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NILPOTENT INJECTORS IN SYMMETRIC GROUPS

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ABSTRACT

An N-Injector in an arbitrary finite group G is defined as a maximal nilpotent subgroup of G, containing a subgroup A of G of maximal order satisfying class$(A) \leqq 2$. Among other results the N-Injectors of Sym(n) are determined and shown to consist of a unique conjugacy class of subgroups of Sym(n).

A. Introduction

N-Injectors in a finite group G are maximal nilpotent subgroups which share many properties with the Sylow subgroups. The theory of Injectors, and in particular N-Injectors has been developed mostly for solvable groups. The aim of this paper is to develop the theory of N-Injectors for arbitrary finite groups, and to determine the N-Injectors of Sym(n).

N-Injectors were first defined in [9] as follows: a subgroup A of G is an N-Injector, if for each $H \lhd\lhd G$, $A \cap H$ is a maximal nilpotent subgroup of H. In [13] it has been proved that if $C(F(G)) \subseteq F(G)$, then G contains N-Injectors, they form a conjugacy class, and they can be characterized as the maximal nilpotent subgroups which contain $F(G)$. The following two observations (a) and (b) show that neither the definition of N-Injectors in [9], nor their characterization in [13] determine in general a unique conjugacy class of maximal nilpotent subgroups in G.

(a) The maximal nilpotent subgroups of Sym(5) are: S_2(Sym(5)), S_5(Sym(5)) and $C(S_3$(Sym(5)). They intersect Alt(5) in S_2(Alt(5)), S_5(Alt(5)) and S_3(Alt(5)), respectively. The intersections are maximal nilpotent subgroups of Alt(5), and since Alt(5) and Sym(5) are the only non-trivial subnormal subgroups of Sym(5), it follows that each maximal nilpotent subgroup of Sym(5) is an N-Injector by the definition in [9].

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(b) As $F(\mathrm{Sym}(5)) = 1$, each maximal nilpotent subgroup of Sym(5) is an N-Injector by the characterization in [13] as well.

Therefore, we will use another characterization of N-Injectors. First a definition.

DEFINITION 1. (i) $d(k, G)$ will denote the maximum of the orders of all nilpotent subgroups of G of class at most k.

(ii) $\mathcal{A}(k, G)$ will denote the set of all nilpotent subgroups of G of class at most k, having order $d(k, G)$.

(iii) $\mathcal{A}(\infty, G)$ $(d(\infty, G))$ will denote the set (order) of nilpotent subgroups of maximal order.

A theorem of Bender proved in [2] states: if $C(F(G)) \subseteq F(G)$, then A is an N-Injector iff A is a maximal nilpotent subgroup of G containing an element of $\mathcal{A}(2, G)$. Now it will be convenient to adopt the following definition of N-Injectors.

DEFINITION 2. A subgroup A of G is called an N-Injector if A is a maximal nilpotent subgroup containing an element of $\mathcal{A}(2, G)$. The set of N-Injectors of G will be denoted by $N\mathrm{I}(G)$.

Our definition assures that any group G contains N-Injectors. If $C(F(G)) \subseteq F(G)$, then by [2] and [13] our definition is equivalent to the original definition in [9].

Section B contains calculations which lead to the evaluation of $d(2, \mathrm{Sym}(n))$, and as a consequence, Theorem B.8, $N\mathrm{I}(\mathrm{Sym}(n))$ is determined and shown to consist of a single conjugacy class.

THEOREM B.8. (a) Sym(n) *contains a single class of N-Injectors.*

(b) *If* $n \not\equiv 3 \pmod 4$, *then each N-Injector is a* 2-*Sylow subgroup of* Sym(n).

If $n = 3 \pmod 4$, *then each N-Injector is the subgroup generated by a* 3-*cycle and a* 2-*Sylow subgroup of* Sym($n - 3$) *on the remaining* $n - 3$ *symbols, and each such subgroup belongs to* $N\mathrm{I}(\mathrm{Sym}(n))$.

In Section C, we have obtained results about $N\mathrm{I}(\mathrm{Sym}(n))$ and the sets $\mathcal{A}(k, \mathrm{Sym}(n))$ similar to known results in groups of odd order, [1], [5]. The main result is that $N\mathrm{I}(\mathrm{Sym}(n)) = \mathcal{A}(\infty, \mathrm{Sym}(n))$.

In Section D, we introduce a generalization of N-Injectors, namely π-N-Injectors. First a definition.

DEFINITION 3. Let π be a set of primes.

(i) $d(\pi, k, G)$ will denote the maximum of orders of all nilpotent π-subgroups of G of class at most k.

(ii) $\mathscr{A}(\pi, k, G)$ will denote the set of all nilpotent π-subgroups of G of class at most k having order $d(\pi, k, G)$.

(iii) $\mathscr{A}(\pi, \infty, G)$ ($d(\pi, \infty, G)$) will denote the set (order) of nilpotent π-subgroups of maximal order.

Definition 4. A subgroup A of G is called a π-N-Injector if A is a maximal nilpotent π-subgroup containing an element of $\mathscr{A}(\pi, 2, G)$. The set of π-N-Injectors of G will be denoted by $N\mathrm{I}(\pi, G)$.

