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THERE ARE SIGNIFICANTLY MORE NONNEGATIVE POLYNOMIALS THAN SUMS OF SQUARES

BY

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ABSTRACT

We study the quantitative relationship between the cones of nonnegative polynomials, cones of sums of squares and cones of sums of even powers of linear forms. We derive bounds on the volumes (raised to the power reciprocal to the ambient dimension) of compact sections of the three cones. We show that the bounds are asymptotically exact if the degree is fixed and number of variables tends to infinity. When the degree is larger than two, it follows that there are significantly more nonnegative polynomials than sums of squares and there are significantly more sums of squares than sums of even powers of linear forms. Moreover, we quantify the exact discrepancy between the cones; from our bounds it follows that the discrepancy grows as the number of variables increases.

1. Introduction

The question of whether nonnegative polynomials admit some sum of squares representation has been of much interest in real algebraic geometry. The investigation was begun by Hilbert, who identified all pairs $(n, 2k)$, with n the number of variables and $2k$ the degree, where any nonnegative polynomial can be written as a sum of squares of polynomials (sos). Hilbert's theorem states that this is the case if the polynomial p is univariate, p is of degree 2, or p is a polynomial in two variables of degree 4. Moreover, in all other cases there exist nonnegative polynomials that are not sos. Hilbert's proof was nonconstructive and he did not exhibit explicit polynomials with this property [18].

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The next step was Hilbert's 17th problem which asked whether a nonnegative polynomial is necessarily a sum of squares of rational functions. It was solved in the affirmative by Artin, leading to the creation of Artin–Schreier theory of real closed fields. However, this representation is not efficiently computable. One of the reasons is that we may be forced to use rational functions with denominators of a very large degree [15].

The first explicit nonnegative polynomials that are not sos were constructed by Motzkin. Since then several more families of such polynomials have been found, but overall the list of explicit examples is very short. All of the known nonnegative polynomials that are not sos seem to lie either on the boundary of the cone of nonnegative polynomials or close to it [5], [18].

In this paper we investigate the quantitative relationship between nonnegative polynomials, sums of squares of polynomials and sums of even powers of linear forms. We show that, for a fixed degree greater than 2, there are significantly more nonnegative polynomials than sums of squares and there are significantly more sums of squares than sums of even powers of linear forms. Moreover, we derive tight asymptotic bounds on the size of these sets as the number of variables grows. The bounds allow us to see the precise quantitative relationship between these sets. It should be noted that our methods do not yield a way to generate examples of nonnegative polynomials that are not sos.

These results can be viewed as a quantitative version of Hilbert's theorem. They also have some computational ramifications. It is NP-hard to decide whether a polynomial is nonnegative [4]. Moreover, there are no practical algorithms for this problem. However, semidefinite programming can be used to decide whether a polynomial is sos, and testing for sos is practically efficient [12]. It has been suggested therefore to substitute testing for nonnegativity with testing for sos [13]. However, since there are much fewer sums of squares than nonnegative polynomials, in general this does not work well.

We now introduce some notation to state our results precisely. Let $P_{n,2k}$ be the vector space of homogeneous polynomials in n variables of degree $2k$. We observe that a nonnegative polynomial can be homogenized without losing its nonnegativity and therefore we deal with the homogeneous case. The main subject of this paper are the following three convex cones in $P_{n,2k}$:

The cone of nonnegative polynomials $Pos_{n,2k}$,

$$Pos_{n,2k} = \{p \in P_{n,2k} \mid p(x) \geq 0 \quad \text{for all } x \in \mathbb{R}^n\}.$$

The cone of sums of squares $Sq_{n,2k}$,

$$Sq_{n,2k} = \{p \in P_{n,2k} |\, p = \sum_i p_i^2 \quad \text{for some } p_i \in P_{n,k}\}.$$

The cone of sums of $2k$-th powers of linear forms $Lf_{n,2k}$,

$$Lf_{n,2k} = \{p \in P_{n,2k} |\, p = \sum_i l_i^{2k} \quad \text{for some linear forms } l_i \in P_{n,1}\}.$$

The cones $Pos_{n,2k}$, $Sq_{n,2k}$ and $Lf_{n,2k}$ are closed, full-dimensional convex cones in $P_{n,2k}$. There is an obvious inclusion relationship,

$$Lf_{n,2k} \subseteq Sq_{n,2k} \subseteq Pos_{n,2k}.$$

In order to compare the relative size of these cones, we take compact sections of these cones with a hyperplane. Let $L_{n,2k}$ be the hyperplane of polynomials of average 1 on the unit sphere S^{n-1}:

$$L_{n,2k} = \left\{p \in P_{n,2k} \,\middle|\, \int_{S^{n-1}} p d\sigma = 1\right\},$$

where σ is the rotation invariant probability measure on S^{n-1}. Let $Pos'_{n,2k}$, $Sq'_{n,2k}$ and $Lf'_{n,2k}$ be the sections of the respective cones with $L_{n,2k}$:

$$Pos'_{n,2k} = Pos_{n,2k} \cap L_{n,2k},\ Sq'_{n,2k} = Sq_{n,2k} \cap L_{n,2k} \text{ and } Lf'_{n,2k} = Lf_{n,2k} \cap L_{n,2k}.$$

These convex bodies are compact full-dimensional sets in $L_{n,2k}$.

For technical reasons we prefer to translate $Pos'_{n,2k}$, $Sq'_{n,2k}$ and $Lf'_{n,2k}$ by subtracting $(x_1^2 + \cdots + x_n^2)^k$ so that polynomials have average 0 on the unit sphere S^{n-1}. We let $M_{n,2k}$ denote the hyperplane of polynomials of average 0 on S^{n-1}. Let $\widetilde{Pos}_{n,2k}$, $\widetilde{Sq}_{n,2k}$ and $\widetilde{Lf}_{n,2k}$ be the respective translations:

$$\widetilde{Pos}_{n,2k} = \{p \in M_{n,2k} |\, p + (x_1^2 + \cdots + x_n^2)^k \in Pos'_{n,2k}\},$$
$$\widetilde{Sq}_{n,2k} = \{p \in M_{n,2k} |\, p + (x_1^2 + \cdots + x_n^2)^k \in Sq'_{n,2k}\},$$
$$\widetilde{Lf}_{n,2k} = \{p \in M_{n,2k} |\, p + (x_1^2 + \cdots + x_n^2)^k \in Lf'_{n,2k}\}.$$

There is the following natural L^2 inner product in $P_{n,2k}$:

$$\langle f, g\rangle = \int_{S^{n-1}} fg d\sigma.$$

We use D_M to denote the dimension of $M_{n,2k}$, S_M to denote the unit sphere in $M_{n,2k}$ and B_M to denote the unit ball in $M_{n,2k}$.

We would like to measure the respective sizes of $\widetilde{Pos}_{n,2k}$, $\widetilde{Sq}_{n,2k}$ and $\widetilde{Lf}_{n,2k}$. For a compact convex set K, a good measure of size of K that takes into account the effect of large dimensions is the volume of K raised to the power reciprocal to the ambient dimension:

$$(\operatorname{Vol} K)^{1/\dim K}.$$

For example, homothetically expanding K by a constant factor leads to an increase by the same factor in this normed volume.

