SUBSERIES CONVERGENCE IN TOPOLOGICAL GROUPS AND VECTOR SPACES

BY

N. J. KALTON

ABSTRACT

Let G be a separable complete metric additive topological group; it is shown that if a series Σx_n is subseries convergent in any weaker Hausdorff group topology on G, then Σx_n converges in G. This result can be used to obtain various extensions of the classical Orlicz-Pettis Theorem on subseries convergence in locally convex spaces.

A series $\sum_{n=1}^{\infty} x_n$ in an additive topological group is said to be *subseries convergent* if whenever (x_{k_n}) is a subsequence of (x_n) then $\sum_{n=1}^{\infty} x_k$ converges. A subseries convergent series is unconditionally convergent, i.e., for every rearrangement $(x_{\rho(n)})$ of (x_n) then the series $\sum_{n=1}^{\infty} x_{\rho(n)}$ converges. The classical Orlicz-Pettis Theorem asserts that in a locally convex topological vector space, subseries convergence in the weak topology implies subseries convergence in the original topology. This theorem was first proved in a special case by Orlicz [10]; since then it has been refined in several papers, see [2] p. 240, [3], [5], [9], [10], [12] and [13].

Recently Thomas [15] and Stiles [14] have extended the theorem to non-locally convex complete metric linear spaces with bases; Stiles poses the question whether the result is true if we only assume the weak topology is Hausdorff.

In this paper, by relating the Orlicz-Pettis Theorem to the Closed Graph Theorem we show that Stiles's conjecture is true if we restrict the space to be separable. The method of proof highlights the important role played by separability in the Orlicz-Pettis Theorem (see Thomas [15]). It also becomes apparent that the result is essentially about topological groups and only happens to assume a slightly simpler form in vector spaces.

Received July 26, 1971

Some improvements are obtained even when we restrict to locally convex spaces, provided we insist on some separability assumptions.

1. The group S.

Let Z be the additive group of integers with the discrete topology; then the product $\prod_{i=1}^{\infty}(Z)_i$ of countably many copies of Z is a topological group under the product topology π. We adopt the following notation:

$$S = \left\{\theta = (\theta_i) \in \prod_{i=1}^{\infty} (Z)_i : \sup_i |\theta_i| < \infty\right\},$$

$$S_n = \{\theta \in S;\ \theta_i = 0 \qquad 1 \leqq i \leqq n\}$$

$$K = \{\theta \in S;\ 0 \leqq \theta_i \leqq 1 \qquad i = 1, 2 \cdots\}$$

$$L_n = \{\theta \in S;\ -n \leqq \theta_i \leqq n \qquad i = 1, 2 \cdots\}.$$

Then S and S_n, $n = 1, 2 \cdots$, are subgroups of $\prod_{i=1}^{\infty}(Z)_i$, and K and L_n $n = 1, 2 \cdots$, are π-compact subsets of S (by an application of Tychonoff's theorem).

We introduce a topology μ on S thus: a base of neighbourhoods of zero is given by sets of the form

$$B\{n_k\} = \{\theta \in S;\ |\theta_k| \leqq n_k\}$$

where $n_k \to \infty$. It is easy to verify that (S, μ) is a topological group, and that $\mu \geqq \pi$.

PROPOSITION 1. *μ is the finest topology agreeing with π on each L_n.*

PROOF. First we show that μ agrees with π on L_n; suppose V is μ-open and $x \in V \cap L_n$. Then for some $n_k \to \infty$

$$x + B\{n_k\} \subset V.$$

Given n, there exists N such that for $k > N$, $n_k \geqq 2n$. Then

$$\begin{aligned}(x + S_N) \cap L_n &\subset (x + B\{n_k\}) \cap L_n \\ &\subset V \cap L_n.\end{aligned}$$

As S_N is a π-open subgroup of S we conclude that $V \cap L_n$ is π-open in L_n.

In order to prove that μ is the finest topology agreeing with π in each L_n, let us suppose $V \cap L_n$ is π-open in L_n for each n. We show first that if $0 \in V$ then 0 is a μ-interior point. The proof is similar to that of the Banach-Dieudonné Theorem on metrizable locally convex spaces (see: Köthe [8] p. 270).

