ISRAEL JOURNAL OF MATHEMATICS 152 (2006), 17–28

A LOGARITHMIC LOWER BOUND FOR MULTI-DIMENSIONAL BOHR RADII

BY

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ABSTRACT

We prove that the Bohr radius K_n of the n-dimensional polydisc in $\mathbb{C}^n$ is up to an absolute constant $\geq \sqrt{\frac{\log n/\log\log n}{n}}$. This improves a result of Boas and Khavinson.

1. Introduction

Let $R \subset \mathbb{C}^n$ be a Reinhardt domain (i.e., a domain in $\mathbb{C}^n$ such that for $u, v \in \mathbb{C}^n$ with $|u_k| \leq |v_k|, 1 \leq k \leq n$, we have that $u \in R$ provided that $v \in R$). Recall that the Bohr radius $K(R)$ of R is the supremum over all $r \geq 0$ such that if $|\sum_\alpha c_\alpha z^\alpha| \leq 1$ for all $z \in R$, then $\sum_\alpha |c_\alpha z^\alpha| \leq 1$ for all $z \in rR$. Bohr's power series theorem from [6] states that

$$K(\mathbb{D}) = \frac{1}{3}, \tag{1.1}$$

where $\mathbb{D}$ as usual denotes the open unit disc. By results of Aizenberg, Boas, Dineen, Khavinson and Timoney (see [1], [2], [3], [12]) for all $1 \leq p \leq \infty$ and all n,

$$\frac{1}{c}\left(\frac{1}{n}\right)^{1-\frac{1}{\min(p,2)}} \leq K(B_{\ell_p^n}) \leq c\left(\frac{\log n}{n}\right)^{1-\frac{1}{\min(p,2)}}, \tag{1.2}$$

Received July 13, 2004

where $c > 0$ denotes a constant independent of p, n. So far, all known non-trivial upper estimates for multi-dimensional Bohr radii use probabilistic methods, and the log-term in (1.2) is a consequence of these methods. Boas in [2, p. 329] conjectures that *presumably this logarithmic factor, an artifact of the proof, should not really be present* (see also [2, section 7, problem 1] and the discussion in [3]). Contrary to this commonly held opinion, the main result of this paper shows that the log-term in the dimension n at least up to a term $\log\log n$ necessarily appears.

THEOREM 1.1: *There is a constant $c > 0$ such that for each $1 \leq p \leq \infty$ and all n,*

$$\frac{1}{c}\Big(\frac{\log n/\log\log n}{n}\Big)^{1-\frac{1}{\min(p,2)}} \leq K(B_{\ell_p^n}).$$

Let us give a brief idea of what might have been Bohr's original motivation for his power series theorem (1.1). It is well known that the domain of convergence of a Dirichlet series $\sum a_n \frac{1}{n^s}$, $s \in \mathbb{C}$ is characterized by three half planes $\{\operatorname{Re} s > a\} \subset \{\operatorname{Re} s > u\} \subset \{\operatorname{Re} s > c\}$ in $\mathbb{C}$; $\{\operatorname{Re} s > c\}$ defines the largest half plane on which the Dirichlet series converges, $\{\operatorname{Re} s > u\}$ the largest half plane such that the series converges uniformly on each strictly smaller half plane, and finally $\{\operatorname{Re} s > a\}$ the largest half plane where the Dirichlet series even converges absolutely. Bohr in [5] did an intensive study of the number

$$S := \sup_{\sum a_n \frac{1}{n^s}} a - u.$$

When he finished his article he did not know a single example $\sum a_n \frac{1}{n^s}$ for which $u \neq a$; note that, in contrast to this, it is relatively easy to show that $\sup a - c = 1$.

In a rather ingenious fashion Bohr translated the problem of finding the precise value of S into a problem formulated entirely in terms of power series in infinitely many variables. His main trick was to consider the following one-to-one correspondence between Dirichlet series and power series in infinitely many variables:

$$(1.3) \qquad \sum_n a_n \frac{1}{n^s} \leftrightsquigarrow \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^\alpha, \quad \text{where } a_n = c_\alpha \text{ if } n = p^\alpha;$$

here, p stands for the sequence $p_1 \leq p_2 \leq \cdots$ of prime numbers. With this translation at hand Bohr in [5] managed to show that $S \leq 1/2$, and he apparently proved his power series theorem (a result in only one variable) while trying to show that this upper estimate for S is best possible. It seems that Bohr was aware of the fact that an optimal multi-dimensional version of (1.1) would have led him to an optimal estimate for S.