We now state the main theorem of this paper:

THEOREM 1.1: *There exist positive constants $c_1(k), c_2(k), c_3(k), c_4(k), c_5(k)$, $c_6(k)$ dependent on k only, such that for any $\epsilon > 0$ and all n sufficiently large*

$$c_1 n^{-1/2} \leq \left(\frac{\operatorname{Vol} \widetilde{Pos}_{n,2k}}{\operatorname{Vol} B_M}\right)^{1/D_M} \leq c_2 n^{-1/2},$$

$$c_3 n^{-k/2} \leq \left(\frac{\operatorname{Vol} \widetilde{Sq}_{n,2k}}{\operatorname{Vol} B_M}\right)^{1/D_M} \leq c_4 n^{-k/2},$$

$$c_5 n^{-k+1/2} \leq \left(\frac{\operatorname{Vol} \widetilde{Lf}_{n,2k}}{\operatorname{Vol} B_M}\right)^{1/D_M} \leq c_6 n^{-k+1/2+\epsilon}.$$

We observe that if the degree $2k$ is 2, then Hilbert's theorem tells us that all three convex bodies $\widetilde{Pos}_{n,2k}$, $\widetilde{Sq}_{n,2k}$ and $\widetilde{Lf}_{n,2k}$ are the same and indeed Theorem 1.1 gives us the same asymptotic behavior of volumes. However, if the degree is at least 4 then we see that the asymptotics are different and it follows that there are significantly more nonnegative polynomials than sums of squares and significantly more sums of squares than sums of even powers of linear forms.

Theorem 1.1 consists of six bounds and we would like to point out the relationship between them. The lower bound for $\widetilde{Pos}_{n,2k}$ and the upper bound for $\widetilde{Sq}_{n,2k}$ are derived using similar techniques. For a generalization of these techniques and applications to other convex objects we refer to [2]. The upper bound for $\widetilde{Pos}_{n,2k}$ is derived using a different approach, involving an analytic inequality of Kellogg on the length of the gradient of a homogeneous polynomial [9]. The cone of sums of powers of linear forms can be identified with the dual cone of the cone of nonnegative polynomial. The lower bound for $\widetilde{Lf}_{n,2k}$ is derived from the upper bound for $\widetilde{Pos}_{n,2k}$ using the Blaschke–Santaló inequality from convexity. The remaining bounds are also proved using duality and convexity inequalities.

We will often omit the subscript $n, 2k$ from $Pos_{n,2k}$, $M_{n,2k}$ and the like when the subscript is clear from the context.

The rest of the paper is structured as follows. In Section 2 we lay out some results from convexity and representation theory that will be useful throughout this paper. In Section 3 we give an informal outline of the proofs. In Section 4 we prove the bounds for nonnegative polynomials. In Section 5 we prove some results that will help us derive bounds using duality. In Section 6 we prove the bounds for sums of squares. In Section 7 we prove the bounds for sums of even powers of linear forms.

2. Preliminaries

For a real Euclidean vector space V with the unit sphere S_V and a function $f\colon V \to \mathbb{R}$, we use $||f||_p$ to denote the L^p norm of f:

$$||f||_p = \left(\int_{S_V} |f|^p d\mu \right)^{1/p} \quad \text{and} \quad ||f||_\infty = \max_{x \in S_V} |f(x)|.$$

For a convex body K with origin in its interior the gauge G_K of K, also known as the norm of K, is a function that measures at a point v how much K needs to be expanded to put v into it:

$$G_K\colon V \to \mathbb{R}, \quad G_K(v) = \text{minimal}, \quad \lambda > 0 \quad \text{such that } \lambda v \in K.$$

2.1. The Action of the Orthogonal Group on $P_{n,2k}$. The special orthogonal group $SO(n)$ acts on $P_{n,2k}$ by rotating the coordinates,

$$A \in SO(n) \quad \text{sends} \quad f \in P_{n,2k} \quad \text{to} \quad Af = f(A^{-1}x).$$

We observe that the cones $Pos_{n,2k}$, $Sq_{n,2k}$ and $Lf_{n,2k}$ are invariant under this action and so is $M_{n,2k}$, the hyperplane of polynomials of average 0 on the unit sphere S^{n-1}. Therefore, the sections $\widetilde{Pos}_{n,2k}$, $\widetilde{Sq}_{n,2k}$ and $\widetilde{Lf}_{n,2k}$ are fixed by $SO(n)$ as well.

Let Δ be the Laplace differential operator:

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

A form f such that

$$\Delta(f) = 0$$

is called **harmonic**. We will need the fact that the irreducible components of this representation are subspaces $H_{n,2l}$ for $0 \leq l \leq k$, which have the following form:

$$H_{n,2l} = \{f \in P_{n,2k} |\; f = (x_1^2 + \cdots + x_n^2)^{k-l} h \text{ where } h \in P_{n,2l} \text{ is harmonic}\}.$$

For $v \in \mathbb{R}^n$, the functional

$$\lambda_v \colon M_{n,2k} \longrightarrow \mathbb{R}, \quad \lambda_v(f) = f(v),$$

is linear and therefore there exists a form $q_v \in M$ such that

$$\lambda_v(f) = \langle q_v, f \rangle.$$

There are explicit descriptions of the polynomials q_v; under a suitable normalization they are the so-called Gegenbauer or ultraspherical polynomials. We will only need the property that for $v \in S^{n-1}$,

$$||q_v||_2 = \sqrt{D_M}.$$

For more details on this representation of $SO(n)$ see [20].

2.2. The Blaschke–Santaló Inequality. Here we introduce an inequality from convexity that allows us to interpolate between volume bounds for a convex body and its polar. Let K be a full-dimensional convex body in $\mathbb{R}^n$ with the origin in its interior and let $\langle , \rangle$ be an inner product. We will use K° to denote the polar of K,

$$K^\circ = \{x \in \mathbb{R}^n |\ \langle x, y \rangle \leq 1 \text{ for all } y \in K\}.$$

Now suppose that a point z is in the interior of K and let K^z be the polar of K when z is translated to the origin:

$$K^z = \{x \in \mathbb{R}^n |\ \langle x - z, y - z \rangle \leq 1 \text{ for all } y \in K\}.$$

The point z at which the volume of K^z is minimal is unique and it is called the Santaló point of K. Moreover, the following inequality on volumes of K and K^z holds:

$$\frac{\operatorname{Vol} K \operatorname{Vol} K^z}{(\operatorname{Vol} B)^2} \leq 1,$$

where B is the unit ball of $\langle , \rangle$ and z is the Santaló point of K. This is known as the Blaschke–Santaló inequality [10].

3. Outline of proofs

Since many of the following proofs are technical we would like to first give an informal outline.

We begin with the description of the proofs for the cone of nonnegative polynomials. We observe that $\widetilde{Pos}_{n,2k}$ is the convex body of forms of integral 0 on

S^{n-1}, such that the minimum of the forms on the unit sphere S^{n-1} is at least -1,

$$\widetilde{Pos}_{n,2k} = \{f \in M_{n,2k} |\ f(x) \geq -1 \text{ for all } x \in S^{n-1}\}.$$

Let B_∞ be the unit ball of L^∞ norm in $M_{n,2k}$,

$$B_\infty = \{f \in M_{n,2k} |\ |f(x)| \leq 1 \text{ for all } x \in S^{n-1}\}.$$

It follows that

$$B_\infty = \widetilde{Pos}_{n,2k} \cap -\widetilde{Pos}_{n,2k} \quad \text{and therefore } B_\infty \subset \widetilde{Pos}_{n,2k}.$$

However, using the Blaschke–Santaló inequality and a theorem of Rogers and Shephard [11] we can show that, conversely,

$$\left(\frac{\operatorname{Vol} B_\infty}{\operatorname{Vol} \widetilde{Pos}_{n,2k}}\right)^{1/D_M} \geq 1/4.$$

Therefore, it suffices to derive upper and lower bounds for the volume of B_∞.