Remarks. (a) If π is the set of all primes, then:

(i) $\mathscr{A}(\pi, k, G) = \mathscr{A}(k, G)$.

(ii) $\mathscr{A}(\pi, \infty, G) = \mathscr{A}(\infty, G)$.

(iii) $N\mathrm{I}(\pi, G) = N\mathrm{I}(G)$.

(b) If π consists of a single prime, then:

$\mathscr{A}(\pi, \infty, G)$, $N\mathrm{I}(\pi, G)$ and the set of p-Sylow subgroups of G, coincide.

(c) If G is solvable and $\{H^x \mid x \in G\}$ is the set of π-Hall subgroups of G then:

(i) $\mathscr{A}(\pi, k, G) = \bigcup_{x \in G} \mathscr{A}(\pi, k, H^x)$.

(ii) $\mathscr{A}(\pi, \infty, G) = \bigcup_{x \in G} \mathscr{A}(\pi, \infty, H^x)$.

(iii) $N\mathrm{I}(\pi, G) = \bigcup_{x \in G} N\mathrm{I}(H^x)$.

We suggest the following two conjectures.

Conjecture 1. *Let G be a finite group and π a set of primes, then $N\mathrm{I}(\pi, G)$ is a conjugacy class.*

Conjecture 2. *Let G be a finite group and π a set of primes, then $\mathscr{A}(\pi, \infty, G)$ is a conjugacy class.*

Remarks. (d) Conjecture 1 holds in solvable groups.

(e) Conjecture 2 holds in groups of odd order, since then $\mathscr{A}(\pi, \infty, G) = N\mathrm{I}(\pi, G)$ [1], [5].

(f) If $\pi = \{p\}$, Conjectures 1 and 2 hold by the Sylow theorem.

(g) In general, $N\mathrm{I}(G)$ and $\mathscr{A}(\infty, G)$ don't coincide. Examples of groups of order $p^\alpha q^\beta$ where it happens can be deduced from [7]. Another example is the Mathieu group M_{11}. By [11] $\mathscr{A}(\infty, M_{11})$ consists of the 2-Sylow subgroups of M_{11}, while $N\mathrm{I}(M_{11})$ consists of the 11-Sylow subgroups of M_{11}.

The main result of Section D is that for $G = \mathrm{Sym}(n)$, Conjectures 1 and 2 hold.

B. The N-Injectors of Sym(n)

NI(Sym(n)) will be determined in three steps.

I. Evaluation of $d(2, S_p(\mathrm{Sym}(n)))$.

II. Evaluation of $d(2, M)$, where M is any maximal nilpotent subgroup of Sym(n).

III. Finding the maximal nilpotent subgroups of Sym(n) which maximize $d(2, M)$, and showing that they constitute a unique conjugacy class. Thus, the N-Injectors are all conjugate.

Step I. Evaluation of $d(2, S_p(\mathrm{Sym}(n)))$

Since $S_p(\mathrm{Sym}(np + r)) = S_p(\mathrm{Sym}(np))$ for $0 \leqq r < p$, it is enough to consider $S_p(\mathrm{Sym}(np))$. We will deal with p odd and $p = 2$ separately.

If p is an odd prime, consider the arithmetic progression $\{kp^2 + p + 1 \mid k = 1, 2, \cdots\}$. By the Dirichlet Theorem, we can find a prime $q = kp^2 + p + 1$ for some k. Clearly, $p \mid q - 1$, but $p^2 \nmid q - 1$. For such q the groups $S_p(\mathrm{Sym}(np))$ [14, p. 11] and $S_p(\mathrm{GL}(n, q))$ [16] are isomorphic (as abstract groups, not as permutation groups). Thus, $d(2, S_p(\mathrm{Sym}(np)))$ will be evaluated as a consequence of Lemma B.1 on p-subgroups of GL(n, q).

If $p = 2$, then the group $Z_2 \wr S_2(\mathrm{Sym}(n)) \cong S_2(\mathrm{Sym}(2n))$ can be represented faithfully as a linear group, acting on a vector space V over GF(3) of dimension n, in the following way. Let $V = V_1 \oplus V_2 \oplus \cdots V_n$ be the direct sum of n 1-dimensional subspaces. Let Z_2 act on each V_i, and let $S_2(\mathrm{Sym}(n))$ permute the subspaces V_i, $1 \leqq i \leqq n$. We note that this embedding of $S_2(\mathrm{Sym}(2n))$ in $S_2(\mathrm{GL}(n, 3))$ is not onto [6]. But still, $d(2, S_2(\mathrm{Sym}(2n)))$ will be evaluated as a consequence of Lemma B.2 on 2-subgroups of GL(n, 3). The final result of Step I will be summarized in Theorem B.3. Results of the nature of Lemma B.1 and Lemma B.2 appear in [10].

