^{m+n}(\chi(m, n) - 1)$  with  $x^n$  and summing over all  $n \geq m$ , we get

$$\begin{aligned}
 F_m(x) &= (-1)^m G_m(-x) - x^m/(1+x) \\
 (4.7) \quad &= (-1)^m \cdot m! \cdot (-x) \cdot (-1)^{m-1} \cdot SF_{m-1}(x)/(1+x) - x^m/(1+x) \\
 &= (m! \cdot x \cdot SF_{m-1}(x) - x^m)/(1+x).
 \end{aligned}$$

Formula (4.3) follows from comparing the coefficients in (4.2).

To prove (4.4), we see that it fits the boundary values and satisfies the recurrence relation (4.1). Verifying (4.1) is straightforward, as is checking (4.4) for  $m = 1$  and  $m = 2$ . Finally, (4.4) is seen for  $n = m \geq 2$  by expanding the expression  $(e^x - 1)^n \cdot e^{-x}$  by the binomial theorem and comparing the coefficient of  $x^n$  on both sides of the expansion. ■

In particular, we have  $f(2, n) = 1$ , for  $n \geq 2$ ,  $f(3, n) = 2^n - 3$ , for  $n \geq 3$ ,  $f(4, n) = 3^n - 4 \cdot 2^n + 6$ , for  $n \geq 4$ ,  $f(5, n) = 4^n - 5 \cdot 3^n + 10 \cdot 2^n - 10$ , for  $n \geq 5$ .

We are now ready to prove the result announced in the beginning of this paper.

*Proof of Theorem 1.7:* If the graph  $G$  is  $(k + m - 1)$ -colorable, then there exists a homomorphism  $\phi: G \rightarrow K_{k+m-1}$ . It induces a  $\mathbb{Z}_2$ -equivariant map

$$\phi^{K_m}: \text{Hom}(K_m, G) \rightarrow \text{Hom}(K_m, K_{k+m-1}).$$

By Proposition 4.5 the space  $\text{Hom}(K_m, K_{k+m-1})$  is homotopy equivalent to a wedge of  $(k-1)$ -spheres, hence, by dimensional reasons,  $\varpi_1^k(\text{Hom}(K_m, K_{k+m-1})) = 0$ . Since the Stiefel–Whitney classes are functorial, the existence of the map  $\phi^{K_m}$  implies that  $\varpi_1^k(\text{Hom}(K_m, G)) = 0$ , which is a contradiction to the assumption of the theorem. ■

### 5. Complexes of homomorphisms from forests and their complements to complete graphs

5.1. THE MINOR NEIGHBOR REDUCTION AND ITS CONSEQUENCES. The next proposition, coupled with Propositions 4.3 and 4.5, will be our workhorse for computing concrete examples.

PROPOSITION 5.1: *If  $G$  and  $H$  are graphs and  $u$  and  $v$  are distinct vertices of  $G$ , such that  $N(v) \subseteq N(u)$ , then the inclusion  $i: G - v \hookrightarrow G$ , resp. the homomorphism  $\phi: G \rightarrow G - v$  mapping  $v$  to  $u$  and fixing other vertices, induce homotopy*

equivalences  $i_H: \text{Hom}(G, H) \rightarrow \text{Hom}(G - v, H)$ , resp.  $\phi_H: \text{Hom}(G - v, H) \rightarrow \text{Hom}(G, H)$ .

*Proof:* Let us apply Proposition 3.2 (1) for the cellular map  $i_H: \text{Hom}(G, H) \rightarrow \text{Hom}(G - v, H)$ . Take  $\eta \in \mathcal{P}(\text{Hom}(G - v, H))$ ,  $\eta: V(G) \setminus \{v\} \rightarrow 2^{V(H)} \setminus \{\emptyset\}$ . We have

$$\mathcal{P}((i_H)^{-1}(\eta)) = \{\tau \in \mathcal{P}(\text{Hom}(G, H)) \mid \tau(w) = \eta(w), \text{ for } w \neq v, w \in V(G)\}.$$