Based on the identification from (1.3) together with a polynomial version of a well-known result of Littlewood, Toeplitz in [16] obtained that $1/4 \leq S \leq 1/2$, and finally Bohnenblust and Hille in [12] completed this approach of Toeplitz by proving that S in fact equals $1/2$.

Within their study of absolute bases in spaces of holomorphic functions on infinite-dimensional spaces, Dineen and Timoney in [12] renewed the interest in multi-dimensional Bohr radii. They proved that for each $\varepsilon > 0$,

$$K(B_{\ell_\infty^n}) \leq c(\varepsilon)\sqrt{\frac{n^\varepsilon}{n}}, \quad n \in \mathbb{N}$$

(a result slightly weaker than the one cited in (1.2)). This in [13] allowed them to reprove the Bohr–Bohnenblust–Hille result $S = 1/2$ with a proof which might be very close to what Bohr himself originally had in mind.

2. Preliminaries

We use standard notations and notions from Banach space theory, as presented, e.g., in [15] and [8]. All considered Banach spaces are assumed to be complex. As usual ℓ_p^n, $1 \leq p \leq \infty$ and $n \in \mathbb{N}$, stands for $\mathbb{C}^n$ together with the p-norm $\|z\|_p := (\sum_{k=1}^n |z_k|^p)^{1/p}$ (with the obvious modification whenever $p = \infty$).

Recall that the Banach–Mazur distance of two n-dimensional Banach spaces X and Y is given by $d(X,Y) := \inf \|R\|\|R^{-1}\|$, the infimum taken over all linear bijections R: $X \longrightarrow Y$ (see [17]). A Schauder basis (x_n) of a Banach space X is said to be unconditional if there is a constant $c \geq 1$ such that $\|\sum_{k=1}^n |a_k| x_k\| \leq c\|\sum_{k=1}^n a_k x_k\|$ for all n and $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$. In this case, the best constant c is denoted by $\chi((x_n))$ and called the unconditional basis constant of (x_n). Moreover, the infimum over all possible constants $\chi(x_n)$ is the unconditional basis constant $\chi(X)$ of X.

See [8], [11], and [14] for all needed background on polynomials and symmetric tensor products. If $X = (\mathbb{C}^n, \|\cdot\|)$ is a Banach space and $m \in \mathbb{N}$, then $\mathcal{P}(^m X)$ stands for the Banach space of all m-homogeneous polynomials $p(z) = \sum_{|\alpha|=m} c_\alpha z^\alpha, z \in \mathbb{C}^n$, together with the norm $\|p\|_{\mathcal{P}(^m X)} := \sup_{\|z\|\leq 1} |p(z)|$. The unconditional basis constant of all monomials z^α, $|\alpha| = m$, is denoted by $\chi_{\mathrm{mon}}(\mathcal{P}(^m X))$.

Sometimes it will be more convenient to think in terms of tensor products instead of spaces of polynomials. We write $\otimes_\varepsilon^m X$ for the mth full injective tensor product, and $\otimes_{\varepsilon_s}^{m,s}X$ for the mth symmetric injective tensor product. Recall that $\otimes^{m,s}X$ can be realized as the range of the symmetrization operator

$$S\colon \otimes^m X \longrightarrow \otimes^m X, \quad S(\otimes y_k) := \frac{1}{m!}\sum_{\sigma\in\Pi_m} \otimes y_{\sigma(k)},$$

where Π_m stands for the group of all permutations of $\{1,\dots,m\}$. For $z \in \otimes^{m,s}X$ we have

$$\|z\|_{\varepsilon_s} \le \|z\|_{\varepsilon} \le \frac{m^m}{m!}\|z\|_{\varepsilon_s} \tag{2.1}$$

