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A NOTE ON STATIONARITY OF SPHERICAL MEASURES

BY

ROMAN MUCHNIK*

Department of Mathematics, University of Chicago
5734 S. University St., Chicago, IL 60637, USA
e-mail: roma@math.uchicago.edu

ABSTRACT

In this short note we show that any smooth probability measure on the boundary $B(G)$ of a semisimple Lie group G is stationary for some probability measure on a lattice Γ. This generalizes a result of Furstenberg about Poisson boundaries for semisimple Lie groups.

1. Introduction

Approximately 35 years ago Furstenberg introduced the notion of Poisson boundary $(B(G), \pi)$ for a given pair (G, μ), where G is a group and $\mu \in P(G)$, the set of probability measures (see [F1]). One of the properties of $(B(G), \pi)$ is that π is μ-stationary. In this short note we describe how, by an appropriate choice of the measure μ, we can make measures $\pi \in P(G, \pi)$ μ-stationary.

Our approach is quite different from the approach of Furstenberg. In contrast to his method, where he uses the fact that the measure on the boundary must be an exit measure of some Brownian motion, we will use a very simple analytical argument. In [F2] Furstenberg proved the following theorem.

THEOREM 1.1: *Let Γ be a lattice in a semisimple connected Lie group G. Let μ_0 be an absolutely continuous measure on G given by $d\mu_0 = \phi_0 dg$, where $\phi_0(e) > 0$ and ϕ_0 is continuous and has compact support. Let $\nu_0 \in P(B(G))$ with $\mu_0 \star \nu_0 = \nu_0$. Then there exists $\mu \in P(\Gamma)$ with $\mu(\gamma) > 0$ for all $\gamma \in \Gamma$ and $\mu \star \nu_0 = \nu_0$.*

It is known that K acts transitively on $B(G)$ ([F2]), where K is a maximal compact subgroup of G. Therefore, the measure ν_0 in the above theorem is

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absolutely continuous with respect to Haar measure on K. With this in mind, we will reprove the above theorem, without the assumption that ν_0 is a stationary measure for some absolutely continuous measure μ_0 with respect to Haar on G. In particular we will prove:

THEOREM 1.2: *Let Γ be a lattice in a semisimple connected Lie group G. Let $B(G)$ be a G-space such that K acts transitively, where K is a maximal compact subgroup of G. Let π_K be the unique K-invariant measure on $B(G)$. If π_1, π_2 are two measures on $B(G)$ equivalent to π_K, then there exists $\mu \in P(\Gamma)$ such that $\mu(\gamma) > 0$ for all $\gamma \in \Gamma$ and $\pi_2 = \mu \star \pi_1 = \sum_{\gamma \in \Gamma} \mu(\gamma)\gamma^\star \pi_1$.*

As a corollary we obtain Furstenberg's Theorem without the assumptions mentioned earlier.

Understanding which measures can arise as a Poisson boundary measure is both desirable and difficult. In most cases, the only fact that we can state about such a measure is that it is a stationary measure for some random walk or Brownian motion. Theorem 1.2 is a first step (even though a small step) towards understanding Poisson boundary measures. In particular, any measure that is equivalent to the K-invariant measure can be a Poisson boundary measure. In particular, the set of Poisson boundaries is a "very rich" set, being at least as large as the set of continuous functions on $B(G)$. (Actually, we believe the space of Poisson boundaries is much bigger than the latter space, but this will be the subject of a future paper.)

In the next section, we introduce the notion of a convolution for K-spaces. In section 2, we prove a few propositions that explain how to make approximations on transitive K-spaces and how to discretize them. The main result is Corollary 3.4. In section 4, we prove Theorem 4.2, which lies at the heart of this note. This theorem is proved for arbitrary G-spaces for which K acts transitively. In section 5, we prove Theorem 1.2, by just checking the conditions of Theorem 4.2

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2. Convolution on X

Let X be a transitive K-space with K-invariant measure π_K and K-invariant metric. For two functions $f, g \in L^1(X)$ and a point $p \in X$, we define

$$(f *_p g)(x) = \int_K f(k^{-1}p)g(kx)dm_K(k),$$

where m_K is Haar measure on K. Also, for a measurable set $U \subset K$ define

$$(f *_p g)|_U(x) = \int_U f(k^{-1}p)g(kx)dm_K(k),$$

where m_K is Haar measure on K.

For a function $f \in L^q$, we denote L^q-norm of f by $\|f\|_q$.