For the lower bound we reduce the proof to bounding the average L^∞ norm of a polynomial in $M_{n,2k}$,

$$\int_{S_M} ||f||_\infty d\mu,$$

where S_M is the unit sphere in $M_{n,2k}$ and μ is the rotation invariant probability measure on S_M. The key idea is to estimate $||f||_\infty$ using L^{2p} norms for some large p. An inequality of Barvinok [1] is used to see that taking $p = n$ suffices for $||f||_{2p}$ to be within a constant factor of $||f||_\infty$. The proof is completed with some further estimates.

The techniques used for the proof of the upper bound are quite different. Let ∇f be the gradient of $f \in P_{n,2k}$,

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right),$$

and let $\langle \nabla f, \nabla f \rangle$ be the following polynomial giving the squared length of the gradient of f,

$$\langle \nabla f, \nabla f \rangle = \left(\frac{\partial f}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial f}{\partial x_n}\right)^2.$$

The key to the proof is the following theorem of Kellogg [9] which tells us that for homogeneous polynomials the maximum length of the gradient on the unit sphere S^{n-1} is equal to the maximum absolute value of the polynomial on S^{n-1} multiplied by the degree of the polynomial:

$$||\langle \nabla f, \nabla f \rangle||_\infty = 4k^2 ||f||_\infty^2.$$

We define a different inner product on $P_{n,2k}$ which we call the gradient inner product,

$$\langle f,g\rangle_G = \frac{1}{4k^2}\int_{S^{n-1}}\langle\nabla f,\nabla g\rangle d\sigma.$$

We denote the norm of f in the gradient metric by $||f||_G$ and the unit ball of the gradient metric in $M_{n,2k}$ by B_G. We observe that

$$||f||_G^2 = \frac{1}{4k^2}\int_{S^{n-1}}\langle\nabla f,\nabla f\rangle d\sigma,$$

and hence it follows that

$$||f||_G \leq ||f||_\infty \quad \text{and therefore } B_\infty \subset B_G.$$

The usual L^2 inner product gives us

$$||f||_2^2 = \int_{S^{n-1}} f^2 d\sigma.$$

The relationship between the gradient metric and the L^2 metric can be calculated precisely by using the fact that both metrics are $SO(n)$-invariant. Therefore, these metrics are multiples of each other in the irreducible subspaces of the $SO(n)$ representation and the multiplication factors can be calculated directly using the Stokes' formula [6]. Hence we obtain an upper bound for the volume of B_∞ in terms of the volume of B_M, the unit ball of the L^2 metric in $M_{n,2k}$.

The proof of the upper bound for the cone of sums of squares is quite similar to the proof of the lower bound for the cone of nonnegative polynomials. We define the following norm on $P_{n,2k}$,

$$||f||_{sq} = \max_{g\in S_{P_{n,k}}} |\langle f,g^2\rangle|,$$

where $S_{P_{n,k}}$ is the unit sphere in $P_{n,k}$. Using inequalities from convexity we can reduce the proof to bounding the average $||f||_{sq}$.

To every form $f \in P_{n,2k}$ we can associate a quadratic form H_f on $P_{n,2k}$ by letting

$$H_f(g) = \langle f,g^2\rangle \quad \text{for } g \in P_{n,k}.$$

It follows that

$$||f||_{sq} = ||H_f||_\infty.$$

Now we can estimate $||H_f||_\infty$ by high L^{2p} norms of H_f and the proof is finished using similar ideas to the proof for the case of nonnegative polynomials.

For the remainder of the proofs we will need to consider yet another metric on $P_{n,2k}$. To a form $f \in P_{n,2k}$,

$$f = \sum_{\alpha=(i_1,\dots,i_n)} c_\alpha x_1^{i_1} \cdots x_n^{i_n},$$

we formally associate the differential operator D_f:

$$D_f = \sum_{\alpha=(i_1,\dots,i_n)} c_\alpha \frac{\partial^{i_1}}{\partial x_1^{i_1}} \cdots \frac{\partial^{i_n}}{\partial x_n^{i_n}}.$$

We define the following metric on $P_{n,2k}$, which we call the differential metric:

$$\langle f, g \rangle_d = D_f(g).$$

It is not hard to check that this indeed defines a symmetric positive definite bilinear form, which is invariant under the action of $SO(n)$. The relationship between the differential metric and the integral metric can be calculated precisely.

For the proof of the lower bound for the cone of sums of squares we show that the dual cone Sq_d^* of $Sq_{n,2k}$ with respect to the differential metric is contained in $Sq_{n,2k}$. We use this to derive a lower bound on the volume of $\widetilde{Sq}_{n,2k}$ by using the Blaschke–Santaló inequality.

It can be shown that the cone of sums of $2k$-th powers of linear forms $Lf_{n,2k}$ is dual to $Pos_{n,2k}$ in the differential metric. The proofs of the bounds for the sums of even powers of linear forms follow from the bounds derived for $\widetilde{Pos}_{n,2k}$ and the Blaschke–Santaló inequality.

4. Nonnegative polynomials

In this section we prove the bounds for nonnegative polynomials. Here is the precise statement of the bounds:

THEOREM 4.1: *There are the following bounds on the volume of $\widetilde{Pos}_{n,2k}$:*

$$\frac{1}{2\sqrt{4k+2}} n^{-1/2} \leq \left(\frac{\operatorname{Vol} \widetilde{Pos}_{n,2k}}{\operatorname{Vol} B_M}\right)^{1/D_M} \leq 4\left(\frac{2k^2}{2k^2+n}\right)^{1/2}.$$

4.1. PROOF OF THE LOWER BOUND. We begin by observing that $\widetilde{Pos}_{n,2k}$ is a convex body in $M_{n,2k}$ with origin in its interior and the boundary of $\widetilde{Pos}_{n,2k}$ consists of polynomials with minimum -1 on the unit sphere S^{n-1}. Therefore, the gauge G_P of $\widetilde{Pos}_{n,2k}$ is given by

$$G_P(f) = \left|\min_{v \in S^{n-1}} f(v)\right|.$$

By using integration in polar coordinates in $M_{n,2k}$ we obtain the following expression for the volume of $\widetilde{Pos}_{n,2k}$,

$$\left(\frac{\operatorname{Vol}\widetilde{Pos}_{n,2k}}{\operatorname{Vol}B_M}\right)^{1/D_M}=\left(\int_{S_M}G_P^{-D_M}\,d\mu\right)^{1/D_M},\tag{4.1.1}$$

where μ is the rotation invariant probability measure on S_M. The relationship (4.1.1) holds for any convex body with origin in its interior [14, p. 91].