LEMMA B.1. *Let p be an odd prime and P a p-subgroup of* GL(n, q) *of class at most $p - 1$. If $p \mid q - 1$ but $p^2 \nmid q - 1$, then $|P| \leqq p^n$.*

PROOF. By induction on n. First we will check the lemma for $n \leqq p$. Since

$$|S_p(\mathrm{GL}(n, q))| = \begin{cases} p^n & \text{if } n < p \\ p^{p+1} & \text{if } n = p \end{cases}$$

it suffices to consider the case $n = p$. However, class $(S_p(\mathrm{GL}(p, q))) = p$, hence $|P| \leqq p^p$. Assume, therefore, that $n > p$ and let P act on V, a vector space of dimension n. By induction, the theorem holds for any p-group of class at most

$p-1$ which acts faithfully on V', where $\dim(V')<n$. If V is reducible under P, say $V=V_1\oplus V_2$, let $P_i=P/C_P(V_i)$ and $|V_i|=q^{n_i}$, $i=1,2$. We obtain:

$$|P|\leqq|P_1|\,|P_2|\leqq p^{n_1}\cdot p^{n_2}=p^{n_1+n_2}=p^n.$$

So, we may assume that V is irreducible. As $P\subseteq S_p(\mathrm{GL}(n,q))\cong S_p(\mathrm{Sym}(np))$, therefore, if P is cyclic, then $|P|\leqq p^{[\log_p(np)]}\leqq p^n$, and we are through. It is left to consider the case where P is non-cyclic. By [14, 19.2] P has a subgroup H of index p such that we can write V as a direct sum, $V=V_1\oplus V_2\oplus\cdots V_p$, where each V_i is an H-invariant subspace, and if $x\in P\setminus H$, then $V_ix=V_{i'}$, where the permutation $i\to i'$ is a p-cycle.

Let $K_i=C_H(V_i)$, $1\leqq i\leqq p$ and take $x\in P\setminus H$. Here, by induction we have $|H/K_i|\leqq p^{n/p}$, hence $|H|\leqq p^n$. If $|H|<p^n$, we obtain $|P|\leqq p^n$, so we may assume $|H|=p^n$. That means $H\cong\Pi_{i=1}^{p}(H/K_i)$, so H is the direct product of its projections on V_i, these projections being conjugate through x. Letting y be an element of order p that H induces on V_i, it follows that P contains $\langle y,x\rangle\cong Cp\wr Cp$, of class p, which contradicts the assumption class $(P)\leqq p-1$, and Lemma B.1 is proved.

LEMMA B.2. *If P is a 2-subgroup of* $\mathrm{GL}(n,3)$ *of class at most two, then* $|P|\leqq 2^{n+[n/2]}$.

PROOF. By induction on n. First, we will check the lemma for $n\leqq 3$. We have:

$$|S_2(\mathrm{GL}(n,3))|=\begin{cases}2 & \text{if } n=1,\\ 2^4 & \text{if } n=2,\\ 2^5 & \text{if } n=3.\end{cases}$$

Following [6] $S_2(\mathrm{GL}(2,3))$ is Semidihedral of class 3 and $S_2(\mathrm{GL}(3,3))$ is the direct product of $S_2(\mathrm{GL}(2,3))$ and Z_2, again a group of class 3, hence if P is of class at most two, then $|P|\leqq 2^3$ if $n=2$, and $|P|\leqq 2^4$ if $n=3$. This completes the checking. Assume, therefore, that $n>3$ and let P act on V, a vector space of dimension n. By induction, the lemma holds for any 2-subgroup of class at most two which acts faithfully on V', where $\dim(V')<n$. If V is reducible under P, say $V=V_1\oplus V_2$, let $P_i=P/C_P(V_i)$ and $|V_i|=3^{n_i}$, $i=1,2$. We obtain:

$$|P|\leqq|P_1|\,|P_2|\leqq 2^{n_1+[n_1/2]}2^{n_2+[n_2/2]}=2^{n+[n_1/2]+[n_2/2]}\leqq 2^{n+[n/2]}.$$

So we may assume that V is irreducible. Since class $(P)\leqq 2$, it follows that if P is Quaternion, Dihedral or Semidihedral, then $|P|=2^3<2^{n+[n/2]}$ for $n>3$ as required.

If P is cyclic, let $[n/2] = \Sigma_{i=1}^{s} 2^{\alpha_i}$ be the 2-adic representation of $[n/2]$. Then $S_2(\mathrm{GL}(n,3))$ is $\Pi_{i=1}^{s} G_i$ for n even and $(\Pi_{i=1}^{s} G_i) \times Z_2$ for n odd, where $G_i = T \wr S_2(\mathrm{Sym}(2^{\alpha_i}))$ and T denotes the Semidihedral group of order 2^4. Since T has a faithful permutation representation of degree 2^3, it follows that G_i has such a representation of degree $2^3 \cdot 2^{\alpha_i}$. Thus $S_2(\mathrm{GL}(n,3))$ has a faithful permutation representation of degree less than or equal to $2^3(\Sigma_{i=1}^{s} 2^{\alpha_i}) + 2 = 2^3[n/2] + 2$. It follows that $|P| \leqq 2^3[n/2] + 2 \leqq 2^{n+[n/2]}$ for $n > 3$, as required.