LEMMA 2.1: *For a function $f(x) \in L^1$ on X we have*

$$\int_X f(kx)d\pi_K(x) = \int_K f(kx)dm_K(k) = \int_X f(x)d\pi_K(x).$$

The proof is just a simple application of the Fubini Theorem and is omitted.

LEMMA 2.2: *For every function $f(x) \in L^\infty(X)$, and $g(x) \in L^1(X)$, we have*

$$\|(f *_p g) - (f *_p g)|_U\|_1 \le m_K(U^c)\|f\|_\infty\|g\|_1.$$

Proof: By the Fubini Theorem

$$\begin{aligned}
\|(f *_p g) - (f *_p g)|_U\|_1 &= \int_X \int_{U^c} f(k^{-1}p)g(kx)dm_K(k)d\pi_K(x) \\
&= \int_{U^c} \int_X f(k^{-1}p)g(kx)d\pi_K(x)dm_K(k) \\
&= \|g\|_1 \int_{U^c} f(k^{-1}p)dm_K(k) \\
&\le m_K(U^c)\|f\|_\infty\|g\|_1.
\end{aligned}$$

This proves the lemma ∎

3. Proximate identities and discretization

PROPOSITION 3.1: *Let p be a point in X. Let $g(x)$ be a function such that $\int_{B_\delta(p)^c} g(x)d\pi_K(x) = t_g(\delta)$. Then for all continuous functions $f(x)$ we have*

$$|f(x) - (f *_p g)(x)| \le 2\|f\|_\infty t_g(\delta) + \|g\|_1 \max_{d(x,y)\le\delta}(|f(x) - f(y)|)$$

such that, for all $n > N$, we have $|F(x) - (f \star g_n)(x)| \le \epsilon$, independently of x.

Proof: Let $M = \max_{x\in X}|F(x)|$. Choose δ such that for all $x, y \in X$ with $d(x,y) \le \delta$, $|F(x) - F(y)| \le \epsilon$.

Let $A = \{k \in K: kx \in B_\delta(p)\} = \{k \in K: k^{-1}p \in B_\delta(x)\}$ for all $x \in A$. Now,

$$\begin{aligned}
|f(x) - (f *_p g)(x)| &= |f(x) - \int_K f(k^{-1}p)g(kx)dm_K(k)| \\
&= |\int_K (f(x) - f(k^{-1}p))g(kx)dm_K(k)| \\
&\le \int_K |f(x) - f(k^{-1}p)||g(kx)|dm_K(k) \\
&= \int_A |f(x) - f(k^{-1}p)||g(kx)|dm_K(k) \\
&\quad + \int_{A^c} |f(x) - f(k^{-1}p)||g(kx)|dm_K(k) \\
&\le \int_A |f(x) - f(k^{-1}p)||g(kx)|dm_K(k) \\
&\quad + 2\|f\|_\infty \int_K \chi_{B_\delta(p)^c}(kx)|g(kx)|dm_K(k) \\
&\le \|g\|_1 \max_{d(y,x)\le\delta} |f(x) - f(y)| + 2\|f\|_\infty t_g(\delta).
\end{aligned}$$

This finishes the proof. ∎

Lemma 3.2 (Discretization): *Let $U \subset K$ and $g(x)$ be a continuous function on X. For every $\epsilon > 0$, there exists $k_1, \ldots, k_s \in U$ and $\alpha_1, \ldots, \alpha_s \ge 0$ such that for every $f \in L^1$ we have*

$$|(f *_p g)|_U(x) - \sum_{i=1}^{s} \alpha_i g(k_i x)| \le \epsilon \|f\|_1,$$

with $\sum_{i=1}^{s} \alpha_i = \int_U f(k^{-1}x_0)dm(k) \le \|f\|_1$.

Proof: The function $H(k,x) = g(kx) - g(x)$ is continuous. Therefore, $H^{-1}(-\epsilon/2, \epsilon/2)$ is a an open set of $K \times X$. It is clear that $id \times X \in H^{-1}(-1/n, 1/n)$.