We interpret the right hand side of (4.1.1) as $||G_P^{-1}||_{D_M}$, and by Hölder's inequality

$$||G_P^{-1}||_{D_M}\geq ||G_P^{-1}||_1.$$

Thus,

$$\left(\frac{\operatorname{Vol}\widetilde{Pos}_{n,2k}}{\operatorname{Vol}B_M}\right)^{1/D_M}\geq\int_{S_M}G_P^{-1}d\mu.$$

By applying Jensen's inequality [8, p. 150], with convex function $y=1/x$ it follows that,

$$\int_{S_M}G_P^{-1}d\mu\geq\left(\int_{S_M}G_P d\mu\right)^{-1}.$$

Hence we see that

$$\left(\frac{\operatorname{Vol}\widetilde{Pos}_{n,2k}}{\operatorname{Vol}B_M}\right)^{1/D_M}\geq\left(\int_{S_M}|\min f|d\mu\right)^{-1}.$$

Clearly, for all $f\in P_{n,2k}$

$$||f||_\infty\geq|\min f|.$$

Therefore,

$$\left(\frac{\operatorname{Vol}\widetilde{Pos}_{n,2k}}{\operatorname{Vol}B_M}\right)^{1/D_M}\geq\left(\int_{S_M}||f||_\infty d\mu\right)^{-1}.$$

The proof of the lower bound of Theorem 4.1 is now completed by the following estimate.

THEOREM 4.2: *Let S_M be the unit sphere in $M_{n,2k}$ and let μ be the rotation invariant probability measure on S_M. Then the following inequality for the average L^∞ norm over S_M holds:*

$$\int_{S_M}||f||_\infty d\mu\leq 2\sqrt{2n(2k+1)}.$$

Proof: It was shown by Barvinok in [1] that for all $f\in P_{n,2k}$,

$$||f||_\infty\leq\binom{2kn+n-1}{2kn}^{1/2n}||f||_{2n}.$$

It is not hard to see that

$$\binom{2kn+n-1}{2kn}^{1/2n} \leq 2\sqrt{2k+1}.$$

Therefore, it suffices to estimate the average L^{2n} norm, which we denote by A:

$$A = \int_{S_M} ||f||_{2n} d\mu.$$

Applying Hölder's inequality we observe that

$$A = \int_{S_M} \left(\int_{S^{n-1}} f^{2n}(x) d\sigma \right)^{1/2n} d\mu \leq \left(\int_{S_M} \int_{S^{n-1}} f^{2n}(x) d\sigma d\mu \right)^{1/2n}.$$

By interchanging the order of integration we obtain

$$A \leq \left(\int_{S^{n-1}} \int_{S_M} f^{2n}(x) d\mu d\sigma \right)^{1/2n}. \tag{4.2.1}$$

We now note that by symmetry of $M_{n,2k}$,

$$\int_{S_M} f^{2n}(x) d\mu$$

is the same for all $x \in S^{n-1}$. Therefore, we see that in (4.2.1) the outer integral is redundant and thus

$$A \leq \left(\int_{S_M} f^{2n}(v) d\mu \right)^{1/2n}, \quad \text{where } v \text{ is any vector in } S^{n-1}. \tag{4.2.2}$$

We recall from Section 2 that for $v \in S^{n-1}$ there exists a form q_v in M such that

$$\langle f, q_v \rangle = f(v) \quad \text{for all } f \in M \quad \text{and} \quad ||q_v||_2 = \sqrt{D_M}.$$

Rewriting (4.2.2) we see that

$$A \leq \left(\int_{S_M} \langle f, q_v \rangle^{2n} d\mu \right)^{1/2n}. \tag{4.2.3}$$

We observe that

$$\int_{S_M} \langle f, q_v \rangle^{2n} d\mu = (D_M)^n \frac{\Gamma(n+\frac{1}{2})\Gamma(\frac{1}{2}D_M)}{\sqrt{\pi}\Gamma(\frac{1}{2}D_M + n)},$$

since this is the integral on the sphere of $2n$-th power of a linear form for which the formula is standard; see, for example, [1]. We substitute this into (4.2.3) to obtain

$$A \leq \left((D_M)^n \frac{\Gamma(n+\frac{1}{2})\Gamma(\frac{1}{2}D_M)}{\sqrt{\pi}\Gamma(\frac{1}{2}D_M+n)}\right)^{1/2n}.$$

Since

$$\left(\frac{\Gamma(\frac{1}{2}D_M)}{\Gamma(\frac{1}{2}D_M+n)}\right)^{1/2n} \leq \sqrt{\frac{2}{D_M}} \quad \text{and} \quad \left(\frac{\Gamma(n+1/2)}{\sqrt{\pi}}\right)^{1/2n} \leq n^{1/2},$$

we see that

$$A \leq (2n)^{1/2}.$$

The theorem now follows. ∎

4.2. Proof of the Upper Bound. We begin by noting that the origin is the only point in $M_{n,2k}$ fixed by $SO(n)$. Let $\widetilde{Pos}^\circ$ be the polar of $\widetilde{Pos}_{n,2k}$ in $M_{n,2k}$,

$$\widetilde{Pos}^\circ = \{f \in M_{n,2k} | \langle f, g\rangle \leq 1 \text{ for all } g \in \widetilde{Pos}_{n,2k}\}.$$

Since $\widetilde{Pos}_{n,2k}$ is fixed by the action of $SO(n)$ and the Santaló point of a convex body is unique, it follows that the origin is the Santaló point of $\widetilde{Pos}_{n,2k}$. We now use Blaschke–Santaló inequality, which applied to $\widetilde{Pos}_{n,2k}$ gives us

$$(\operatorname{Vol}\widetilde{Pos}_{n,2k})(\operatorname{Vol}\widetilde{Pos}^\circ) \leq (\operatorname{Vol} B_M)^2.$$

Therefore, it would suffice to show that

$$\left(\frac{\operatorname{Vol}\widetilde{Pos}^\circ}{\operatorname{Vol} B_M}\right)^{1/D_M} \geq \frac{1}{4}\left(\frac{2k^2+n}{2k^2}\right)^{1/2}. \tag{4.2.4}$$

Let B_∞ be the unit ball of the L^∞ metric in $M_{n,2k}$,

$$B_\infty = \{f \in M_{n,2k} | \, ||f||_\infty \leq 1\}.$$

We observe that B_∞ is clearly the intersection of $\widetilde{Pos}_{n,2k}$ with $-\widetilde{Pos}_{n,2k}$:

$$B_\infty = \widetilde{Pos}_{n,2k} \cap -\widetilde{Pos}_{n,2k}.$$

By taking polars it follows that

$$B_\infty^\circ = \operatorname{ConvexHull}\{\widetilde{Pos}^\circ, -\widetilde{Pos}^\circ\} \subset \widetilde{Pos}^\circ \oplus (-\widetilde{Pos}^\circ),$$

where $\oplus$ denotes Minkowski addition. By the theorem of Rogers and Shephard, ([11] p. 78), it follows that

$$\operatorname{Vol} B_\infty^\circ \leq \binom{2D_M}{D_M} \operatorname{Vol} \widetilde{Pos^\circ}.$$

Since

$$\binom{2D_M}{D_M} \leq 4^{D_M},$$

we obtain

$$\left(\frac{\operatorname{Vol} \widetilde{Pos^\circ}}{\operatorname{Vol} B_\infty^\circ}\right)^{1/D_M} \geq \frac{1}{4}.$$

Combining with (4.2.4) we see that we have reduced the proof of the lower bound of Theorem 4.1 to showing that

$$\left(\frac{\operatorname{Vol} B_\infty^\circ}{\operatorname{Vol} B_M}\right)^{1/D_M} \geq \left(\frac{2k^2+n}{2k^2}\right)^{1/2}. \tag{4.2.5}$$

For a form f we use ∇f to denote the gradient of f:

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

We also define a different Euclidean metric on $P_{n,2k}$ which we call the gradient metric:

$$\langle f, g\rangle_G = \frac{1}{4k^2}\int_{S^{n-1}} \langle \nabla f, \nabla g\rangle d\sigma.$$

We denote the unit ball in this metric by B_G and the norm of f by $||f||_G$. For $f \in P_{n,2k}$ let $\langle \nabla f, \nabla f\rangle$ be the following polynomial:

$$\langle \nabla f, \nabla f\rangle = \left(\frac{\partial f}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial f}{\partial x_n}\right)^2.$$

It was shown by Kellogg in [9] that

$$||\langle \nabla f, \nabla f\rangle||_\infty = 4k^2||f||_\infty^2.$$

It clearly follows that

$$||f||_\infty \geq ||f||_G,$$

and therefore

$$B_\infty \subseteq B_G.$$

Polarity reverses inclusion and thus we see that

$$B_G^\circ \subseteq B_\infty^\circ \quad \text{and} \quad \operatorname{Vol} B_G^\circ = \frac{(\operatorname{Vol} B_M)^2}{\operatorname{Vol} B_G},$$

since B_G is an ellipsoid. Hence we see that

$$\operatorname{Vol} B_\infty^\circ \geq \frac{(\operatorname{Vol} B_M)^2}{\operatorname{Vol} B_G}.$$

Thus (4.2.5) and consequently the upper bound of Theorem 4.1 will follow from the following lemma.

LEMMA 4.3:
$$\left(\frac{\operatorname{Vol} B_M}{\operatorname{Vol} B_G}\right)^{1/D_M} \geq \left(\frac{2k^2+n}{2k^2}\right)^{1/2}.$$

Proof: It will suffice to show that for all $f \in M_{n,2k}$,

$$\langle f, f\rangle_G \geq \frac{2k^2+n}{2k^2}\langle f, f\rangle. \tag{4.3.1}$$

By the invariance of both inner products under the action of $SO(n)$, it is enough to prove (4.3.1) in the irreducible components of the representation. From Section 2, we know that the irreducible components are $H_{n,2l}$ for $0 \leq l \leq k$, which have the following form:

$$H_{n,2l} = \{f \in P_{n,2k} |\ f = (x_1^2 + \cdots + x_n^2)^{k-l}h \text{ where } h \in P_{n,2l} \text{ is harmonic}\}.$$

First let f be a harmonic form of degree $2d$ in n variables. It is not hard to show using Stokes' formula that

$$\langle f, f\rangle = \frac{2d}{4d+n-2}\langle f, f\rangle_G;$$

see [6] p. 488.

Now suppose that $f = (x_1^2 + \cdots + x_n^2)^{k-d}h$, where h is a harmonic form of degree $2d \leq 2k$. It is easy to check that

$$\langle f, f\rangle_G = \frac{d^2}{k^2}\langle h, h\rangle_G + \frac{k^2-d^2}{k^2}\langle h, h\rangle.$$

Since h is harmonic we know that

$$\langle h, h\rangle_G = \frac{4d+n-2}{2d}\langle h, h\rangle \quad \text{and} \quad \langle f, f\rangle = \langle h, h\rangle.$$

Thus

$$\langle f, f\rangle_G = \frac{2k^2 + d(n-2) + 2d^2}{2k^2}\langle f, f\rangle.$$

Since $f \in M_{n,2k}$ we know that $1 \leq d \leq k$. The minimum clearly occurs when $d = 1$ and we see that

$$\langle f, f\rangle_G \geq \frac{2k^2+n}{2k^2}\langle f, f\rangle.$$

The lemma now follows. ∎

5. The differential metric

Before we proceed with the proofs for sums of squares and sums of powers of linear forms, we will need some preparatory results that involve switching to a different Euclidean metric on $P_{n,2k}$.

To a form $f \in P_{n,2k}$,

$$f = \sum_{\alpha=(i_1,\ldots,i_n)} c_\alpha x_1^{i_1} \cdots x_n^{i_n},$$

we formally associate the differential operator D_f:

$$D_f = \sum_{\alpha=(i_1,\ldots,i_n)} c_\alpha \frac{\partial^{i_1}}{\partial x_1^{i_1}} \cdots \frac{\partial^{i_n}}{\partial x_n^{i_n}}.$$

We define the following metric on $P_{n,2k}$, which we call the differential metric:

$$\langle f, g \rangle_d = D_f(g).$$

It is not hard to check that this indeed defines a symmetric positive definite bilinear form, which is invariant under the action of $SO(n)$. For a point $v \in S^{n-1}$ we will use v^{2k} to denote the polynomial

$$v^{2k} = (v_1 x_1 + \cdots + v_n x_n)^{2k}.$$

We also define an important linear operator T: $P_{n,2k} \to P_{n,2k}$, which to a form $f \in P_{n,2k}$ associates a weighted average of forms v^{2k} with the weight $f(v)$:

$$T(f) = \int_{S^{n-1}} f(v) v^{2k} d\sigma(v).$$

We take our definition for T from [3]; it can be shown that this operator was first introduced in a very different form by Reznick in [17]. The operator T acts as a switch between our standard L^2 metric and the differential metric in the following sense:

LEMMA 5.1: *The following identity relating the operator T and the two metrics holds,*

$$\langle Tf, g \rangle_d = (2k)! \langle f, g \rangle.$$

Proof: We observe that

$$\langle Tf, g \rangle_d = \langle \int_{S^{n-1}} f(v) v^{2k} d\sigma(v), g \rangle_d = \int_{S^{n-1}} \langle f(v) v^{2k}, g \rangle_d d\sigma(v).$$

Since

$$\langle v^{2k}, g\rangle_d = (2k)!g(v),$$

it follows that

$$\langle Tf, g\rangle_d = (2k)! \int_{S^{n-1}} f(v)g(v)d\sigma(v) = (2k)!\langle f, g\rangle. \qquad \blacksquare$$

Let L be a full-dimensional cone in $P_{n,2k}$ such that $(x_1^2 + \cdots + x_n^2)^k$ is in the interior of L and $\int_{S^{n-1}} f d\sigma > 0$ for all non-zero f in L. We define $\widetilde{L}$ as the set of all forms f in $M_{n,2k}$ such that $f + (x_1^2 + \cdots + x_n^2)^k$ lies in L,

$$\widetilde{L} = \{f \in M_{n,2k} | \ f + (x_1^2 + \cdots + x_n^2)^k \in L\}.$$

We let L^* be the dual cone of L in the L^2 metric and L_d^* be the dual cone of L in the differential metric:

$$L^* = \{f \in P_{n,2k} | \ \langle f, g\rangle \geq 0 \text{ for all } g \in L\},$$
$$L_d^* = \{f \in P_{n,2k} | \ \langle f, g\rangle_d \geq 0 \text{ for all } g \in L\}.$$

We observe that $(x_1^2 + \cdots + x_n^2)^k$ is in the interior of both L^* and L_d^* and also $\int_{S^{n-1}} f d\sigma > 0$ for all non-zero f in both of the dual cones. Therefore, we can similarly define $\widetilde{L^*}$ and $\widetilde{L_d^*}$ as sets of all forms f in M such that $f + (x_1^2 + \cdots + x_n^2)^k$ lies in the respective cone.