It is left to consider the case where P is not cyclic Quaternion, Semidihedral, or Dihedral. Now the proof can be continued as in Lemma B.1. By [14, 19.2] P has a subgroup H of index 2 such that $V = V_1 \oplus V_2$ where V_i, $i = 1,2$ are H-invariant subspaces of V and each $x \in P \setminus H$ permutes V_1 and V_2. Let $K_i = C_H(V_i)$, it can be assumed that $H = K_1 \times K_2$ and $P \cong K_1 \wr C_2$. Here $\mathrm{class}(P) \leqq 2$ implies that K_1 is abelian. But K_1 is irreducible on V_2 (otherwise P is reducible), so K_1 is cyclic, and now $\mathrm{class}(P) \leqq 2$ is possible only for $|K_1| = 2$, $|P| = 8$, and the lemma certainly holds for this case. Lemma B.2 is proved.

Theorem B.3. *If $m = np + r$, where $0 \leqq r < p$, then*:

$$d(2, S_p(\mathrm{Sym}(m))) = d(2, S_p(\mathrm{Sym}(np))) = \begin{cases} p^n & \text{if } p \text{ is odd,} \\ 2^{n+[n/2]} & \text{if } p = 2. \end{cases}$$

Proof. In view of our discussion preceding Lemma B.1, it follows from Lemmas B.1 and B.2 that:

$$d(2, S_p(\mathrm{Sym}(np))) \leqq \begin{cases} p^n & \text{if } p \text{ is odd,} \\ 2^{n+[n/2]} & \text{if } p = 2. \end{cases}$$

In fact, the equality holds. If p is odd, then the group generated by n disjoint p-cycles is elementary abelian of order p^n. If $p = 2$, divide $2n$ into $[n/2]$ sets of 4 elements each (in the case where n is odd there is a remainder of a two element set). Since the Dihedral group of order 8 has class two and acts faithfully on a set of 4 elements, we can construct a direct product of $[n/2]$ such groups, adding a transposition to the product if n is odd. In any case, a group of order $2^{n+[n/2]}$ is obtained.

Corollary B.4.

$$d(2, S_p(\mathrm{Sym}(p^n))) = \begin{cases} p^{p^{n-1}} & \text{if } p \text{ is odd,} \\ 2 & \text{if } p = 2 \text{ and } n = 1, \\ 2^{3 \cdot 2^{n-2}} & \text{if } p = 2 \text{ and } n > 1. \end{cases}$$

Step II. Evaluation of $d(2, M)$

First, we will describe the structure of the maximal nilpotent subgroups of $\mathrm{Sym}(n)$, [15, I.5]. Let $n = \Sigma_{i=1}^{k} n_i$ be a partition of n into k positive integers and assume it satisfies:

(i) The integer 1 occurs in the partition at most once.

(ii) The integer p^α (p is a prime) occurs in the partition at most $p-1$ times.

Then, to each such partition, there corresponds a unique conjugacy class of maximal nilpotent subgroups. Let $\mathrm{Sym}(n)$ act on $\Omega = \{1, 2, \cdots, n\}$. Take any partition $\{\Omega_i \mid 1 \leqq i \leqq k\}$ of Ω such that $|\Omega_i| = n_i$ for $1 \leqq i \leqq k$. Each subgroup in the class is a direct product of k transitive nilpotent subgroups G_i of degree n_i, acting on Ω_i, $1 \leqq i \leqq k$. Each G_i is isomorphic to a direct product of p-Sylow subgroups of $\mathrm{Sym}(p^\alpha)$ [12, p. 379], where p^α is the highest power of p dividing n_i and p runs over the primes which divide n_i.

REMARKS. (a) The structure of the subgroups of $\mathrm{Sym}(n)$ maximal with respect to being nilpotent and transitive, can be deduced from the above description. These are the subgroups which correspond to the trivial partition $n = n_1$ and they form a conjugacy class.

(b) To each partition of n there corresponds a nilpotent subgroup as described above, but conditions (i) and (ii) imply its maximality.

NOTATION. Though the n_i's in a partition of n need not be distinct, we will use the set notation $\{n_1, \cdots, n_r\}$ in order to denote partitions.

THEOREM B.5. *If M is a maximal nilpotent subgroup of* $\mathrm{Sym}(n)$ *which corresponds to the partition $\{n_1, \cdots, n_r\}$, then $d(2, M) = \Pi_{i=1}^{r} \varphi_2(n_i)$, where*

$$\varphi_2(m) = \begin{cases} 1 & \text{if } m = 1, \\ 2 & \text{if } m = 2, \\ 2^{3\cdot 2^{\alpha-2}} & \text{if } m = 2^\alpha,\ \alpha \geqq 2, \\ p^{p^{\alpha-1}} & \text{if } m = p^\alpha,\ p \neq 2,\ \alpha > 0, \\ \prod_{i=1}^{l} \varphi_2(p_i^{\alpha_i}) & \text{if } m = \prod_{i=1}^{l} p_i^{\alpha_i}, \\ & \text{where } p_i \text{ are distinct primes for } 1 \leqq i \leqq l. \end{cases}$$

PROOF. By the discussion above $M = \Pi_{i=1}^{r} M_i$, where each M_i is a nilpotent subgroup of M of degree n_i. By Corollary B.4, $d(2, M) \geqq \Pi_{i=1}^{r} \varphi_2(n_i)$ and since every subgroup of M of class at most two is contained in a product of its projections on the M_i's, we obtain an equality.

LEMMA B.6. *If* $p^\alpha > 3$, *where* p *is an odd prime and* $p^\alpha = 2^{\alpha_1} + \cdots 2^{\alpha_s}$ *is the 2-adic representation of* p^α, *then*: $\prod_{i=1}^s \varphi_2(2^{\alpha_i}) > \varphi_2(p^\alpha)$.