We define $m(0) = 0$ and construct a sequence $m(n)$ inductively such that if $\theta \in L_n$ and

$$|\theta_k| \leqq p \qquad m(p-1) < k \leqq m(p)$$
$$p = 1, 2, \cdots n-1$$

then $\theta \in V$. Let us suppose $m(1), \cdots, m(n-1)$ have been defined; for $q \geqq 1$ we define F_q as the set of all $\theta \in S$ with

$$|\theta_k| \leqq p \qquad m(p-1) < k \leqq m(p)$$

and

$$p = 1, 2 \cdots, n-1$$

$$|\theta_k| \leqq n \qquad m(n-1) < k \leqq m(n-1) + q.$$

Then F_q is π-closed for each q. Let $D_{n+1} = L_{n+1} - (L_{n+1} \cap V)$; D_{n+1} is π-closed in L_{n+1}, and therefore π-compact; furthermore

$$\bigcap_{q=1}^{\infty} F_q \subset L_n$$

so that

$$D_{n+1} \cap \bigcap_{q=1}^{\infty} F_q = \varnothing$$

by the inductive hypothesis. Hence there exists q_0 such that

$$D_{n+1} \cap \bigcap_{q=1}^{q_0} F_q = \varnothing.$$

We define $m(n) = m(n-1) + q_0$, and it is clear that the inductive hypothesis is satisfied.

If we now set

$$p_k = r \quad \text{where} \quad m(r-1) < k \leqq m(r)$$

then

$$B\{p_k\} \subset V$$

so that 0 is a μ-interior point of V.

Now suppose $\theta \in V$; then $\theta \in L_m$ for some m and so for any fixed n we have $L_n \subset L_{m+n} - \theta$, and as $(V - \theta) \cap (L_{m+n} - \theta)$ is π-open in $L_{m+n} - \theta$, we obtain that $(V - \theta) \cap L_n$ is π-open in L_n for each n. Hence 0 is a μ-interior point of $V - \theta$ and so θ is a μ-interior point of V.

A group homomorphism $\alpha: G \to H$ between two topological groups is *almost* (or *nearly*) *continuous* if for every neighbourhood U of 0 in H, $\overline{\alpha^{-1}(U)}$ is a neighbourhood of 0 in G.

PROPOSITION 2. *Let α be a group homomorphism of S into an additive topological group G; then*

(i) *α is continuous if and only if the restriction of α to K is continuous.*

(ii) *α is almost continuous if and only if for every neighbourhood U of 0 in G, $\overline{\alpha^{-1}(U)} \cap K$ is a neighbourhood of 0 in K.*

PROOF. The proofs of (i) and (ii) are identical; we prove the latter. To prove (ii) we choose a sequence V_n of neighbourhoods of 0 in G such that

$$V_0 - V_0 \subset U$$

$$V_n + V_n \subset V_{n-1}$$

Then for some increasing sequence $m(n)$, $n \geqq 1$,

$$S_{m(n)} \cap K \subset \overline{\alpha^{-1}(V_n)}$$

We define $p_k = p$ where $m(p) \leqq k < m(p+1)$, with $p_k = 0$ for $k < m(1)$. Then for $\theta \in B\{p_k\} \cap L_n$

we have

$$\theta \in W_n - W_n$$

where

$$\begin{aligned} W_n &= (S_{m(1)} \cap K) + (S_{m(2)} \cap K) + \cdots + (S_{m(n)} \cap K) \\ &\subset \overline{\alpha^{-1}(V_1)} + \overline{\alpha^{-1}(V_2)} + \cdots + \overline{\alpha^{-1}(V_n)} \\ &\subset \overline{\alpha^{-1}(V_1 + \cdots + V_n)} \subset \overline{\alpha^{-1}(V_0)} \end{aligned}$$

so that

$$W_n - W_n \subset \overline{\alpha^{-1}(V_0 - V_0)} \subset \overline{\alpha^{-1}(U)}$$

and

$$B\{p_k\} \subset \overline{\alpha^{-1}(U)}.$$

2. The closed graph theorem in S

THEOREM 1. *Let α be a group homomorphism of S into the additive topological group G. Suppose $\alpha(S)$ is separable; then α is almost continuous.*

PROOF. Suppose U is a neighbourhood of 0 in G and that V is a neighbourhood with

$$V - V \subset U.$$

Then as $\alpha(S)$ is separable

$$\alpha(S) \subset \bigcup_{i=1}^{\infty} (\alpha(\theta^{(i)}) + V)$$

for some sequence $\theta^{(i)}$in S. Hence

$$S \subset \bigcup_{i=1}^{\infty} (\theta^{(i)} + \alpha^{-1}(V))$$

and so, as $K \subset S$ is compact, by the Baire category theorem there exists an n such that $\overline{\theta^{(n)} + \alpha^{-1}(V) \cap K}$ has non-zero interior in K. Thus for some N and some $\phi \in K$