(see, e.g., [14, p. 167]). If e_k, $1 \le k \le n$, denotes the standard basis in $\mathbb{C}^n$, then the monomials $e_i = e_{i_1}\otimes\cdots\otimes e_{i_m}$, where $i = (i_1,\dots,i_m) \in M(m,n) := \{1,\dots,n\}^m$, form a (linear) basis of $\otimes^m X$, and its unconditional basis constant with respect to the injective norm ε is denoted by $\chi_{\mathrm{mon}}(\otimes_\varepsilon^m X)$. Analogously, all $S(e_{i_1}\otimes\cdots\otimes e_{i_n})$, $(i_1,\dots,i_m) \in J(m,n) := \{(i_1,\dots,i_m) \in M(m,n) \mid i_1 \le \cdots \le i_m\}$, form a basis of $\otimes^{m,s}X$, and clearly $\chi_{\mathrm{mon}}(\otimes_{\varepsilon_s}^{m,s})$ stands for its unconditional basis constant with respect to ε_s.

Finally, we recall that for $X = (\mathbb{C}^n, \|\cdot\|)$ the following isometric equality holds:

$$\otimes_{\varepsilon_s}^{m,s}X^* = \mathcal{P}(^mX), \quad x^*\otimes\cdots\otimes x^* \rightsquigarrow [z \rightsquigarrow x^*(z)^m]; \tag{2.2}$$

here, X^* denotes the dual of X (see again, e.g., [14, p. 168] or [11]).

3. The proof of Theorem 1.1

The proof of Theorem 1.1 is based on the following basic link between the study of multi-dimensional Bohr radii and local Banach space theory from [9, Theorem 2.2]: For each Banach space $X = (\mathbb{C}^n, \|\cdot\|)$ for which the e_k's form a 1-unconditional basis (i.e., $\chi(e_k) = 1$) we have

$$\frac{1}{3\sup_m \chi_{\mathrm{mon}}(\mathcal{P}(^mX))^{1/m}} \le K(B_X) \le \min\left(\frac{1}{3}, \frac{1}{\sup_m \chi_{\mathrm{mon}}(\mathcal{P}(^mX))^{1/m}}\right); \tag{3.1}$$

note that this result is an abstract extension of Bohr's power series theorem from (1.1).

The following three lemmata concentrate on upper estimates for the unconditional basis constants $\chi_{\mathrm{mon}}(\mathcal{P}(^mX))$ (with "good" constants in the degree m and the dimension n). The first one improves part of [9, 6.1].

LEMMA 3.1: *Let $X = (\mathbb{C}^n, \|\cdot\|)$ be a Banach space such that the e_k's form a 1-unconditional basis. Then for each m*

$$\chi_{\mathrm{mon}}(\mathcal{P}(^m X)) \leq \frac{m^m}{m!} 2^{m+1} d(X, \ell_1^n)^{m-1}.$$

Equivalently, we have by (2.2) and duality that

$$\chi_{\mathrm{mon}}(\otimes_{\varepsilon_s}^{m,s} X)) \leq \frac{m^m}{m!} 2^{m+1} d(X, \ell_\infty^n)^{m-1}. \tag{3.2}$$

Proof: Take $(\alpha_j) \in \mathbb{C}^{J(m,n)}$. For the proof of (3.2) we have to show that

$$\Big\| \sum_{J(m,n)} |\alpha_j| S(e_j) \Big\|_{\varepsilon_s} \leq \frac{m^m}{m!} 2^{m+1} d(X, \ell_\infty^n)^{m-1} \Big\| \sum_{J(m,n)} \alpha_j S(e_j) \Big\|_{\varepsilon_s}.$$

From [7, pp. 134, 136] we know that

$$\chi_{\mathrm{mon}}(\otimes_\varepsilon^m X) \leq 2^{m+1} \chi(\otimes_\varepsilon^m X) \tag{3.3}$$

and

$$\chi(\otimes_\varepsilon^m X) \leq d(X, \ell_\infty^n)^{m-1}. \tag{3.4}$$

Moreover, we will use the fact (see [7, p. 124]) that

$$S(e_j) = \frac{1}{\mathrm{card}[j]} \sum_{\substack{i \in M(m,n) \\ i \in [j]}} e_i, \quad j \in J(m,n); \tag{3.5}$$

recall that $[j]$ denotes the equivalence class in $M(m,n)$ defined by the equivalence relation: $j \sim i$: $\iff$ there is a permutation σ of $\{1, \ldots, m\}$ such that $j(k) = i(\sigma(k))$, $1 \leq k \leq m$.