PROOF. The lemma can be easily checked up to $p^\alpha \leqq 17$, so assume $p^\alpha > 17$. If $p^\alpha = 3 \pmod 4$, then $\prod_{i=1}^s \varphi_2(2^{\alpha_i}) = 2^{-\frac{5}{4}} \prod_{i=1}^s 2^{3 \cdot 2\alpha_i - 2}$ and if $p^\alpha = 1 \pmod 4$, then $\prod_{i=1}^s \varphi_2(2^{\alpha_i}) = 2^{-\frac{3}{4}} \prod_{i=1}^s 2^{3 \cdot 2\alpha_i - 2}$. So in any case we get in view of $p^\alpha > 17$ and $p \geqq 3$:

$$\prod_{i=1}^s \varphi_2(2^{\alpha_i}) \geqq 2^{-\frac{5}{4}} \prod_{i=1}^s 2^{3 \cdot 2\alpha_i - 2} = 2^{-\frac{5}{4}} \cdot 2^{\frac{3}{4}\sum_{i=1}^s 2\alpha_i} = 2^{\frac{3}{4}p^\alpha - \frac{5}{4}} > 2^{\frac{2}{3}p^\alpha} > p^{p^{\alpha-1}} = \varphi_2(p^\alpha).$$

LEMMA B.7. *Let* $\{n_1, \cdots, n_r\}$ *be a partition of* n *which maximizes the product* $\prod_{i=1}^r \varphi_2(n_i)$. *Then either all the* n_i*'s are powers of* 2, *or* $n_1 = 3$ *and all the other* n_i*'s (if* $r > 1$) *are powers of* 2.

PROOF. If one of the n_i's equals $p^\alpha t$, where p is an odd prime, $p^\alpha > 3$ and $(t, p) = 1$, then replacing n_i by the set $\{2^{\alpha_i} t \mid 1 \leqq i \leqq s\}$, where $p^\alpha = 2^{\alpha_1} + \cdots + 2^{\alpha_s}$ yields in view of Lemma B.6:

$$\prod_{i=1}^s \varphi_2(2^{\alpha_i} t) \geqq \prod_{i=1}^s (\varphi_2(2^{\alpha_i})\varphi_2(t)) = (\varphi_2(t))^s \prod_{i=1}^s \varphi_2(2^{\alpha_i})$$

$$\geqq \varphi_2(t) \prod_{i=1}^s \varphi_2(2^{\alpha_i}) > \varphi_2(t)\varphi_2(p^\alpha) = \varphi_2(tp^\alpha) = \varphi_2(n_i)$$

contradicting the maximality of $\prod_{i=1}^r \varphi_2(n_i)$. Hence each n_i is of the form $3^\alpha \cdot 2^\beta$, where $0 \leqq \alpha \leqq 1$ and $\beta \geqq 0$. If $\alpha = 1$ and $\beta > 0$ then replacing $3 \cdot 2^\beta$ by 2^β and $2^{\beta+1}$ again yields a contradiction since $\varphi_2(3 \cdot 2^\beta) < \varphi_2(2^\beta)\varphi_2(2^{\beta+1})$. So $\alpha = 1$ implies $\beta = 0$. We conclude that each n_i is either 3 or 2^β, $\beta \geqq 0$. Since $\varphi_2(3)\varphi_2(3) = 9 < 16 = \varphi_2(4)\varphi_2(2)$, $n_i = 3$ can occur at most once and the lemma is proved.

THEOREM B.8. (a) Sym(n) *contains a single class of* N*-Injectors.*

(b) *If* $n \neq 3 \pmod 4$, *then each* N*-Injector is a* 2*-Sylow subgroup of* Sym(n).

If $n = 3 \pmod 4$, *then each* N*-Injector is the subgroup generated by a* 3*-cycle and a* 2*-Sylow subgroup of* Sym($n - 3$) *on the remaining* $n - 3$ *symbols, and each such subgroup belongs to* NI(Sym(n)).

PROOF. Let M be an N-Injector of Sym(n) corresponding to a partition $\{n_1, \cdots, n_r\}$. By Lemma B.7, the n_i's are powers of 2 and possibly one of them is 3. Since M is a maximal nilpotent subgroup of Sym(n), conditions (i) and (ii) imply that all the n_i's are distinct. Hence, either the n_i's are the terms in the 2-adic representation of n, or 3 occurs and the other n_i's are the terms in the 2-adic representation of $n - 3$, whichever yields a larger $d(2, M)$. Throughout the proofs of Lemmas B.6 and B.7 strict inequalities were involved hence for each n a unique partition corresponds to an N-Injector of Sym(n). Thus by [15], NI(Sym(n)) consists of a unique conjugacy class and (a) follows. A simple checking determines the n_i's in the two cases where $n \not\equiv 3 \pmod 4$ and where $n \equiv 3 \pmod 4$. Applying [14, p. 11] (b) follows.

REMARK. The 2-Sylow subgroups of Sym(n) are the only self-normalizing nilpotent subgroups, namely the Carter subgroups of Sym(n) [3]. By Theorem B.8, the Carter subgroups and the N-Injectors of Sym(n) coincide for $n \not\equiv 3$ (mod 4) and are similar, but different, for $n \equiv 3 \pmod 4$.