An element in  $\mathcal{P}((i_H)^{-1}(\eta))$  is determined by its value on  $v$ . Take

$$\tau \in \mathcal{P}((i_H)^{-1}(\eta))$$

such that

$$\tau(v) = \bigcap_{y \in \mathbf{N}(v)} \mathbf{N}(\eta(y)) \supseteq \bigcap_{y \in \mathbf{N}(u)} \mathbf{N}(\eta(y)) \supseteq \eta(u) \neq \emptyset.$$

Clearly,  $\tau$  is the maximal element of  $\mathcal{P}((i_H)^{-1}(\eta))$ , hence  $\Delta(\mathcal{P}((i_H)^{-1}(\eta)))$  is contractible, so Condition (A) is satisfied.

Let us now check Condition (B). Take

$$\tau \in \mathcal{P}(\text{Hom}(G, H)), \quad \eta \in \mathcal{P}(\text{Hom}(G - v, H)),$$

such that for any  $x \in V(G) \setminus \{v\}$  we have  $\tau(x) \supseteq \eta(x)$ . The set

$$\mathcal{P}((i_H)^{-1}(\eta)) \cap \mathcal{P}(\text{Hom}(G, H))_{\leq \tau}$$

consists of all  $\nu \in \mathcal{P}(\text{Hom}(G, H))$ , such that for any  $x \in V(G)$  we have  $\tau(x) \supseteq \nu(x)$ , and for any  $x \in V(G) \setminus \{v\}$  we have  $\eta(x) = \nu(x)$ . Thus, it has a maximal element defined by

$$\nu(x) = \begin{cases} \eta(x), & \text{for } x \neq v, x \in V(G); \\ \tau(x), & \text{for } x = v. \end{cases}$$

Conditions (A) and (B) being satisfied, we now get that  $\text{Bd}(i_H)$ , hence also  $i_H$ , is a homotopy equivalence.

To see that  $\phi_H$  is also a homotopy equivalence note first that  $i_H \circ \phi_H = \text{id}_{\text{Hom}(G-v, H)}$ . Let  $j$  be the homotopy inverse of  $i_H$ ; then

$$\phi_H \circ i_H \simeq j \circ i_H \circ \phi_H \circ i_H = j \circ i_H \simeq \text{id}_{\text{Hom}(G, H)}. \quad \blacksquare$$

If  $G$  is a graph, and  $u, v \in V(G)$ ,  $u \neq v$ , such that  $\mathbf{N}(v) \subseteq \mathbf{N}(u)$ , then we say that  $G$  reduces to  $G - v$ . We shall also say that  $u$  dominates  $v$ , or that  $v$  is

dominated by  $u$ . If in addition  $N(v) \neq N(u)$ , we say that  $u$  strongly dominates  $v$ . We call  $u$  and  $v$  equivalent if  $N(v) = N(u)$ . The strong domination defines a partial order  $P(G)$  on the set of equivalence classes. We call a graph irreducible if it does not reduce to any subgraph.

We note a simple, but useful property of the vertex domination: if  $u, v \in S \subseteq V(G)$ ,  $u \neq v$ , and  $u$  dominates  $v$  in  $G$ , then  $u$  dominates  $v$  in  $G[S]$ . If  $u$  strongly dominates  $v$  in  $G$ , it is not true in general that  $u$  strongly dominates  $v$  in  $G[S]$ .

As the example of the tree already shows, the minimal subgraph of  $G$  to which it reduces is not unique. However, the following weaker version of uniqueness is true.

**PROPOSITION 5.2:** *Let  $G$  be a graph and  $S, S' \subseteq V(G)$ , such that  $G$  reduces both to  $G[S]$  and to  $G[S']$ , and both  $G[S]$  and  $G[S']$  are irreducible. Then  $G[S]$  is isomorphic to  $G[S']$ .*

*Proof:* We prove the statement by induction on the number of vertices in  $G$ . If  $|V(G)| = 1$ , then  $S = S' = V(G)$ , so the result is trivially true. Assume now that  $|V(G)| \geq 2$ .