$$(\phi + S_N) \cap K \subset \theta^{(n)} + \overline{\alpha^{-1}(V)}$$

Let $\psi \in S_N \cap K$; then we let

$$\chi_i = 1 \quad \text{if } \psi_i = \phi_i = 1$$

and

$$\chi_i = 0 \quad \text{otherwise.}$$

Then $\chi \in S_N, \psi - \chi \in S_N$

and

$$\phi - \chi \in K, \ \phi + (\psi - \chi) \in K.$$

Therefore

$$\phi - \chi - \theta^{(n)} \in \overline{\alpha^{-1}(V)}$$

$$\phi + \psi - \chi - \theta^{(n)} \in \overline{\alpha^{-1}(V)}$$

and so

$$\psi \in \overline{\alpha^{-1}(V)} - \overline{\alpha^{-1}(V)} \subset \overline{\alpha^{-1}(U)}.$$

Hence

$$S_N \cap K \subset \overline{\alpha^{-1}(U)}$$

and by Proposition 2, α is almost continuous.

COROLLARY. *We may replace the assumption "$\alpha(S)$ is separable" by the assumption "$\alpha(K)$ is separable".*

THEOREM 2. *Let G be a separable complete metrizable additive topological group; then any homomorphism $\alpha: S \to G$ with closed graph is continuous.*

PROOF. This follows immediately from Theorem 1 and Kelley [7] p. 213 Example R.

3. The Orlicz-Pettis Theorem in complete metric topological groups.

PROPOSITION 3. *Let (G,ρ) be an additive Hausdorff topological group; a homomorphism $\alpha: S \to G$ is continuous for ρ if and only if it takes the form*

$$\alpha(\theta) = \sum_{i=1}^{\infty} \theta_i x$$

where the infinite sums converge in ρ.

PROOF. Let $\varepsilon_k^{(n)} = \delta_{nk}$ (the Kronecker delta); then for $\theta \in S$

$$\theta = \sum_{n=1}^{\infty} \theta_n \varepsilon^{(n)}.$$

Let $\alpha(\varepsilon^{(n)}) = x_n$; if α is continuous we obtain

$$\alpha(\theta) = \sum_{n=1}^{\infty} \theta_n x_n \qquad (\rho).$$

Conversely suppose

$$\alpha(\theta) = \sum_{i=1}^{\infty} \theta_i x_i.$$

Then given a closed neighbourhood V of 0, we may show, by a simple reductio ad absurdum argument, that there exists n such that for all $\theta \in K$ with

$$\theta_i = 0 \qquad i \leqq n$$
$$\theta = 0 \qquad \text{eventually}$$

we have $\alpha(\theta) \in V$. It follows since for any $\theta \in K$

$$\sum_{i=1}^{\infty} \theta_i x_i = \lim_{m\to\infty} \sum_{i=1}^{m} \theta_i x_i$$

that

$$\alpha(S_n \cap K) \subset V$$

and so α is continuous on K. By Proposition 2, α is continuous.

THEOREM 3. *Let G be an additive group and ρ and τ be two Hausdorff topologies on G such that (G,ρ) and (G,τ) are separable topological groups: suppose $\rho \leqq \tau$ and τ is complete and metrizable. Then if Σx_i is ρ-subseries convergent it follows that Σx_i is τ-subseries convergent.*

PROOF. We define $\alpha: S \to G$ by

$$\alpha(\theta) = \sum_{i=1}^{\infty} \theta_i x_i$$

where the infinite sum converges in ρ. Then by Proposition 3, α is continuous as a map $\alpha: (S, \mu) \to (G, \rho)$; hence α has closed graph as a map $\alpha: (S, \mu) \to (G, \tau)$ and by Theorem 2 α is continuous for these topologies. The result follows by applying Proposition 2 again.

COROLLARY. *Let (E, τ) be a separable complete metric linear space, and suppose F is a separating family of continuous linear functionals on E. If Σx_i is $\sigma(E,F)$ subseries convergent, then Σx_i is τ-subseries convergent.*

This corollary improves the result of Thomas [15], who assumes that E is a Banach space and F has "positive index", or Stiles [14] who assumes that E has a basis, and that F is linear space generated by the co-ordinate functionals for the given basis.

A number of intriguing results follow from Theorem 3; the following theorem was first proved for Banach spaces by Bachelis and Rosenthal [1] and a different proof applicable to locally convex Frechet spaces was given by Bennett and Kalton [3]. We do not here assume local convexity.