Lemma 5.2: *Let L be a full-dimensional cone in $P_{n,2k}$ such that $(x_1^2 + \cdots + x_n^2)^k$ is the interior of L and $\int_{S^{n-1}} f d\sigma > 0$ for all f in L. Then there is the following relationship between the volumes of $\widetilde{L^*}$ and $\widetilde{L_d^*}$,*

$$\frac{k!}{(n/2 + 2k)^k} \leq \left(\frac{\operatorname{Vol} \widetilde{L_d^*}}{\operatorname{Vol} \widetilde{L^*}}\right)^{1/D_M} \leq \left(\frac{k!}{(n/2 + k)^k}\right)^{\alpha},$$

where

$$\alpha = 1 - \left(\frac{2k - 1}{2k + n - 2}\right)^2.$$

Proof: From Lemma 5.1 we see that

$$\langle f, g\rangle \geq 0 \quad \text{if and only if } \langle Tf, g\rangle_d \geq 0 \quad \text{for all } f, g \in P_{n,2k}.$$

Therefore, it follows that T maps L^* to L_d^*,

$$T(L^*) = L_d^*.$$

It is not hard to show that T commutes with the natural action of $SO(n)$. Therefore, after a suitable complexification, Schur's Lemma [7] tells us that T

acts by contraction in each irreducible subspace of $P_{n,2k}$. In particular, it follows that

$$T((x_1^2+\cdots+x_n^2)^k)=c(x_1^2+\cdots+x_n^2)^k.$$

In order to compute c, we note that it is enough to compute $T((x_1^2+\cdots+x_n^2)^k)$ at a particular point on the unit sphere, say e_1. It follows that

$$\begin{aligned}c=T((x_1^2+\cdots+x_n^2)^k)(e_1)&=\int_{S^{n-1}}\langle x,e_1\rangle^{2k}d\sigma(x)\\&=\int_{S^{n-1}}x_1^{2k}d\sigma=\frac{\Gamma(\frac{2k+1}{2})\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n+2k}{2})}.\end{aligned}$$

The equality of the second line is standard; see, for example, [1].

Since $\frac{1}{c}T$ commutes with the action of $SO(n)$ and fixes $(x_1^2+\cdots+x_n^2)^k$, it follows that it also fixes the orthogonal complement of $(x_1^2+\cdots+x_n^2)^k$, which is the hyperplane of all forms of integral 1 on the sphere. Therefore, $\frac{1}{c}T$ maps the section $\widetilde{L^*}$ to $\widetilde{L^*_d}$.

It is possible to describe precisely the action of $\frac{1}{c}T$ on $M_{n,2k}$; see [3]. From Schur's Lemma, it follows that $\frac{1}{c}T$ is a contraction operator and the exact coefficients of contraction can be computed. We only need the following estimate, which follows from [3] Lemma 7.4 by estimating the change in volume to be at worst the smallest contraction coefficient:

$$\left(\frac{\operatorname{Vol}\widetilde{L^*_d}}{\operatorname{Vol}\widetilde{L^*}}\right)^{1/D_M}\geq\frac{k!\Gamma(k+n/2)}{\Gamma(2k+n/2)}.$$

We observe that

$$\frac{k!\Gamma(k+n/2)}{\Gamma(2k+n/2)}\geq\frac{k!}{(n/2+2k)^k},$$

and therefore

$$\left(\frac{\operatorname{Vol}\widetilde{L^*_d}}{\operatorname{Vol}\widetilde{L^*}}\right)^{1/D_M}\geq\frac{k!}{(n/2+2k)^k}.$$

Also, from Lemma 7.4 of [3], it follows that contraction by the largest coefficient occurs in the space of all harmonic polynomials of degree $2k$ which has dimension

$$D_H=\binom{n+2k-1}{2k}-\binom{n+2k-3}{2k-2}.$$

Since the dimension of the ambient space M is

$$D_M=\binom{n+2k-1}{2k}-1,$$

we can estimate that
$$\frac{D_H}{D_M} \geq 1 - \left(\frac{2k-1}{n+2k-2}\right)^2.$$
Since we can also estimate the largest contraction coefficient from above,
$$\frac{k!\Gamma(k+n/2)}{\Gamma(2k+n/2)} \leq \frac{k!}{(n/2+k)^k},$$
the lemma now follows. ∎

We also show the following lemma, which allows us to compare the cone of sums of squares to its dual.

Lemma 5.3: *The dual cone Sq_d^* to the cone of sums of squares in the differential metric is contained in the cone of sums of squares $Sq_{n,2k}$,*
$$Sq_d^* \subseteq Sq_{n,2k}.$$

Proof: In this proof we will work exclusively with the differential metric on $P_{n,k}$ and $P_{n,2k}$. Let W be the space of quadratic forms on $P_{n,k}$. For A, B in W, with corresponding symmetric matrices M_A, M_B the inner product of A and B is given by
$$\langle A, B\rangle = \operatorname{tr} M_A M_B.$$
For $q \in P_{n,k}$, let A_q be the rank one quadratic form giving the square of the inner product with q:
$$A_q(p) = \langle p, q\rangle_d^2.$$
Then for any $B \in W$,
$$\langle A_q, B\rangle = B(q).$$

For $f \in P_{n,2k}$, let H_f be the following quadratic form on $P_{n,k}$:
$$H_f(p) = \langle p^2, f\rangle_d.$$

Now suppose that $f \in Sq_d^*$. Then the quadratic form H_f is clearly positive semidefinite. Therefore, H_f can be written as a nonnegative linear combination of forms of rank 1:
$$H_f = \sum A_q \quad \text{for some } q \in P_{n,k}. \tag{5.3.1}$$
Let V be the subspace of W given by the linear span of the forms H_f for all $f \in P_{n,2k}$. Let $\mathbb{P}$ be the operator of orthogonal projection onto V. We claim that
$$\mathbb{P}(A_q) = \binom{2k}{k}^{-1} H_{q^2}.$$

It suffices to show that $A_q - \binom{2k}{k}^{-1} H_{q^2}$ is orthogonal to the forms $H_{v^{2k}}$, since these forms span V. We observe that

$$H_{v^{2k}}(p) = (2k)!p(v)^{2k} = \frac{(2k)!A_{v^k}(p)}{(k!)^2} = \binom{2k}{k} A_{v^k}(p).$$

Therefore, we see that

$$\begin{aligned}\langle A_q - \binom{2k}{k}^{-1} H_{q^2}, H_{v^{2k}} \rangle &= H_{v^{2k}}(q) - \langle H_{q^2}, A_{v^k} \rangle \\ &= H_{v^{2k}}(q) - H_{q^2}(v^k) = 0.\end{aligned}$$

Now we apply $\mathbb{P}$ to both sides of (5.3.1). It follows that

$$H_f = \mathbb{P}\Big(\sum A_q\Big) = \sum \binom{2k}{k}^{-1} H_{q^2} = \binom{2k}{k}^{-1} H_{\sum q^2}.$$

Therefore f is a sum of squares. ∎

6. Sums of squares

In this section we prove the bounds for the sums of squares. The full statement of the bounds is the following,

THEOREM 6.1: *There are the following bounds for the volume of $\widetilde{Sq}_{n,2k}$:*

$$\frac{(k!)^2}{4^{2k}(2k)!\sqrt{24}} \frac{n^{k/2}}{(n/2+2k)^k} \le \left(\frac{\operatorname{Vol} \widetilde{Sq}_{n,2k}}{\operatorname{Vol} B_M}\right)^{1/D_M} \le \frac{4^{2k}(2k)!\sqrt{24}}{k!} n^{-k/2}.$$