Thus, there exists an open set $W \subset K$ such that

$$W \times \{x\} \subset H^{-1}(-\epsilon/2, \epsilon/2)$$

for all $x \in X$. It is easy to observe that if $k, h \in W$ we have

$$|g(hx) - g(kx)| \le |g(hx) - g(x)| + |g(kx) - g(x)| \le \epsilon.$$

Since K is compact we can choose finitely many $h_1, h_2, \ldots, h_s$ such that $Wh_1, Wh_2, \ldots, Wh_s$ cover K. Now in each Wh_i such that $Wh_i \cap U \ne \emptyset$ we choose an element $k_i \in U$. Observe that for $t_i \in Wk_i$ we have

$$g(t_i x) - g(k_i x) = g_n((t_1 h_i^{-1})h_i x) - g_n((k_i h_i^{-1})(h_i x)) \leq \epsilon,$$

because $t_1 h_i{}^{-1}, k_i h_i^{-1} \in W$.

We can find a measurable disjoint partition $V_1, \ldots, V_s$ of K such that $V_i \subset Wk_i$ for all $i \in \{1, \ldots, s\}$.

Now consider the function

$$\sum_{i=1}^{s} g(k_i x) \int_{V_i \cap U} f(k^{-1}x_0) dm_K(k).$$

Now compare this sum to $(f *_p g)|_U(x)$.

$$\begin{aligned} &\left| \int_U f(k^{-1}p)g(kx)dm_K(k) - \sum_{i=1}^{s} g(k_i x) \int_{V_i \cap U} f(k^{-1}p)dm_K(k) \right| \\ =&\sum_{i=1}^{s} \int_{V_i \cap U} |(g(kx) - g(k_i x))||f(k^{-1}p)|dm_K(k) \\ \leq&\sum_{i=1}^{s} \int_{V_i \cap U} \epsilon |f(kx)|dm_K(k) \leq \epsilon |f(x)|_1. \end{aligned}$$

Set $\alpha_i = \int_{V_i \cap U} f(k^{-1}x_0)dm_K(k)$. It is clear that

$$\sum_{i=1}^{s} \alpha_i = \int_U f(k^{-1}x_0)dm_K(k) \leq \int_K f(k^{-1}x_0)dm_K(k) = \|f\|_1. \quad \blacksquare$$

Corollary 3.3: *Let f be a positive function on X with $\inf_{x \in X} f(x) = a > 0$. Assume that g is a continuous positive function. Then for all $C < 1$ there exist $k_1, \ldots, k_s$ and $\alpha'_1, \ldots, \alpha'_s$ such that for all $x \in X$,*

$$C(f \star_p g)|_U(x) \leq \sum_{i}^{s} \alpha'_i g(k_i x) \leq (f \star_p g)|_U(x).$$

Proof: Assume that $(f \star_p g)(x) \geq b > 0$ for all x. Set

$$\epsilon = \frac{1-C}{1+C} \frac{b}{\|f\|_1},$$

so that by Lemma 3.2 we have $\alpha_1, \ldots, \alpha_s$ and $k_1, \ldots, k_s \in U$ such that

$$|(f \star_p g)|_U(x) - \sum_{i}^{s} \alpha_i g(k_i x)| \leq \frac{1-C}{1+C} b.$$

In other words,

$$\sum_i^s \alpha_i g(k_i x) \le \frac{1-C}{1+C} b + f(x) \le \frac{2}{1+C} f(x),$$

and

$$\sum_i^s \alpha_i g(k_i x) \ge f(x) - \frac{1-C}{1+C} b \ge f(x) \frac{2C}{1+C}.$$

Setting $\alpha_i = (1+C)\alpha_i/2$ we obtain the corollary.

If $0 = (f \star_p g)|_U(x) = \int_U f(k^{-1}p) g(kx) dm_K(k) \ge a \int_U g(kx) dm_K(k)$, then $m_K(U) = 0$ as g is positive. Therefore, $(f \star_p g)|_U(x) = 0$ for all x. In this case, the corollary is obvious. ∎

COROLLARY 3.4: *Let f and g be continuous positive functions with $\inf_{x\in X} f(x) = a > 0$. Assume that*

$$R = R(f, g, \delta) = 2\|f\|_\infty t_g(\delta) + \|g\|_1 \max_{d(x,y)\le\delta} |f(x) - f(y)|$$

for some $\delta \ge 0$. Fix $U \subset K$. Then there exist $k_1, \dots, k_s \in U$ and $\alpha_1, \dots, \alpha_s \ge 0$ such that

$$\sum_{i=1}^s \alpha_i g(k_i x) \le f(x),$$

and

$$\|f(x) - \sum_{i=1}^s \alpha_i g(k_i x)\|_1 \le \frac{R(a + \|f\|_1) + a(m_K(U^c) + (1-C))\|f\|_\infty \|g\|_1}{R+a}.$$