C. Further properties of NI(Sym(n))

We will investigate the sets $\mathcal{A}(k, \mathrm{Sym}(n))$, where k is a positive integer or equals ∞, and obtain results similar to known results for groups of odd order [1], [5]. The problem of evaluation of $d(1, \mathrm{Sym}(n))$ was solved by Bercov and Moser [4], who reduced it to an arithmetical problem: find a partition $\{n_1, \cdots, n_r\}$ of n which maximizes the product $\Pi_{i=1}^r n_i$. Then they used [8] and obtained the following formula:

$$d(1, \mathrm{Sym}(n)) = \begin{cases} 3^k & \text{if } n = 3k, \\ 4 \cdot 3^{k-1} & \text{if } n = 3k+1, \\ 2 \cdot 3^k & \text{if } n = 3k+2. \end{cases}$$

In Section B, we have already evaluated $d(2, \mathrm{Sym}(n))$. Now we will prove:

THEOREM C.1. *Let k denote ∞ or a positive integer different from* 1 *or* 3 *and let* $A \in \mathcal{A}(k, \mathrm{Sym}(n))$.

(a) *If* $n \not\equiv 3 \pmod 4$, *then A is a* 2-*subgroup.*

If $n \equiv 3 \pmod 4$, *then A is generated by a* 3-*cycle and a* 2-*subgroup on the remaining* $n-3$ *symbols.*

(b) $\mathcal{A}(\infty, \mathrm{Sym}(n)) = N\mathrm{I}(\mathrm{Sym}(n))$.

(c) *There is a* $B \in \mathcal{A}(\infty, \mathrm{Sym}(n))$ *such that* $B \supseteq A$, *and any maximal nilpotent subgroup of* Sym(n) *which contains A belongs to* $\mathcal{A}(\infty, \mathrm{Sym}(n))$.

PROOF. We first prove (a). The case $k = 2$ has already been proved, so assume $k \geqq 4$ or $k = \infty$. For each prime power p^α, define

$$\varphi_k(p^\alpha) = d(k, S_p(\mathrm{Sym}(p^\alpha)))$$

and if $m = \Pi_{i=1}^s p_i^{\alpha_i}$ where p_i are distinct primes, let $\varphi_k(m) = \Pi_{i=1}^s \varphi_k(p_i^{\alpha_i}) = d(k, \Pi_{i=1}^s S_{p_i}(\mathrm{Sym}(p_i^{\alpha_i})))$. If $A \in \mathcal{A}(k, \mathrm{Sym}(n))$, then $A \subseteq M$ where M is a maximal nilpotent subgroup of Sym(n). Thus, $d(k, \mathrm{Sym}(n)) = \max\{d(k, M) \mid M$ is a maximal nilpotent subgroup of Sym(n)} and let the maximum value be obtained on M, which corresponds to the partition $\{n_1, \cdots, n_r\}$. Clearly [12, p. 379] $\varphi_k(p^\alpha) \leqq \varphi_\infty(p^\alpha) = p^{(p^\alpha - 1)/(p-1)}$, $\varphi_k(1) = 1$, $\varphi_k(2) = 2$, $\varphi_k(4) = 8$ and if $\alpha \geqq 3$, then $\varphi_k(2^\alpha) \geqq \varphi_4(2^\alpha) \geqq 2^{7 \cdot 2^{\alpha-3}}$, since $S_2(\mathrm{Sym}\, 2^3)$ is of order 2^7 and of class 4 [12, p. 379].

We claim now that an analogue of Lemma B.6 holds: if $p^\alpha > 3$, where p is an odd prime and $p^\alpha = 2^{\alpha_1} + \cdots + 2^{\alpha_s}$ is the 2-adic representation of p^α, then: $\prod_{i=1}^s \varphi_k(2^{\alpha_i}) > \varphi_k(p^\alpha)$ for $k \geqq 4$. It is easy to check the claim for $p^\alpha \leqq 31$, so assume, since p^α is a power of an odd prime, that $p^\alpha \geqq 37$. Using our estimates it suffices to prove:

$$2^{-\frac{17}{8}} \prod_{i=1}^{s} 2^{7 \cdot 2^{\alpha_i - 3}} > p^{(p^\alpha - 1)/(p-1)} \qquad \text{or} \qquad 2^{\frac{7}{8}p^\alpha - \frac{17}{8}} > p^{(p^\alpha - 1)/(p-1)}.$$

Since $p^\alpha \geqq 37$, hence $\frac{7}{8}p^\alpha - \frac{17}{8} > \frac{13}{16}p^\alpha$ and the problem reduces to proving

$$2^{\frac{13}{16}p^\alpha} > p^{(p^\alpha - 1)/(p-1)} \qquad \text{or} \qquad \frac{13}{16}p^\alpha > \frac{p^\alpha - 1}{p-1}\log_2 p.$$