6.1. PROOF OF THE UPPER BOUND. Let us begin by considering the support function of $\widetilde{Sq}_{n,2k}$, which we call $L_{\widetilde{Sq}}$:

$$L_{\widetilde{Sq}}(f) = \max_{g \in \widetilde{Sq}_{n,2k}} \langle f, g \rangle.$$

The average width $W_{\widetilde{Sq}}$ of $\widetilde{Sq}_{n,2k}$ is given by

$$W_{\widetilde{Sq}} = 2 \int_{S_M} L_{\widetilde{Sq}} d\mu.$$

We now recall Urysohn's Inequality [19, p. 318], which applied to $\widetilde{Sq}_{n,2k}$ gives

$$\left(\frac{\operatorname{Vol} \widetilde{Sq}_{n,2k}}{\operatorname{Vol} B_M}\right)^{1/D_M} \le \frac{W_{\widetilde{Sq}}}{2}. \tag{6.1.1}$$

Therefore, it suffices to obtain an upper bound for $W_{\widetilde{Sq}}$.

Let $S_{P_{n,k}}$ denote the unit sphere in $P_{n,k}$. We observe that extreme points of $\widetilde{Sq}$ have the form

$$g^2 - (x_1^2 + \cdots + x_n^2)^k \quad \text{where } g \in P_{n,k} \quad \text{and} \quad \int_{S^{n-1}} g^2 d\sigma = 1.$$

For $f \in M_{n,2k}$,

$$\langle f, (x_1^2 + \cdots + x_n^2)^k \rangle = \int_{S^{n-1}} f d\sigma = 0,$$

and therefore,

$$L_{\widetilde{Sq}}(f) = \max_{g \in S_{P_{n,k}}} \langle f, g^2 \rangle.$$

We now introduce a norm on $P_{n,2k}$, which we denote $|| \ ||_{sq}$:

$$||f||_{sq} = \max_{g \in S_{P_{n,k}}} |\langle f, g^2 \rangle|.$$

It is clear that

$$L_{\widetilde{Sq}}(f) \leq ||f||_{sq}.$$

Therefore, by (6.1.1) it follows that

$$\left(\frac{\operatorname{Vol} \widetilde{Sq}_{n,2k}}{\operatorname{Vol} B_M} \right)^{1/D_M} \leq \int_{S_M} ||f||_{sq} d\mu.$$

The proof of the upper bound of Theorem 6.1 is reduced to the estimate below.

THEOREM 6.2: *There is the following bound for the average $|| \ ||_{sq}$ over S_M:*

$$\int_{S_M} ||f||_{sq} d\mu \leq \frac{4^{2k}(2k)!\sqrt{24}}{k!} n^{-k/2}.$$

Proof: For $f \in P_{n,2k}$ we introduce a quadratic form H_f on $P_{n,k}$:

$$H_f(g) = \langle f, g^2 \rangle \quad \text{for } g \in P_{n,k}.$$

We note that

$$||f||_{sq} = \max_{g \in S_{P_{n,k}}} |\langle f, g^2 \rangle| = ||H_f||_\infty.$$

We bound $||H_f||_\infty$ by a high L^{2p} norm of H_f. Since H_f is a form of degree 2 on the vector space $P_{n,k}$ of dimension $D_{n,k}$, it follows by the inequality of Barvinok in [1] applied in the same way as in the proof of Theorem 4.2 that

$$||H_f||_\infty \leq 2\sqrt{3}||H_f||_{2D_{n,k}}.$$

Therefore, it suffices to estimate

$$A = \int_{S_M} ||H_f||_{2D_{n,k}} d\mu = \int_{S_M} \left(\int_{S_{P_{n,k}}} \langle f, g^2 \rangle^{2D_{n,k}} d\sigma(g) \right)^{1/2D_{n,k}} d\mu(f).$$

We apply Hölder's inequality to see that

$$A \leq \left(\int_{S_M} \int_{S_{P_{n,k}}} \langle f, g^2 \rangle^{2D_{n,k}} d\sigma(g) d\mu(f) \right)^{1/2D_{n,k}}.$$

By interchanging the order of integration we obtain

$$A \leq \left(\int_{S_{P_{n,k}}} \int_{S_M} \langle f, g^2 \rangle^{2D_{n,k}} d\mu(f) d\sigma(g) \right)^{1/2D_{n,k}}. \tag{6.2.1}$$

Now we observe that the inner integral

$$\int_{S_M} \langle f, g^2 \rangle^{2D_{n,k}} d\mu(f)$$

clearly depends only on the length of the projection of g^2 into $M_{n,2k}$. Therefore, we have

$$\int_{S_M} \langle f, g^2 \rangle^{2D_{n,k}} d\mu(f) \leq ||g^2||_2^{2D_{n,k}} \int_{S_M} \langle f, p \rangle^{2D_{n,k}} d\mu(f),$$

for any $p \in S_M$.

We observe that

$$||g^2||_2 = (||g||_4)^2 \quad \text{and} \quad ||g||_2 = 1.$$

By a result of Duoandikoetxea [6] Corollary 3 it follows that

$$||g^2||_2 \leq 4^{2k}.$$

Hence we obtain

$$\int_{S_M} \langle f, g^2 \rangle^{2D_{n,k}} d\mu(f) \leq 4^{4kD_{n,k}} \int_{S_M} \langle f, p \rangle^{2D_{n,k}} d\mu(f).$$

We note that this bound is independent of g and substituting into (6.2.1) we get

$$A \leq 4^{2k} \left(\int_{S_M} \langle f, p \rangle^{2D_{n,k}} d\mu(f) \right)^{1/2D_{n,k}}.$$

Since $p \in S_M$ we have

$$\int_{S_M} \langle f, p\rangle^{2D_{n,k}} d\mu(f) = \frac{\Gamma(D_{n,k}+\frac{1}{2})\Gamma(\frac{1}{2}D_M)}{\sqrt{\pi}\Gamma(D_{n,k}+\frac{1}{2}D_M)}.$$

We use the following easy inequalities:

$$\Big(\frac{\Gamma(\frac{1}{2}D_M)}{\Gamma(D_{n,k}+\frac{1}{2}D_M)}\Big)^{1/2D_{n,k}} \leq \sqrt{\frac{2}{D_M}}$$

and

$$\Big(\frac{\Gamma(D_{n,k}+\frac{1}{2})}{\sqrt{\pi}}\Big)^{1/2D_{n,k}} \leq \sqrt{D_{n,k}},$$

to see that

$$A \leq 4^{2k}\sqrt{\frac{2D_{n,k}}{D_M}}.$$

We now recall that

$$D_{n,k} = \binom{n+k-1}{k} \quad \text{and} \quad D_M = \binom{n+2k-1}{2k} - 1.$$

Therefore

$$\sqrt{\frac{D_{n,k}}{D_M}} \leq \frac{(2k)!}{k!} n^{-k/2}.$$

Thus

$$A \leq \frac{4^k(2k)!\sqrt{2}}{k!} n^{-k/2}.$$

Theorem 6.2 now follows. ∎

6.2. Proof of the Lower Bound. We begin with a corollary of Theorem 6.2. Let B_{sq} be the unit ball of the norm $|| \, ||_{sq}$,

$$B_{sq} = \{f \in M_{n,2k} | \, ||f||_{sq} \leq 1\}.$$

From Theorem 6.2 we know that

$$\int_{S_M} ||f||_{sq} d\mu \leq \frac{4^{2k}(2k)!\sqrt{24}}{k!} n^{-k/2}.$$

It follows in the same way as in Section 3.1 that

$$\Big(\frac{\operatorname{Vol} B_{sq}}{\operatorname{Vol} B_M}\Big)^{1/D_M} \geq \frac{k!}{4^{2k}(2k)!\sqrt{24}} n^{k/2}.$$