Choose  $M \subseteq V(G)$  containing exactly one vertex from each maximal equivalence class in  $P(G)$ , and no other vertices. If  $M = V(G)$ , then  $G$  is irreducible, so we can assume that  $M \neq V(G)$ . Let us show that there exists  $\tilde{S} \subseteq M$ , such that  $G$  reduces to  $G[\tilde{S}]$ , and  $G[S]$  is isomorphic to  $G[\tilde{S}]$ .

Assume that no such  $\tilde{S}$  exists. Consider all the reduction sequences  $(\tilde{v}_1, \dots, \tilde{v}_{|V(G)|-|S|})$  leading from  $G$  to a graph isomorphic to  $G[S]$ . Set  $\{\tilde{w}_i\}_{i \in I} := M \cap \{\tilde{v}_1, \dots, \tilde{v}_{|V(G)|-|S|}\}$ , and choose the sequence which minimizes  $\sum_{i \in I} (|V(G)| - i)$ . Denote this sequence by  $(w_1, \dots, w_{|V(G)|-|S|})$ .

Set

$$\tilde{S} := V(G) \setminus \{w_1, \dots, w_{|V(G)|-|S|}\} \quad \text{and} \quad \{w_i\}_{i \in I} := M \cap \{w_1, \dots, w_{|V(G)|-|S|}\}.$$

If each vertex of  $G[\tilde{S}]$  is either in  $\tilde{S} \cap M$  or is dominated in  $G[\tilde{S}]$  by some vertex in  $\tilde{S} \cap M$ , then, since  $G[\tilde{S}]$  is irreducible, we conclude that  $\tilde{S} \subseteq M$ , yielding a contradiction.

Thus we may pick the *smallest*  $i$ , such that there exists  $v \in \tilde{S} \setminus M$ , which is not dominated by any vertex of  $M \setminus \{w_1, \dots, w_i\}$  in  $G_i = G - \{w_1, \dots, w_i\}$ . By the choice of  $M$ , and what is said above, we have  $i \in [|V(G)| - |S|]$ . Clearly, since  $v$  was dominated by some vertex of  $M \setminus \{w_1, \dots, w_i\}$  in  $G_{i-1} = G - \{w_1, \dots, w_{i-1}\}$ , we have that  $w_i \in M$ , and  $w_i$  is the only vertex of  $M \setminus \{w_1, \dots, w_i\}$  which dominates  $v$  in  $G_{i-1}$ . In particular,  $w_i$  itself is not dominated by any other vertex of  $M \setminus \{w_1, \dots, w_i\}$  in  $G_{i-1}$ .

By the choice of  $i$ , every vertex in  $G_{i-1}$ , which is not in  $M \setminus \{w_1, \dots, w_i\}$ , is dominated by some vertex in  $M \setminus \{w_1, \dots, w_i\}$ , hence  $w_i$  is not strongly dominated by any other vertex. Since  $G_{i-1} \rightarrow G_{i-1} - \{w_i\} = G_i$  is a legal reduction, there must exist a vertex  $w$  equivalent to  $w_i$  in  $G_{i-1}$ . We have  $w \notin M$ , since either  $w = v$ , or  $w$  dominates  $v$ .

Consider a graph isomorphism  $\varphi: G_{i-1} \rightarrow G_{i-1}$ , which swaps the vertices  $w_i$  and  $w$ , and fixes every other vertex. It is easy to see that

$$(w_1, \dots, w_{i-1}, \varphi(w_i), \varphi(w_{i+1}), \dots, \varphi(w_{|V(G)|-|S|}))$$

is a legal reduction sequence leading from  $G$  to  $G[\widehat{S}]$ , such that  $G[\widehat{S}]$  is isomorphic to  $G[S]$ .

Furthermore, since removal of  $w_i \in M$  was either replaced by or swapped with the removal of  $w \notin M$ , the invariant, which we minimized over the sequences, is actually smaller for this sequence than for  $(w_1, \dots, w_{|V(G)|-|S|})$ . This is again a contradiction.