In particular,

$$\|f(x) - \sum_{i=1}^s \alpha_i g(k_i x)\|_1 \le \frac{2R}{R+a}\|f\|_1 + \frac{a}{R+a} m_K(U^c)\|f\|_\infty \|g\|_1.$$

Proof: By Corollary 3.3 there exist $\alpha'_1, \dots, \alpha_{s'} \ge 0$ and $k_1, \dots, k_s \in K$ such that

$$C(f \star_p g)|_U(x) \le \sum_{i=1}^s \alpha'_i g(k_i x) \le (f \star_p g)|_U(x) \le (f \star_p g)(x).$$

By Proposition 3.1, we have

$$(f \star_p g)(x) \le f(x) + R \le (1 + R/a) f(x).$$

So $\sum_{i=1}^{s}\alpha'_i g(k_i x) \le (1+R/a)f(x)$. Observe that

$$\begin{aligned}
&\Big\|\Big(1+\frac{R}{a}\Big)f(x)-\sum_{i=1}^{s}\alpha'_i g(k_i x)\Big\|_1\\
\le&\Big\|\Big(1+\frac{R}{a}\Big)f(x)-(f\star_p g)(x)\Big\|_1+\|(f\star_p g)(x)-(f\star_p g)|_U(x)\|_1\\
&+\|(f\star_p g)|_U(x)-\sum_{i=1}^{s}\alpha'_i g(k_i x)\|_1\\
\le&\Big(\frac{R}{a}\|f\|_1+R\Big)+m_K(U^c)\|f\|_\infty\|g\|_1+(1-C)\|(f\star_p g)|_U\|_1\\
\le&R+\frac{R}{a}\|f\|_1+(m_K(U^c)+(1-C))\|f\|_\infty\|g\|_1.
\end{aligned}$$

Setting $\alpha_i=\frac{a}{R+a}\alpha'_i$ and observing that $a\le\|f\|_1$, we obtain the first estimate.

Set $C=2m_K(U^c)$ and use that $a\le\|f\|_1$. This gives us the second estimate. ∎

4. Main theorem

LEMMA 4.1: *Let G be a connected semisimple Lie group. Γ is a lattice in G and K is a maximal compact subgroup. Let m_K be the Haar measure on K. For an open set U define*

$$f_U(g)=m_K\{h\in K\colon g^{-1}h\Gamma\cap U\Gamma\neq\emptyset\}.$$

Then, for any sequence $g_i\in G$ such that $g_i^{-1}\to\infty$ in G/K, we have

$$\lim_{i\to\infty}f_U(g_i)=m_{G/\Gamma}(U/\Gamma),$$

where $m_{G/\Gamma}$ is the G-invariant measure on G/Γ.

Proof: Observe that $f_U(g)=\int_K\chi_{U/\Gamma}(g^{-1}h\Gamma)dm_K(h)$ and

$$m_{G/\Gamma}(U/\Gamma)=\int_{G/\Gamma}\chi_{U/\Gamma}(g\Gamma)dm_{G/\Gamma}(g\Gamma).$$

The result easily follows from the equidistribution result proved in [EM] because

$$\int_K\chi_{U/\Gamma}(g_i^{-1}h\Gamma)dm_K(h)\to m_{G/\Gamma}(U/\Gamma)$$

as $g_i\to\infty$. The lemma follows. ∎

THEOREM 4.2: *Let G be a semisimple Lie group, and X be a G-space on which K acts transitively. Let π_K be a K-invariant measure on X. Assume that for each $g \in G$,*

$$\alpha(g,x) = \frac{dg_\star \pi_K}{d\pi_K}(x)$$

is a positive continuous function. Assume that there exists a sequence g_i such that $(g_i)_\star \pi_K = \delta_p$ for some $p \in X$. Let Γ be a lattice in G. Assume that $\{f_\gamma(x)\}_{\gamma \in \Gamma}$ is a set of functions such that there exists $D > 0$ with the property that

$$D^{-1} \leq f_\gamma(x) \leq D,$$

for all $\gamma \in \Gamma$. Then for all positive continuous functions F on X, there exist $\lambda_\gamma > 0$ so that we have $F(x) = \sum_{\gamma\in\Gamma} \lambda_\gamma f_\gamma(x)\alpha(\gamma,x)$ in $L^1(X,\pi_K)$ with $\lambda_\gamma \geq 0$. Moreover, if Γ is co-compact, we can find $\lambda_\gamma > 0$ such that the convergence is uniform.