Define $(\xi_i) \in \mathbb{C}^{M(m,n)}$ by $\xi_i := \frac{\alpha_j}{\mathrm{card}[j]}$, $i \in [j]$, $j \in J(m,n)$. Then we get

$$\begin{aligned}
\Big\| \sum_{j \in J(m,n)} |\alpha_j| S(e_j) \Big\|_{\varepsilon_s} &\overset{(3.5)}{=} \Big\| \sum_{j \in J(m,n)} |\alpha_j| \frac{1}{\mathrm{card}[j]} \sum_{\substack{i \in M(m,n) \\ i \in [j]}} e_i \Big\|_{\varepsilon_s} \\
&\overset{(2.1)}{\leq} \Big\| \sum_{j \in J(m,n)} \sum_{\substack{i \in M(m,n) \\ i \in [j]}} |\alpha_j| \frac{1}{\mathrm{card}[j]} e_i \Big\|_{\varepsilon} \\
&= \Big\| \sum_{i \in M(m,n)} |\xi_i| e_i \Big\|_{\varepsilon} \\
&\leq \chi_{\mathrm{mon}}(\otimes_\varepsilon^m X) \Big\| \sum_{i \in M(m,n)} \xi_i e_i \Big\|_{\varepsilon}
\end{aligned}$$

$$\overset{(3.3)}{\leq} 2^{m+1}\chi(\otimes_\varepsilon^m X)\Big\|\sum_{i\in M(m,n)}\xi_i e_i\Big\|_\varepsilon$$
$$\overset{(3.4)}{\leq} 2^{m+1}d(X,\ell_\infty^n)^{m-1}\Big\|\sum_{i\in M(m,n)}\xi_i e_i\Big\|_\varepsilon$$
$$\overset{(3.5)}{\leq} 2^{m+1}d(X,\ell_\infty^n)^{m-1}\Big\|\sum_{j\in J(m,n)}\alpha_j S(e_j)\Big\|_\varepsilon$$
$$\overset{(2.1)}{\leq} \frac{m^m}{m!}2^{m+1}d(X,\ell_\infty^n)^{m-1}\Big\|\sum_{j\in J(m,n)}\alpha_j S(e_j)\Big\|_{\varepsilon_s},$$

the conclusion. ∎

The proof of the following alternative estimate for $\chi_{\mathrm{mon}}(\mathcal{P}(^mX))$ combines techniques from [5, Satz III], [3, Theorem 2, 3] and [10].

LEMMA 3.2: *Let $X = (\mathbb{C}^n, \|\cdot\|)$ be a Banach space such that the e_k's form a 1-unconditional basis. Then for each m*

$$\chi_{\mathrm{mon}}(\mathcal{P}(^mX)) \leq \chi_{\mathrm{mon}}(\mathcal{P}(^m\ell_\infty^n)) \leq c^m\Big(1+\frac{n}{m}\Big)^{\frac{m}{2}},$$

$c > 0$ an absolute constant.

Proof: Let us start with the proof of the first inequality: For $r_1,\ldots,r_n > 0$ it can be shown easily that

$$\chi_{\mathrm{mon}}(\mathcal{P}(^m\ell_\infty^n(r_1,\ldots,r_n)) \leq \chi_{\mathrm{mon}}(\mathcal{P}(^m\ell_\infty^n)),$$

where $\ell_\infty^n(r_1,\ldots,r_n)$ stands for $\mathbb{C}^n$ together with the norm

$$\|z\| := \sup_{1\leq k\leq n}\Big|\frac{z_k}{r_k}\Big|, \quad z\in\mathbb{C}^n;$$

clearly, if $|\sum_{|\alpha|=m}c_\alpha z^\alpha| \leq 1$, $z \in B_{\ell_\infty^n(r_1,\ldots,r_n)}$, then

$$\Big|\sum_{|\alpha|=m}c_\alpha r^\alpha z^\alpha\Big| \leq 1, \quad z\in B_{\ell_\infty^n},$$

and hence

$$\Big|\sum_{|\alpha|=m}|c_\alpha| z^\alpha\Big| \leq \chi_{\mathrm{mon}}(\mathcal{P}(^m\ell_\infty^n)), \quad z\in B_{\ell_\infty^n(r_1,\ldots,r_n)}.$$