But

$$\frac{13}{16}p^\alpha = \frac{13}{16}(p-1)\frac{p^\alpha}{p-1} > \frac{p^\alpha - 1}{p-1}\log_2 p$$

since $\frac{13}{16}(p-1) > \log_2 p$ for $p \geqq 3$. The claim has been proved. Now applying the arguments of the proofs of Lemma B.7 and Theorem B.8, we deduce that M is either $S_2(\mathrm{Sym}(n))$ or a subgroup generated by a 3-cycle and $S_2(\mathrm{Sym}(n-3))$. Thus, $d(k, \mathrm{Sym}(n)) = \max\{3d(k, S_2(\mathrm{Sym}(n-3))), d(k, S_2(\mathrm{Sym}(n)))\}$. Let $n = 4m + \varepsilon$, where $\varepsilon = 0, 1, 2, 3$. Then for $\varepsilon = 0, 1, 2$ we get:

$$\begin{aligned} 3d(k, S_2(\mathrm{Sym}(n-3))) &\leqq 3 \cdot 2d(k, S_2(\mathrm{Sym}(n-4-\varepsilon))) \\ &\leqq \frac{6}{8}\, d(k, S_2(\mathrm{Sym}(n-\varepsilon))) < d(k, S_2(\mathrm{Sym}(n))) \end{aligned}$$

and if $\varepsilon = 3$, then

$$d(k, S_2(\mathrm{Sym}(n))) \leqq 2d(k, S_2(\mathrm{Sym}(4m))) < 3d(k, S_2(\mathrm{Sym}(4m))).$$

These inequalities complete the proof of (a). Parts (b) and (c) follow directly from the proof of part (a).

The problem of $\mathcal{A}(3, (\mathrm{Sym}(n))$ will be dealt with elsewhere. The following example shows that the case $k = 3$ (as well as the case $k = 1$) is exceptional.

Example. $S_3(\mathrm{Sym}(9))$ is of order 3^4 and class 3, while $S_2(\mathrm{Sym}(9))$ is of order 2^7 and class 4, [12, p. 379] and it is easy to see that $\mathcal{A}(3, \mathrm{Sym}(9))$ is the set of 3-Sylow subgroups of Sym(9).

D. π-N-**Injectors in** Sym(n)

π-N-Injectors have been defined in Section A. We will now prove that onjectures 1 and 2 of Section A hold in Sym(n).

THEOREM D.1. *Let π be any set of primes, then $NI(\pi, \mathrm{Sym}(n))$ is a conjugacy lass.*

PROOF. If $2, 3 \in \pi$, then $NI(\pi, \mathrm{Sym}(n)) = NI(\mathrm{Sym}(n))$ and hence is a conjugcy class. If $2 \in \pi$ but $3 \not\in \pi$, then using the same arguments as in the proof of heorem B.8 it follows that $NI(\pi, \mathrm{Sym}(n))$ is the set of the 2-Sylow subgroups of ym(n) and, hence again, is a conjugacy class. So let us assume that $2 \not\in \pi$. We ill prove that any element of $NI(\pi, \mathrm{Sym}(n))$ has the following form: it is the -Hall subgroup of a maximal nilpotent subgroup of Sym(n) which corresponds o a uniquely defined partition $\{m, n_1, \cdots, n_s\}$ of n satisfying the following onditions:

(a) $m < \min\{p \mid p \in \pi\}$ or $m = 0$.

(b) For each i, $1 \leqq i \leqq s$, $n_i = p_i^{\alpha_i}$ where $p_i \in \pi$ not necessarily distinct and $\alpha_i \geqq 1$.

Since each subgroup in the set $\mathscr{A}(\pi, 2, \mathrm{Sym}(n))$ is contained in a maximal ilpotent subgroup M,

$$d(\pi, 2, \mathrm{Sym}(n)) = \max\{d(\pi, 2, M) \mid M \text{ is maximal nilpotent in } \mathrm{Sym}(n)\}.$$

et $\{m_1, \cdots, m_r\}$ be a partition of n which corresponds to an M_1 such that $d(\pi, 2, M_1) = d(\pi, 2, \mathrm{Sym}(n))$. Then define $m = \Sigma\, m_i$, where the summation runs ver those i's for which $d(\pi, 2, \mathrm{Sym}(m_i)) = 1$. Clearly $m < \min\{p \mid p \in \pi\}$. Let M e a maximal nilpotent subgroup of Sym(n) corresponding to the partition $\{m, n_1, \cdots, n_s\}$ where the n_i's are those m_j's which satisfy $d(\pi, 2, \mathrm{Sym}(m_j)) > 1$. bviously $d(\pi, 2, M) = d(\pi, 2, M_1) = d(\pi, 2, \mathrm{Sym}(n))$. It follows from the maxinality of $d(\pi, 2, M)$ that only primes from π divide each n_i, $1 \leqq i \leqq s$. We will ow prove that $n_i = p_i^{\alpha_i}$ where $p_i \in \pi$ and $\alpha_i \geqq 1$. Suppose that $n_i = p^{\alpha} q^{\beta}$, where $p, q \in \pi$ and W.L.O.G. $p > q$. By splitting n_i into q^{β} parts of cardinality p^{α} ach, we get a contribution of $p^{p^{\alpha-1}q^{\beta}}$ instead of $p^{p^{\alpha-1}} q^{q^{\beta-1}}$, thus increasing $d(\pi, 2, \mathrm{Sym}(n))$. So n_i cannot be a product of powers of two distinct primes, and sing induction, we obtain $n_i = p_i^{\alpha_i}$.