Proof: Recall that $\alpha(g,x)$ is a K-invariant co-cycle, i.e.

$$\alpha(gh,x) = \alpha(g,hx)\alpha(h,x)$$

and $\alpha(k,x) = 1$ for all $k \in K$. For any $U \subset G$ we define

$$E(U) = \{h \in K \colon g_i h\Gamma \cap U\Gamma \neq \emptyset\} \subset K.$$

Set $C(U) = \max_{g\in U, x\in X}(\alpha(g,x), \alpha(g^{-1},x)) + 1$. (Recall that $\alpha(g^{-1},x) = \alpha(g,g^{-1}x)^{-1}$.)

Fix a sequence of open bounded subsets $U_n \subset G$ (not necessary distinct) such that $\lim_{n\to\infty} m_{G/\Gamma}(U_n/\Gamma) = 1$ and $\sum_{n=1}^{\infty} 1/C(U_n)^2 = \infty$. (In the co-compact case we set $U \subset G$ to be a bounded open set such that $U/\Gamma = G/\Gamma$.)

Now we will construct a sequence $\{L_n(x)\}$ of positive continuous functions such that

$$L_n(x) = F(x) - \sum_{\gamma\in\Gamma} c_\gamma f_\gamma(x)\alpha(\gamma,x)$$

with $c_\gamma \geq 0$ and only finitely many of them are positive and

$$\lim_{n\to\infty} \|L_n\|_1 = 0.$$

(In the co-compact case $\lim_{n\to\infty} L_n(x) = 0$ uniformly.) It is clear that this will prove the theorem.

Set $L_0 = F$. Assume that we have constructed $L_n(x)$. Let us describe L_{n+1}. Let $a_n = \inf_{x\in X}(L_n(x))$.

It is clear that $\|L_n\|_\infty \le \|F\|_\infty = M$. Set $\delta_n > 0$ to be a number such that $\max_{d(x,y)\le\delta_n} |L_n(x) - L_n(y)| \le a_n/4$.

Since $(g_i)_\star\pi_K \to \delta_p$, there exists I_n such that for all $i > I_n$ we have

$$(g_i)_\star\pi_K(B(p,\delta_n)) \ge 1 - a_n/8M,$$

i.e.,

$$t_{\delta_n}(\alpha(g_i,x)) \le a_n/8M$$

for all $i > I_n$.

Define

$$W_n = \{h \in K\colon g_{j_n}h\Gamma \cap U_n\Gamma \ne \emptyset\}.$$

(In the co-compact case, $W_n = K$.) By Lemma 4.1 there exists $j_n > I_n$ such that $m_K(W_n) = f_{U_n}(g_{j_n}) \ge m_{G/\Gamma}(U_n/\Gamma) - 1/n$.

Apply Corollary 3.4 to the functions $L_n(x)$ and $\alpha(g_{j_n},x)$. We have

$$R(L_n, \alpha(g,\cdot), \delta_n) \le 2\|F\|_\infty \frac{a_n}{8M} + \frac{a_n}{4} \le \frac{a_n}{2}.$$

By Corollary 3.4, there exists a sequence of $k_1^{(n)},\dots,k_{s_n}^{(n)} \in W_n$ and $c_1^{(n)},\dots,c_{s_n}^{(n)} > 0$ such that

$$\sum_{i=1}^{s_n} c_i^{(n)}\alpha(g_{j_n},k_ix) \le L_n(x),$$

and

$$\left\| L_n(x) - \sum_{i=1}^{s_n} c_i^{(n)}\alpha(g_{j_n},k_ix)\right\|_1 \le \frac{2}{3}\|L_n\|_1 + \frac{2}{3}m_K(W_n^c)M.$$

(In the co-compact case, again we have

$$R(L_n, \alpha(g,\cdot), \delta_n) \le 2\|F\|_\infty \frac{a_n}{8M} + \frac{a_n}{4} \le \frac{a_n}{2}.$$