THEOREM 4. *Let (E, τ) be a separable complete metric linear space and let $(x ; f_i)$ be a biorthogonal system in E, (with each f_i continuous), such that $\{f_i\}$ is total over E. Suppose $x \in E$ is such that for any sequences $\varepsilon_i = \pm 1$, there exists $y(\varepsilon) \in E$ with*

$$f_i(y(\varepsilon)) = \varepsilon_i f_i(x).$$

Then

$$x = \sum_{i=1}^{\infty} f_i(x) x_i.$$

PROOF. The conditions insure that Σx_i is $\sigma(E,F)$ subseries convergent where F is the linear subspace of E^* generated by $(f_n; n = 1, 2 \cdots)$; the result then follows by Theorem 3 or its Corollary.

An FK-space is a space of complex-valued sequences with a complete metric vector space topology τ such that the co-ordinate maps $x \to |\delta^n(x)| = |x_n|$ are continuous.

THEOREM 5. *Let E be a separable FK-space and suppose $x^{(n)} \in E$; then $\Sigma x^{(n)}$ converges unconditionally if and only if for every sequence $\varepsilon_n = \pm 1$ and for each k the series $\Sigma \varepsilon_n x_k^{(n)}$ converges and if $y_k = \Sigma \varepsilon_n x_k^{(n)}$ then $y \in E$.*

PROOF. Let F be the linear span of the functions $\{\delta^n\}$; we have that $\Sigma x^{(n)}$ is $\sigma(E,F)$ subseries convergent and the result again follows by Theorem 3 and its corollary.

The following theorem is well known (it follows easily, for example, from Theorem 4 of [4]).

THEOREM 6. *Let X be a separable Banach space which is the dual of a Banach space. Then X contains no subspace isomorphic to c_0.*

PROOF. Let $X = Y^*$ and suppose $x_n \in X$ is a basis of a subspace Z of X equivalent to the usual basis of c_0. Then for any sequence $\theta_n = 1$ or 0, we have that $\Sigma_{i=1}^n \theta_i x_i$ is a bounded sequence in X, and so is $\sigma(Y^*, Y)$ bounded in Y^*. Thus for $y \in Y$

$$\sum_{i=1}^{\infty} |x_i(y)| < \infty$$

and so $\Sigma_{i=1}^n \theta_i x_i$ is a $\sigma(Y^*, Y)$-Cauchy sequence; by the weak*-compactness of closed balls in Y^* we have that $\Sigma \theta_i x_i$ converges $\sigma(Y^*, Y)$, i.e. Σx_i is $\sigma(Y^*, Y)$ subseries convergent. By Theorem 3 Σx_i is norm convergent and this contradicts the choice of (x_n).

4. The Orlicz-Pettis Theorem in general topological groups

The restriction that G be complete and metrizable in Theorem 3 may be reduced if we include further restrictions on the topologies τ and ρ.

THEOREM 7. *Let G be additive group which becomes a Hausdorff topological group under the topologies ρ and τ; suppose $\rho \leqq \tau$ and τ has a base of ρ-closed neighbourhoods of 0. Suppose further that (G,τ) is separable; then if Σx_i is ρ-subseries convergent it follows that Σx_i is τ-subseries convergent.*

PROOF. Once again we define the map

$$\alpha: S \to G$$

by

$$\alpha(\theta) = \rho - \sum_{i=1}^{\infty} \theta_i x_i$$

and by Proposition 3, α is continuous for ρ.

Now let V be a ρ-closed τ-neighbourhood of 0 in G; let $\{V_n\}$, chosen inductively be ρ-closed τ-neighbourhoods of zero with

$$V_1 - V_1 \subset V$$

$$V_n - V_n \subset V_{n-1}.$$

Then $H = \cap_{n=1}^{\alpha} V_n$ is a ρ-closed subgroup of G; consider the quotient space G/H, and the quotient map $\pi: G \to G/H$. Let γ be a topology on G/H with a base of neighbourhoods $\{\pi V_n; n = 1, 2 \cdots\}$; then $(G/H, \gamma)$ is metrizable. Furthermore the map $\pi: (G, \tau) \to (G/H, \gamma)$ is continuous and surjective, so that G/H is γ-separable. Let M be the γ-completion of G/H; we identify G/H as a subspace of M, and consider the map $\pi\alpha: S \to M$.