It is left to prove that $d(\pi, 2, \mathrm{Sym}(n))$ determines the n_i's uniquely. Denote by $w(p)$ the sum of all the n_i's which are powers of the prime p, and let $w(p) = \Sigma_{k=1}^{t} a_k p^{\alpha_k}$, $1 \leqq a_k \leqq p-1$, $\alpha_k \geqq 1$ be the p-adic representation of $w(p)$. By the maximality of M each p^{α_k} appears exactly a_k times as an n_i for some i. Since $2 \not\in \pi$, Theorem B.3 implies that the exponent of p in $d(\pi, 2, M)$ is

$\Sigma_{k=1}^{t} a_k p^{\alpha_k - 1}$ and hence the a_k's and the α_k's are uniquely determined by $d(\pi, 2, M)$. We conclude that all the n_i's are uniquely determined and hence the elements of $NI(\pi, \mathrm{Sym}(n))$ form a single conjugacy class.

THEOREM D.2. *Let π be any set of primes, then the set $\mathcal{A}(\pi, \infty, \mathrm{Sym}(n))$ is a conjugacy class.*

PROOF. The proof is quite similar to the proof of Theorem D.1. We will prove that if $H \in \mathcal{A}(\pi, \infty, \mathrm{Sym}(n))$ then it is a π-Hall subgroup of a maximal nilpotent subgroup of $\mathrm{Sym}(n)$ which corresponds to a uniquely defined partition $\{m, n_1, \cdots, n_s\}$ of n satisfying the following conditions:

(a) $m < \min\{p \mid p \in \pi\}$ or $m = 0$.

(b) For each i, $1 \leqq i \leqq s$, $n_i = p_i^{\alpha_i}$ where $p_i \in \pi$ not necessarily distinct and $\alpha_i \geqq 1$.

Define $\{m_1, \cdots, m_r\}$, $\{m, n_1, \cdots, n_s\}$ and M similarly to their definitions in the proof of Theorem D.1. We will now prove that $n_i = p_i^{\alpha_i}$ for some $p_i \in \pi$ and $\alpha_i \geqq 1$. If $p \mid n_i$ then clearly $p \in \pi$. Suppose that $n_i = p^\alpha q^\beta$ where $p, q \in \pi$ and W.L.O.G. $p > q$. Splitting n_i into q^β parts of cardinality p^α each, we get since $|S_p(\mathrm{Sym}(p^\alpha))| = p^{(p^\alpha - 1)/(p-1)}$ a contribution of

$$p^{((p^\alpha - 1)/(p-1))q^\beta} > p^{(p^\alpha - 1)/(p-1)} q^{(q^\beta - 1)/(q-1)},$$

thus increasing $d(\pi, \infty, \mathrm{Sym}(n))$. We have proved that n_i cannot be a product of powers of two distinct primes, and using induction, we obtain $n_i = p_i^{\alpha_i}$.

It is left to prove that $d(\pi, \infty, \mathrm{Sym}(n))$ determines the n_i's uniquely. Denote by $w(p)$ the sum of all the n_i's which are powers of the prime p, and let $w(p) = \Sigma_{k=1}^{t} a_k p^{\alpha_k}$, $1 \leqq a_k \leqq p-1$, $\alpha_k \geqq 1$ be the p-adic representation of $w(p)$. By the maximality of M each p^{α_k} appears exactly a_k times as n_i for some i. But by [14, p. 11]

$$|S_p(M)| = \left| \prod_{k=1}^{t} (S_p(\mathrm{Sym}(p_k^{\alpha_k})))^{a_k} \right| = |S_p(\mathrm{Sym}(w(p)))|$$

and since p divides $w(p)$, $|S_p(M)|$ and hence $d(\pi, \infty, \mathrm{Sym}(n))$ uniquely determine $w(p)$. Thus $d(\pi, \infty, \mathrm{Sym}(n))$ uniquely determines the a_k's and the α_k's, and consequently also the n_i's. The uniqueness of the n_i's implies that the elements of $\mathcal{A}(\pi, \infty, \mathrm{Sym}(n))$ form a single conjugacy class.

We will conclude with some examples:

EXAMPLES. (a) For $k \geqq 2$ or $k = \infty$ $\mathcal{A}(\{3\}', k, \mathrm{Sym}(n))$ is a set of 2-subgroups of $\mathrm{Sym}(n)$.

(b) For $k \geqq 1$ or $k = \infty$ $\mathcal{A}(\{3, 5\}', k, \mathrm{Sym}(n))$ is a set of 2-subgroups of $\mathrm{Sym}(n)$.

(c) $NI(\{2\}', \mathrm{Sym}(11))$ consists of groups generated by the disjoint 3-cycles and a disjoint 5-cycle, while $\mathscr{A}(\{2\}', \infty, \mathrm{Sym}(11))$ is the set of 3-Sylow subgroups of Sym(11). Thus, the fact that $NI(G) = \mathscr{A}(\infty, G)$ for groups of odd order cannot be extended to $NI(\{2\}', G) = \mathscr{A}(\{2\}', \infty, G)$ for arbitrary groups. However, it holds if G is solvable.

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