Now since the e_k's form a 1-unconditional basis of X, we have

$$B_X = \bigcup B_{\ell_\infty^n(r_1,\ldots,r_n)},$$

where the union is taken over all $r_1, \ldots, r_n$ for which $B_{\ell_\infty^n(r_1,\ldots,r_n)} \subset B_X$. Therefore, for each choice of coefficients c_α, $|\alpha| = m$ we see that

$$\sup_{z \in B_X} \Big| \sum_{|\alpha|=m} |c_\alpha| z^\alpha \Big| \leq \sup_{r_1,\ldots,r_n>0} \chi_{\mathrm{mon}}(\mathcal{P}(^m\ell_\infty^n(r_1,\ldots,r_n))) \sup_{z\in B_X} \Big| \sum_{|\alpha|=m} c_\alpha z^\alpha \Big|$$
$$\leq \chi_{\mathrm{mon}}(\mathcal{P}(^m\ell_\infty^n)) \sup_{z\in B_X} \Big| \sum_{|\alpha|=m} c_\alpha z^\alpha \Big|,$$

which gives our first conclusion. Let us prove the second inequality: Take an m-homogeneous polynomial $\sum_{|\alpha|=m} c_\alpha z^\alpha$ such that

$$\Big\| \sum_{|\alpha|=m} c_\alpha z^\alpha \Big\|_{\mathcal{P}(^m\ell_\infty^n)} \leq 1.$$

An easy calculation using Stirling's formula shows that

$$\sum_{\substack{\alpha\in\mathbb{N}_0^n \\ |\alpha|=m}} 1 = \binom{n+m-1}{n-1} \leq c^m \Big(1 + \frac{n}{m}\Big)^m, \tag{3.6}$$

where $c > 0$ is an absolute constant. Hence by the Cauchy–Schwarz inequality

$$\Big\| \sum_{|\alpha|=m} |c_\alpha| z^\alpha \Big\|_{\mathcal{P}(^m\ell_\infty^n)} \leq \sum_{|\alpha|=m} |c_\alpha| \leq \Big(\sum_{|\alpha|=m} |c_\alpha|^2 \Big)^{\frac{1}{2}} c^{m/2} \Big(1 + \frac{n}{m}\Big)^{\frac{m}{2}}.$$

By orthonormability of the monomials z^α, $|\alpha| = m$ in $L_2(T_n, \lambda)$, T_n the n-dimensional Torus $[|z| = 1]^n \subset \mathbb{C}^n$ and λ the nth product measure of the normalized Lebesgue measure on the sphere $[|z| = 1]$, we obtain

$$\Big(\sum_{|\alpha|=m} |c_\alpha|^2 \Big)^{\frac{1}{2}} = \Big(\int_{T_n} \Big| \sum_{|\alpha|=m} c_\alpha z^\alpha \Big|^2 d\lambda(z) \Big)^{\frac{1}{2}} \leq 1,$$

and hence

$$\Big\| \sum_{|\alpha|=m} |c_\alpha| z^\alpha \Big\|_{\mathcal{P}(^m\ell_\infty^n)} \leq c^{m/2} \Big(1 + \frac{n}{m}\Big)^{\frac{m}{2}}$$

finishes the proof. ∎

For $X = \ell_p^n$ and $1 \leq p \leq 2$ this estimate can be improved.