Finally, consider the case  $S, S' \subseteq M$ . Since  $|M| < |V(G)|$ , we can use the induction assumption to prove the theorem, as long as we can show that  $G[M]$  reduces to  $G[S]$  and to  $G[S']$ . By the argument above, we can choose  $S$  so that, if  $(w_1, \dots, w_{|V(G)|-|S|})$  is the reduction sequence leading to  $G[S]$ , and  $\{w_i\}_{i \in I} = M \cap \{w_1, \dots, w_{|V(G)|-|S|}\}$ , then, for any  $i = 1, \dots, |V(G)| - |S|$ , every vertex in  $V(G) \setminus \{w_1, \dots, w_i\}$  is dominated by some vertex from  $M \setminus \{w_1, \dots, w_i\}$  in  $G - \{w_1, \dots, w_i\}$ . It is then immediate that  $\{w_{i_1}, \dots, w_{i_t}\}$  is the reduction sequence from  $G[M]$  to  $G[S]$ , where  $I = \{i_1, \dots, i_t\}$ ,  $i_1 < \dots < i_t$ .

Indeed, for any  $i \in I$ ,  $w_i$  is dominated by some vertex in  $G - \{w_1, \dots, w_{i-1}\}$ , hence it is dominated by some vertex from  $M \setminus \{w_1, \dots, w_{i-1}\}$  in  $G - \{w_1, \dots, w_{i-1}\}$ . It follows that  $w_i$  is dominated by some vertex in  $G[M \setminus \{w_j | j \in I, j < i\}]$ , allowing one to reduce the latter graph to  $G[M \setminus \{w_j | j \in I, j \leq i\}]$ . ■

For future reference we explicitly state the following consequence of Proposition 5.1.

**COROLLARY 5.3:** *Let  $G$  be a graph, and  $S \subseteq V(G)$ , such that  $G$  reduces to  $G[S]$ . Assume  $S$  is  $\Gamma$ -invariant for some  $\Gamma \subseteq \text{Aut}(G)$ . Then the inclusion  $i: G[S] \hookrightarrow G$  induces a  $\Gamma$ -invariant homotopy equivalence  $i_H: \text{Hom}(G, H) \rightarrow \text{Hom}(G[S], H)$  for an arbitrary graph  $H$ .*

Note also that Proposition 5.1 cannot be generalized to encompass arbitrary graph homomorphisms  $\phi$  of  $G$  onto  $H$ , where  $H$  is a subgraph of  $G$ , and  $\phi$

is identity on  $H$ . As an example in subsection 2.3 showed,  $\text{Hom}(C_6, K_3) \not\cong \text{Hom}(K_2, K_3)$  despite the existence of the folding map of  $C_6$  onto  $K_2$ .

5.2. THE HOMOTOPY TYPE OF  $\text{Hom}(F, K_n)$  AND  $\text{Hom}(\overline{F}, K_n)$ . Next, we use Proposition 5.1 to compute homotopy types of the complexes of maps from finite forests to complete graphs.

PROPOSITION 5.4: *If  $T$  is a tree with at least one edge, then the map  $i_{K_n}: \text{Hom}(T, K_n) \rightarrow \text{Hom}(K_2, K_n)$  induced by any inclusion  $i: K_2 \hookrightarrow T$  is a homotopy equivalence, in particular  $\text{Hom}(T, K_n) \simeq S^{n-2}$ . As a consequence, if  $F$  is a forest, and  $T_1, \dots, T_k$  are all its connected components consisting of at least 2 vertices, then  $\text{Hom}(F, K_n) \simeq \prod_{i=1}^k S^{n-2}$ .*

*Proof:* Let  $T$  be a tree with  $k$  vertices,  $k \geq 2$ . Note the general fact, that if  $v$  is a leaf of a tree,  $u$  is the vertex adjacent to  $v$ , and  $w \neq v$  is a vertex adjacent to  $u$ , then  $N(w) \supseteq N(v) = \{u\}$ , hence  $T$  reduces to  $T - v$ .