Now by Proposition 3.1, $|L_n(x) - (L_n \star_p \alpha(g_{j_n},\cdot))(x)| \le a_n/2$. This implies that $L_n(x)/2 \le (L_n \star_p \alpha(g_{j_n},\cdot))(x) \le 2L_n(x)$. Now we apply Corollary 3.3 to the functions $L_n(x)$ and $\alpha(g_{j_n},x)$ and $C = \frac{1}{2}$, to obtain a sequence $k_1^{(n)},\dots,k_{s_n}^{(n)} \in W_n$ and $c_1^{(n)},\dots,c_{s_n}^{(n)} > 0$ such that

$$\frac{(L_n \star_p \alpha(g_{j_n},\cdot))(x)}{2} \le \sum_{i=1}^{s_n} c_i^{(n)}\alpha(g_{j_n},k_ix) \le (L_n \star_p \alpha(g_{j_n},\cdot))(x).$$

Dividing by 2 and renaming $c_i^{(n)}$, we obtain $L_n(x)/4 \le \sum_{i=1}^{s_n} c_i^{(n)}\alpha(g_{j_n},k_ix) \le L_n(x)$.)

Since $k_i^{(n)} \in W_n$, we have that there exists $\gamma_i^{(n)} \in \Gamma$ and $u_i^{(n)} \in U_n$ such that $g_{j_n} k_i^{(n)} = u_i^{(n)} \gamma_i^{(n)}$.

So we have

$$\alpha(g_{j_n}, k_i x) = \alpha(u_i^{(n)} \gamma_i^{(n)} k_i^{-1}, k_i x) = \alpha(u_i^{(n)}, \gamma_i^{(n)} x) \alpha(\gamma_i^{(n)}, x).$$

Here we have that

$$\frac{1}{C(U_n)} < \frac{\alpha(g_{j_n}, k_i x)}{\alpha(\gamma_i^{(n)}, x)} < C(U_n).$$

So set

$$L_{n+1}(x) = L_n(x) - \frac{1}{DC(U_n)} \sum_{i=1}^{s_n} c_i f_\gamma(x) \alpha(\gamma_i^{(n)}, x).$$

Now it is easy to see that

$$L_n(x) - \frac{1}{D^2 C(U_n)^2} \sum_{i=1}^{s_n} c_i^{(n)} f_\gamma(x) \alpha(g_{j_n}, k_i x) \geq L_{n+1}(x)$$

and

$$L_{n+1}(x) > L_n(x) - \sum_{i=1}^{s_n} c_i^{(n)} f_\gamma(x) \alpha(g_{j_n}, k_i x) \geq 0.$$

Now it is clear that L_{n+1} is a positive continuous function. Also

$$\begin{aligned} \|L_{n+1}\|_1 \leq & \left\| L_n(x) - \frac{1}{D^2 C(U_n)^2} \sum_{i=1}^{s_n} c_i^{(n)} \alpha(g_{j_n}, k_i x) \right\|_1 \\ \leq & \left\| L_n(x) - \frac{L_n(x)}{D^2 C(U_n)^2} \right\|_1 \\ & + \frac{1}{D^2 C(U_n)^2} \left\| L_n(x) - \sum_{i=1}^{s_n} c_i^{(n)} \alpha(g_{j_n}, k_i x) \right\|_1 \\ \leq & \left(1 - \frac{2}{3D^2 C(U_n)^2}\right) \|L_n\|_1 + \frac{2 m_K(W_n^c) M}{3 D^2 C(U_n)^2}. \end{aligned}$$

(In the co-compact case,

$$L_{n+1}(x) = L_n(x) - \frac{\sum_{i=1}^{s_n} c_i^{(n)} \alpha(g_{j_n}, k_i x)}{D^2 C(U)^2} \leq L_n(x) \Big(1 - \frac{1}{4D^2 C(U)^2}\Big).$$

This proves that $\lim_{n\to\infty} L_n(x) = 0$ uniformly.)

Now we need the following lemma.

LEMMA 4.3: *Let $0 \le \delta_n \le 1$ for all n and $\lim_{n\to\infty} \epsilon_n = 0$. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of non-negative numbers such that*

$$a_{n+1} \le (1-\delta_n)a_n + \delta_n \epsilon_n.$$

If $\lim_{n\to\infty} \prod_{k=1}^{n}(1-\delta_k) = 0$, then $\lim_{n\to\infty} a_n = 0$.