Now let $\widetilde{Sq}^{\circ}$ be the polar of $\widetilde{Sq}_{n,2k}$ in $M_{n,2k}$. It follows easily that B_{sq} is the intersection of $\widetilde{Sq}^{\circ}$ and $-\widetilde{Sq}^{\circ}$:

$$B_{sq} = \widetilde{Sq}^{\circ} \cap -\widetilde{Sq}^{\circ}.$$

Let Sq^* be the dual cone of $Sq_{n,2k}$ in the integral metric and let $\widetilde{Sq^*}$ be defined in the same way as for the previous cones. It is not hard to check that $\widetilde{Sq}^{\circ}$ is the negative of $\widetilde{Sq^*}$:

$$\widetilde{Sq}^{\circ} = -\widetilde{Sq^*}.$$

Therefore, we see that

$$\left(\frac{\operatorname{Vol}\widetilde{Sq^*}}{\operatorname{Vol}B_M}\right)^{1/D_M} \geq \frac{k!}{4^{2k}(2k)!\sqrt{24}}n^{k/2}. \tag{6.2.2}$$

Now we observe that $(x_1^2 + \cdots + x_n^2)^k$ is in the interior of $Sq_{n,2k}$, and also for all non-zero f in $Sq_{n,2k}$ we have $\int_{S^{n-1}} f d\sigma > 0$. Therefore, we can apply Lemma 5.2 to $Sq_{n,2k}$ and it follows that

$$\left(\frac{\operatorname{Vol}\widetilde{Sq_d^*}}{\operatorname{Vol}\widetilde{Sq^*}}\right)^{1/D_M} \geq \frac{k!}{(n/2+2k)^k}.$$

Combining with (6.2.2) we see that

$$\left(\frac{\operatorname{Vol}\widetilde{Sq_d^*}}{\operatorname{Vol}B_M}\right)^{1/D_M} \geq \frac{(k!)^2}{4^{2k}(2k)!\sqrt{24}}\frac{n^{k/2}}{(n/2+2k)^k}.$$

By Lemma 5.3, we know that Sq_d^* in contained in $Sq_{n,2k}$ and therefore

$$\widetilde{Sq_d^*} \subseteq \widetilde{Sq}_{n,2k}.$$

The lower bound of Theorem 6.1 now follows.

7. Sums of $2k$-th powers of linear forms

In this section we prove the bounds for sums of powers of linear forms. Here is the precise statement of the bounds,

THEOREM 7.1: *There are the following bounds for the volume of $\widetilde{Lf}_{n,2k}$:*

$$\frac{k!(2k^2+n)^{1/2}}{4k\sqrt{2}(n/2+2k)^k} \leq \left(\frac{\operatorname{Vol}\widetilde{Lf}_{n,2k}}{\operatorname{Vol}B_M}\right)^{1/D_M} \leq 2\sqrt{n(4k+2)}\left(\frac{k!}{(n/2+k)^k}\right)^{\alpha}$$

where

$$\alpha = 1 - \left(\frac{2k-1}{n+2k-2}\right)^2.$$

7.1. Proof of the Lower Bound. We observe that the cone of sums of $2k$-th powers of linear forms is dual to the cone of nonnegative polynomials in the differential metric,

$$Lf_{n,2k} = Pos_d^*,$$

since in the differential metric,

$$\langle f, v^{2k}\rangle_d = (2k)!f(v) \quad \text{for all } f \in P_{n,2k}.$$

Therefore, it follows that

$$\widetilde{Lf}_{n,2k} = \widetilde{Pos_d^*}.$$

We first consider the dual cone Pos^* of $Pos_{n,2k}$ in the integral metric. Similarly to the situation with the cone of sums of squares, it is not hard to check that the dual $\widetilde{Pos}^\circ$ of $\widetilde{Pos}_{n,2k}$ in $M_{n,2k}$ with respect to the integral metric is $-\widetilde{Pos}^*$,

$$\widetilde{Pos}^\circ = -\widetilde{Pos}^*.$$

We recall that in Section 4.2 we have shown (4.2.4):

$$\left(\frac{\operatorname{Vol}\widetilde{Pos}^\circ}{\operatorname{Vol}B_M}\right)^{1/D_M} \geq \frac{1}{4}\left(\frac{2k^2+n}{2k^2}\right)^{1/2}.$$

Since $Pos_{n,2k}$ has $(x_1^2+\cdots+x_n^2)^k$ in its interior and $\int_{S^{n-1}} f d\sigma > 0$ for all non-zero f in $Pos_{n,2k}$, we can apply Lemma 5.2 to $Pos_{n,2k}$ and obtain

$$\left(\frac{\operatorname{Vol}\widetilde{Pos}_d^*}{\operatorname{Vol}\widetilde{Pos}^*}\right)^{1/D_M} \geq \frac{k!}{(n/2+2k)^k}.$$

Since $\widetilde{Lf}_{n,2k} = \widetilde{Pos}_d^*$ and $\widetilde{Pos}^\circ = -\widetilde{Pos}^*$ we can combine with (4.2.4) and get

$$\left(\frac{\operatorname{Vol}\widetilde{Lf}_{n,2k}}{\operatorname{Vol}B_M}\right)^{1/D_M} \geq \frac{k!}{4k\sqrt{2}}\frac{(2k^2+n)^{1/2}}{(n/2+2k)^k}.$$

7.2. Proof of the Upper Bound. We begin by applying the Blaschke–Santaló inequality to $\widetilde{Pos}_{n,2k}$ as in Section 3.2 to obtain

$$\frac{\operatorname{Vol}\widetilde{Pos}_{n,2k}\operatorname{Vol}\widetilde{Pos}^\circ}{(\operatorname{Vol}B_M)^2} \leq 1.$$

Since $\widetilde{Pos}^\circ = -\widetilde{Pos}^*$ we can rewrite this to get

$$\left(\frac{\operatorname{Vol}\widetilde{Pos}^*}{\operatorname{Vol}B_M}\right)^{1/D_M} \leq \left(\frac{\operatorname{Vol}B_M}{\operatorname{Vol}\widetilde{Pos}_{n,2k}}\right)^{1/D_M}.$$

We observe that by the lower bound of Theorem 4.1 it follows that

$$\left(\frac{\operatorname{Vol}\widetilde{Pos^*}}{\operatorname{Vol}B_M}\right)^{1/D_M} \leq 2\sqrt{n(4k+2)}. \tag{7.1.1}$$

Now we apply the upper bound of Lemma 5.2 to $Pos_{n,2k}$ and get

$$\left(\frac{\operatorname{Vol}\widetilde{Pos}_d^*}{\operatorname{Vol}\widetilde{Pos}^*}\right)^{1/D_M} \leq \left(\frac{k!}{(n/2+k)^k}\right)^{\alpha},$$

where

$$\alpha = 1 - \left(\frac{2k-1}{n+2k-2}\right)^2.$$

The upper bound now follows by combining with (7.1.1).

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