LEMMA 3.3: *For $1 \leq p \leq \infty$ and n, m,*

$$\chi_{\mathrm{mon}}(\mathcal{P}(^m\ell_p^n)) \leq c^m \Big[\Big(1 + \frac{n}{m}\Big)^{1 - \frac{1}{\min(p,2)}} \Big]^m,$$

$c > 0$ *an absolute constant.*

Proof: Take an m-homogeneous polynomial $p(z) = \sum_{|\alpha|=m} c_\alpha z^\alpha$ with

$$\Big\| \sum_{|\alpha|=m} c_\alpha z^\alpha \Big\|_{\mathcal{P}(^m\ell_p^n)} \leq 1.$$

Then for all multi-indices α with $|\alpha| = m$ we have

$$|c_\alpha| \leq \Big(\frac{m^m}{\alpha^\alpha}\Big)^{\frac{1}{p}} \leq e^{m/p}\Big(\frac{m!}{\alpha!}\Big)^{\frac{1}{p}}; \tag{3.7}$$

here the second inequality is obvious and the first one follows from an apparently standard "Cauchy estimate" — for the sake of completeness we repeat the short argument: For each $r \in B_{\ell_p^n}$ with $r_k > 0$ we know that

$$c_\alpha = \frac{1}{(2\pi i)^n} \int_{|z_n|=r_n} \cdots \int_{|z_1|=r_1} \frac{p(z)}{z^\alpha z_1 \cdots z_n} dz_1 \cdots dz_n$$

(see, e.g., [11, p. 145]); therefore $|c_\alpha| \leq 1/r^m$, and hence (3.7) is a consequence of

$$\sup_{z \in B_{\ell_p^n}} |z^\alpha| = \Big(\frac{\alpha^\alpha}{m^m}\Big)^{\frac{1}{p}}$$

(see again, e.g., [11, p. 43]). Now by Hölder's inequality for $z \in \mathbb{C}^n$,

$$\begin{aligned}\Big| \sum_{|\alpha|=m} |c_\alpha| z^\alpha \Big| &\leq e^{m/p} \sum_{|\alpha|=m} \Big(\frac{m!}{\alpha!}\Big)^{\frac{1}{p}} |z^\alpha| \\ &\leq e^{m/p}\Big(\sum_{|\alpha|=m} 1\Big)^{\frac{1}{p'}} \Big(\sum_{|\alpha|=m} \frac{m!}{\alpha!}(|z_1|^p, \ldots, |z_n|^p)^\alpha\Big)^{\frac{1}{p}} \\ &\overset{(3.6)}{\leq} e^{m/p} c^{m/p'} \Big(1 + \frac{n}{m}\Big)^{\frac{m}{p'}} \Big(\sum_{k=1}^n |z_k|^p\Big)^{\frac{m}{p}} \\ &= e^{m/p} c^{m/p'} \Big(1 + \frac{n}{m}\Big)^{\frac{m}{p'}} \|z\|_{\ell_p^n}^m.\end{aligned}$$

Thus for each p, m, n,

$$\chi_{\mathrm{mon}}(\mathcal{P}(^m\ell_p^n)) \leq e^{m/p} c^{m/p'} \Big(1 + \frac{n}{m}\Big)^{\frac{m}{p'}},$$

which together with Lemma 3.2 yields the conclusion. ∎

We are now prepared to give a proof of Theorem 1.1.

Proof: Define for $n \in \mathbb{N}$

$$\ell(n) := \sup\{\ell \in \mathbb{N} \mid \ell^\ell \le n\}.$$

Then it can be shown easily that there is $c > 0$ such that for all n

$$\frac{1}{c}\frac{\log n}{\log\log n} \le \ell(n) \le c\frac{\log n}{\log\log n}.$$

Hence, it suffices to check that for $1 \le p \le \infty$ and each n

$$\frac{1}{c}\Big(\frac{\ell(n)}{n}\Big)^{1-\frac{1}{\min(p,2)}} \le K(B_{\ell_p^n}),$$

or, by (3.1), equivalently

$$\sup_m \chi_{\mathrm{mon}}(\mathcal{P}(^m\ell_p^n))^{\frac{1}{m}} \le c\Big(\frac{n}{\ell(n)}\Big)^{1-\frac{1}{\min(p,2)}},$$

$c > 0$ an absolute constant.