Proof: For $\epsilon > 0$, there exists N such that for all $n > N$, $\epsilon_n \le \epsilon$. Let $a_n = b_n + \epsilon$, so we have

$$b_{n+1} + \epsilon \le (1-\delta_n)(b_n+\epsilon) + \delta_n\epsilon_n \le (1-\delta_n)b_n + \epsilon,$$

for all $n > N$. This shows that $b_{n+1} \le (1-\delta_n)b_n$. So we conclude that $\lim_{n\to\infty} b_n = 0$ and thus $0 < \lim_{n\to\infty} a_n \le \epsilon$. However, taking $\epsilon \to 0$ proves the lemma. ∎

So to obtain $\lim_{n\to\infty} \|L_n\|_1 = 0$ it suffices to have

$$\prod_{i=1}^{\infty}\Big(1 - \frac{2}{3C(U_n)^2}\Big) = 0,$$

which is equivalent to $\sum_{i=1}^{\infty} 1/C(U_n)^2 = \infty$ and

$$\lim_{n\to\infty} m_k(W_n^c) \le \lim_{n\to\infty} m_{G/\Gamma}((U_n/\Gamma)^c) + 1/n = 0.$$

This finishes the proof of Theorem 4.2. ∎

COROLLARY 4.4: *Let π_1, π_2 be two probability measures on X such that $\frac{d\pi_i}{d\pi_K}(x)$ is a positive continuous function for $i = 1, 2$. Then there exists $\mu \in P(\Gamma)$ such that $\mu \star \pi_1 = \pi_2$.*

Proof: The equation $\mu \star \pi_1 = \pi_2$ is equivalent to

$$\frac{d\pi_2}{d\pi_K}(x) = \sum_{\gamma\in\Gamma} \mu(\gamma)\frac{d\pi_1}{d\pi_K}(\gamma x)\Big(\frac{d\pi_1}{d\pi_K}(x)\Big)^{-1}\alpha(\gamma, x),$$

where convergence is in the $L^1(X, \pi_K)$-norm. This proves the corollary. ∎

Example: One of the consequences of Theorem 4.2 happens in the case of $n = 2$ and $G = \mathrm{PSL}(2, \mathbb{R})$. In this case, G/P can be identified with the circle S^1 and can be viewed as the boundary of the 2-dimensional disk G/K. The

K-invariant measure is just the Lebesgue measure m on S^1, and the Radon–Nikodym derivative of this measure with respect to the transformation $g \in G$ is just the Poisson kernel at the point go, where $o = [K]$ is the origin.

In this case, we obtain that every positive continuous function can be represented as a sum with positive coefficients of Poisson kernels at the points γo, where $\gamma \in \Gamma$ for a lattice Γ in $\mathrm{PSL}(2,\mathbb{R})$ (i.e., $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$). It was known by the results of F. Bonsall (see [B] or [R]) that any L^1 function can be approximated by a sum of Poisson kernels (in particular, in the case of $\mathrm{PSL}(2,\mathbb{Z})$). However, his result does not imply that the coefficients are positive for a positive continuous function. So Theorem 4.2 is an improvement of this result in the cases of lattices. I should note that W. Hayman and T. Lyons [HL] and later F. Bonsall and D. Walsh [BW] have improved Bonsall's result and have shown that every continuous (lower semi-continuous) function can be approximated uniformly as a sum of Poisson kernels if and only if

$$\sum_{g\in\Gamma} \left\| \frac{dg^\star m}{dm} \right\|_\infty^{-1} \frac{dg^\star m}{dm}(\zeta) = +\infty,$$

for all $\zeta \in S^1$ (this is in the case of $\mathrm{PSL}(2,\mathbb{R})$).

However, this condition fails even for $\mathrm{PSL}(2,\mathbb{Z})$. In the special case of a cocompact lattice Γ in $\mathrm{PSL}(2,\mathbb{R})$, our result does follow from the work of W. Hayman and T. Lyons as the above sum diverges at every point.

Even though Theorem 4.2 (in case $n = 2$) is not as general as the framework considered by the authors mentioned above, it does provide a unifying scheme for higher dimensions and work of Bonsall (as well as Walsh, Lyons, Hayman), in which every point (or almost every point) on the boundary is non-tangentially dense.*

5. Proof of Theorem 1.2

We just need to verify the conditions of Theorem 4.2 for $B(G)$. It is known that for any measure, $\pi \in P(B(G))$, we have $\delta_{B(G)} \subset \overline{G\pi}$. For the proof of continuity of the Radon–Nikodym derivatives, see Lemma 5.6 of [F1]. However, this can be observed from the smoothness of the action by G on the manifold representing the homogeneous space G.

* These cases are considered in [CM].

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