Let us now number the vertices  $v_1, \dots, v_k$  so that for any  $i \in [k - 1]$ ,  $v_i$  is a leaf in  $T - \{v_{i+1}, \dots, v_k\}$ . By the previous observation

$$T \rightarrow T - \{v_k\} \rightarrow T - \{v_{k-1}, v_k\} \rightarrow \dots \rightarrow T - \{v_3, \dots, v_k\} = T[\{v_1, v_2\}] = K_2$$

is a valid reduction sequence. Thus the first part of the statement follows by Proposition 5.1.

That  $\text{Hom}(T, K_n) \simeq S^{n-2}$  follows from Proposition 4.3. Finally, the formula for the homotopy type of  $\text{Hom}(F, K_n)$  follows from (3) in subsection 2.4. ■

Let  $S_a^n$  denote the  $n$ -dimensional sphere equipped with the antipodal action of  $\mathbb{Z}_2$ ; in the same way,  $S_t^n$  denotes the  $n$ -dimensional sphere equipped with the trivial action of  $\mathbb{Z}_2$ .

Given two spaces  $X$  and  $Y$  with  $\mathbb{Z}_2$ -action, we let  $X \simeq_{\mathbb{Z}_2} Y$  denote the  $\mathbb{Z}_2$ -equivariant homotopy equivalence.

PROPOSITION 5.5: *Let  $T$  be a tree with at least one edge and a  $\mathbb{Z}_2$ -action determined by an invertible graph homomorphism  $\gamma: T \rightarrow T$ . If  $\gamma$  flips an edge in  $T$ , then  $\text{Hom}(T, K_n) \simeq_{\mathbb{Z}_2} S_a^{n-2}$ , otherwise  $\text{Hom}(T, K_n) \simeq_{\mathbb{Z}_2} S_t^{n-2}$ .*

*Proof:* Assume  $\gamma$  flips an edge, that is there exist  $a, b \in V(G)$ , such that  $(a, b) \in E(G)$ ,  $\gamma(a) = b$ , and  $\gamma(b) = a$ . By Corollary 5.3 the inclusion map  $i: T[\{a, b\}] \hookrightarrow T$  induces a  $\mathbb{Z}_2$ -equivariant homotopy equivalence  $\text{Hom}(T, K_n) \simeq_{\mathbb{Z}_2} \text{Hom}(K_2, K_n)$ , where the last space has the natural  $\mathbb{Z}_2$ -action induced by the flipping  $\mathbb{Z}_2$ -action on  $K_2$ . By Proposition 4.3 we get  $\text{Hom}(T, K_n) \simeq_{\mathbb{Z}_2} S_a^{n-2}$ .



Assume now, there is no edge flipped by  $\gamma$ . Since  $T$  is a contractible finite CW complex (the topology is generated by fixing homeomorphisms between edges of  $T$  and a standard unit interval) it follows from [8, p. 257] that  $\gamma$  must have a fixed point. Denote this point by  $x$ . Clearly, either  $x$  is a vertex of  $T$  or  $x$  is the middle-point of some edge  $e \in E[G]$ . In the latter case, if the edge is not fixed pointwise, then it is flipped, which contradicts our assumptions on  $\gamma$ .

Thus we found a vertex  $v \in V(G)$  fixed by  $\gamma$ . If there exists  $e = (a, b) \in E(G)$ , such that  $\gamma(a) = a, \gamma(b) = b$ , then  $i: T[\{a, b\}] \hookrightarrow T$  induces a  $\mathbb{Z}_2$ -equivariant homotopy equivalence  $\text{Hom}(T, K_n) \simeq_{\mathbb{Z}_2} \text{Hom}(K_2, K_n)$ , where the  $\mathbb{Z}_2$ -action on the last space is the trivial one. It follows that  $\text{Hom}(T, K_n) \simeq_{\mathbb{Z}_2} S_t^{n-2}$ .