We show that $\pi\alpha$ has a closed graph; suppose θ_λ is a net in S such that

$$\theta_\lambda \to \theta$$

$$\pi\alpha\theta_\lambda \to m$$

Then $(\pi\alpha\theta_\lambda)$ is Cauchy in G/H and so given n there exists λ_0 such that for λ, $\lambda' \geqq \lambda_0$

$$\pi\alpha\theta_\lambda' - \pi\alpha\theta_\lambda \in \pi V_n$$

$$\alpha\theta_\lambda' - \alpha\theta_\lambda \in V_n$$

As V_n is ρ-closed and $\alpha\theta_\lambda \to \alpha\theta$ in ρ

$$\alpha\theta - \alpha\theta_\lambda \in V_n \qquad \lambda \geqq \lambda_0$$

and we deduce that $\pi\alpha\theta_\lambda \to \pi\alpha\theta$ in γ, i.e. $m = \pi\alpha\theta$. Hence $\pi\alpha$ is continuous by Theorem 2, and so for $\theta \in S$ by Proposition 3 $\Sigma\theta_i\pi(x_i)$ converges in γ to $\pi\alpha\theta$. Thus for $n \geqq n_0$ (for some n_0) we have

$$\alpha\theta - \sum_{i=1}^{n} \theta_i x_i \in V$$

and this is true for any such V

$$\sum_{i=1}^{\infty} \theta_i x_i = \alpha(\theta) \text{ in } \tau.$$

THEOREM 8. *Let $\langle E,F \rangle$ be a dual pair of vector spaces and suppose that τ is an $\langle E,F \rangle$-polar topology on E such that (E,τ) is separable. Then if Σx_i is $\sigma(E,F)$-subseries convergent, then Σx_i is τ-subseries convergent.*

PROOF. As τ has a basis of $\sigma(E,F)$-closed neighbourhoods of zero this follows directly from Theorem 7.

COROLLARY. *If $(E,\beta(E,F))$ is separable then Σx_i is $\beta(E,F)$ convergent.*

The following theorem is essentially a generalization of the classical Orlicz-Pettis theorem for locally convex spaces; it improves for example the result of Thomas [15].

THEOREM 9. *Let $\langle E,F\rangle$ be a dual pair of vector spaces and suppose τ is an $\langle E,F\rangle$-polar topology on E; then if Σx_i is $\sigma(E,F)$ subseries convergent, it is necessary and sufficient for Σx_i to be τ-subseries convergent that the set*

$$X = (\Sigma\theta_i x_i;\, \theta \in K)$$

is contained in a τ-separable subspace of E.

PROOF. One direction is an immediate consequence of Theorem 8. Conversely if Σx_i is τ-subseries convergent then X is τ-separable and so lin X is τ-separable.

It is to be observed that this includes the classical Orlicz-Pettis theorem; for if τ is an $\langle E,F\rangle$-dual topology then τ and $\sigma(E,F)$ will define the same separable subspaces.

REFERENCES

1. G. F. Bachelis and H. P. Rosenthal, *On unconditionally converging series and biorthogonal systems in a Banach space*, Pacific J. Math., **37** (1971), 1–6.

2 S. Banach, *Théorie des Operations Linéares*, Warsaw, 1932.

3. G. Bennett and N. J. Kalton, *FK-spaces containing c_0*, to appear.

4. C. Bessaga and A. Pelczynski, *On bases and unconditional convergence of series in Banach spaces*, Studia Math. **17** (1958), 151–164.

5. A. Grothendieck, *Sur les applications linéares faiblement compactes d'espaces $C(K)$*, Canad. J. Math. 5 (1953), 129–173.

6. N. J. Kalton, *Some forms of the closed graph theorem*, Proc. Cambridge Philos. Soc., **70** (1971), 401–408.

7. J. L. Kelley, *General Topology*, New York, 1955.

8. G. Köthe, *Topological Vector Spaces*, Berlin, 1969.

9. C. W. McArthur, *A note on subseries convergence*, Proc. Amer. Math. Soc. **12** (1961), 540–545.

10. C. W. McArthur, *On a theorem of Orlicz and Pettis*, Pacific J. Math. **22** (1967), 297–302.

11. W. Orlicz, *Beitrage zur Theorie der Orthogonalentwicklungen* II, Studia Math. **1** (1929), 241–255.

12. B. J. Pettis, *On integration in vector spaces*, Trans. Amer. Math. Soc. **44** (1938), 277–304.

13. A. P. Robertson, *On unconditional convergence in topological vector spaces*, Proc. Roy. Soc. Edinburgh, Sect. A **68** (1969), 145–157.

14. W. J. Stiles, *On subseries convergence in F-spaces*, Israel J. Math. **8** (1970), 53–56.

15. E. Thomas, *Sur le théoreme d'Orlicz et un problème de M. Laurent Schwartz*, C. R. Acad. Sci. Paris. Sér. A. **267** (1968), A7–A10.

DEPARTMENT OF PURE MATHEMATICS
UNIVERSITY COLLEGE, SWANSEA