Fix p, m and n. Consider first the case $\ell(n) \le m$. Then by Lemma 3.3 (clearly also $\ell(n) \le n$),

$$\begin{aligned}\chi_{\mathrm{mon}}(\mathcal{P}(^m\ell_p^n))^{1/m} &\le c\Big(1+\frac{n}{m}\Big)^{1-\frac{1}{\min(p,2)}}\\ &\le c\Big(1+\frac{n}{\ell(n)}\Big)^{1-\frac{1}{\min(p,2)}}\\ &\le 2c\Big(\frac{n}{\ell(n)}\Big)^{1-\frac{1}{\min(p,2)}}.\end{aligned}$$

If $\ell(n) > m$, then we have that $n^{1/\ell(n)} \le n^{1/m}$, and by definition $\ell(n) \le n^{1/\ell(n)}$. Recall that

$$d(\ell_p^n, \ell_1^n) \le n^{1-\frac{1}{\min(p,2)}} \tag{3.8}$$

(see, e.g., [17, p. 280]). Hence we conclude from Lemma 3.1 that

$$\begin{aligned}\chi_{\mathrm{mon}}(\mathcal{P}(^m\ell_p^n))^{1/m} &\le 4ed(\ell_p^n, \ell_1^n)^{1-1/m}\\ &= 4en^{1-\frac{1}{\min(p,2)}}\frac{1}{(n^{1/m})^{1-\frac{1}{\min(p,2)}}}\\ &\le 4en^{1-\frac{1}{\min(p,2)}}\frac{1}{(n^{1/\ell(n)})^{1-\frac{1}{\min(p,2)}}}\\ &\le 4e\Big(\frac{n}{\ell(n)}\Big)^{1-\frac{1}{\min(p,2)}}.\end{aligned}$$

This completes the proof of Theorem 1.1. ∎

4. Bohr radii versus Banach–Mazur distances

It was conjectured in [9] that there is a constant $c > 0$ such that for all Banach spaces $X = (\mathbb{C}^n, \|\cdot\|)$ for which the e_k's form a 1-unconditional basis,

$$\frac{1}{c}\frac{1}{d(X,\ell_1^n)} \leq K(B_X) \leq c\frac{1}{d(X,\ell_1^n)};$$

it was proved that for a rich class of n-dimensional spaces X the upper part of this conjecture holds up to a log-term in the dimension n (e.g., whenever the Euclidean ball is contained in B_X). Theorem 1.1 now shows that the logarithmic factor is not superfluous—the upper part of the conjecture does not hold for the ℓ_p^n's.

On the other hand, (3.1) combined with Lemma 3.1 confirms the lower part for all X.

COROLLARY 4.1: *Let $X = (\mathbb{C}^n, \|\cdot\|)$ be a Banach space such that the e_k's form a 1-unconditional basis. Then*

$$\frac{1}{4e}\frac{1}{d(X,\ell_1^n)} \leq K(B_X).$$

Apart from the better constant, this is a proper extension of Aizenberg's result from [1]:

$$\frac{1}{3\sqrt[3]{e}} \leq K(B_{\ell_1^n}).$$

We finish with the following remark which, in the context of unconditionality, quantifies the "gap" between symmetric injective tensor products and full injective tensor products of ℓ_p^n's.

Remark 1: There is a constant $c > 0$ such that the following estimates hold for each $1 \leq p \leq \infty$ and n:

(1) $$\frac{1}{c}\Big(\frac{n}{\log n}\Big)^{\frac{1}{\max(p,2)}} \leq \sup_m \chi_{\mathrm{mon}}(\otimes_{\varepsilon_s}^{m,s}\ell_p^n)^{1/m} \leq c\Big(\frac{n}{\log n/\log\log n}\Big)^{\frac{1}{\max(p,2)}}.$$

(2) $$\frac{1}{c}n^{\frac{1}{\max(p,2)}} \leq \sup_m \chi_{\mathrm{mon}}(\otimes_{\varepsilon}^{m}\ell_p^n)^{1/m} \leq cn^{\frac{1}{\max(p,2)}}.$$

Clearly, (1) is a reformulation of Theorem 1.1 with the help of (3.1). The upper estimate in (2) is an immediate consequence of (3.3), (3.4) and (3.8), for the lower estimate analyzes the probabilistic argument given for [7, (5.4)].

In both remarks the unconditional basis constant of the monomials can be replaced by the unconditional basis constant itself.

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