Finally, consider the case when there is no edge in  $T$  which is fixed by  $\gamma$  pointwise. Let  $u$  be any vertex of  $T$  adjacent to  $v$ , and let  $w = \gamma(u) \neq u$ . Since the set  $\{u, w, v\}$  is  $\gamma$ -invariant, we see by Corollary 5.3 that the inclusion map  $i: T[\{u, w, v\}] \hookrightarrow T$  induces a  $\mathbb{Z}_2$ -equivariant homotopy equivalence  $i_H: \text{Hom}(T, K_n) \rightarrow \text{Hom}(T[\{u, w, v\}], K_n) = \text{Hom}(L_3, K_n)$ , where the  $\mathbb{Z}_2$ -action on the last space is induced from the  $\mathbb{Z}_2$ -action on  $L_3$  which swaps  $u$  and  $w$ .

Let  $\phi: L_3 \rightarrow K_2$  be any of the two elements of  $\text{Hom}_0(L_3, K_2)$ . Clearly,  $\phi$  is  $\mathbb{Z}_2$ -equivariant with the  $\mathbb{Z}_2$ -action on  $K_2$  being trivial. This shows that  $\phi_H: \text{Hom}(L_3, K_n) \rightarrow \text{Hom}(K_2, K_n) \simeq_{\mathbb{Z}_2} S_t^{n-2}$  is a  $\mathbb{Z}_2$ -equivariant homotopy equivalence, which finishes the proof. ■

Since taking the unlooped complement reverses neighbor set inclusions, we see that  $G$  reduces if and only if  $\overline{G}$  reduces. The next proposition describes what happens if  $G$  is a forest.

**PROPOSITION 5.6:** *If  $F$  is a forest, then  $\text{Hom}(\overline{F}, K_n) \simeq \text{Hom}(K_m, K_n)$ , where  $m$  is the maximal cardinality of an independent set in  $F$ .*

*Proof:* We use induction on the number of edges in  $F$ . If  $E(F) = \emptyset$ , then  $\overline{F} = K_{|V(F)|}$ , the maximal cardinality of an independent set in  $F$  is  $|V(F)|$ , and the statement is obvious. So assume  $|E(F)| \geq 1$ .

Let  $v \in V(F)$  be an arbitrary leaf, and let  $u \in V(F)$  be the vertex adjacent to  $v$ . We have  $N_{\overline{F}}(u) \subseteq V(F) \setminus \{u, v\} = N_{\overline{F}}(v)$ . Hence  $\overline{F}$  reduces to  $\overline{F} - u$ . Clearly,  $\overline{F} - u = \overline{F - u}$ , so by combining the induction assumption with Proposition 5.1 we get  $\text{Hom}(\overline{F}, K_n) \simeq \text{Hom}(K_{\tilde{m}}, K_n)$ , where  $\tilde{m}$  is the maximal cardinality of an independent set in  $F - u$ .

Let  $I$  be an independent set in  $F$  of maximal cardinality. Either  $u$  or  $v$  must be in  $I$ , since otherwise  $I \cup \{v\}$  is independent, and larger than  $I$ . If  $u \in I$ , then  $(I \setminus \{u\}) \cup \{v\}$  is also an independent set in  $F$  of maximal cardinality. Either way,

we have an independent set  $J$  in  $F$  of maximal cardinality containing  $v$ . Since any independent set in  $F - u$  is also independent in  $F$ , and  $J$  is independent in  $F - u$ , we can conclude that  $m = \tilde{m}$ , hence the result. ■

It follows from Proposition 4.5 that  $\text{Hom}(\overline{F}, K_n)$  is homotopy equivalent to a wedge of  $(n - m)$ -dimensional spheres.

## 6. Notes added in proof

After the appearance of the initial series of papers, [4, 5], and this paper, there has been a lot of further research activities pertaining to  $\text{Hom}$ -complexes. We refer the reader to the recent survey [21].

We update here on a few concrete developments which have taken place by the time of publication of this article.

- Conjecture 2.5 has been proved; see [13].
- Proposition 4.2 has been strengthened. It has been proved that  $\text{Hom}(K_2, G)$  and  $\mathcal{N}(G)$  have the same simple homotopy type. See [20] for the proof, and a description of an explicit formal deformation.
- Proposition 5.1 has been strengthened. It has been proved that the folds are admissible on both sides of the  $\text{Hom}(-, -)$ , and that one can replace the homotopy equivalences by formal deformations; see [19] for an explicit construction.

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