TREES AND THE BIREFLECTION PROPERTY

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ABSTRACT

The group of automorphisms of a tree (partially ordered set where the set of predecessors of an element is well ordered) with no infinite levels enjoys the property that every member is a product of two elements of order $\leqq 2$. It is shown that this property — called the bireflection property — fails for some trees having infinite levels. In fact, every subtree of a tree T has the bireflection property if and only if the tree of all zero-one sequences of length $\leqq \omega$ with finitely many ones is not embeddable in T.

0. Introduction

We call an element φ of a group G a *reflection* if $\varphi = \varphi^{-1}$. A *bireflection* of $\theta \in G$ is an ordered pair (φ, ψ) of reflections satisfying $\theta = \varphi\psi$. G is called *bireflectional* iff every element of G has some bireflection. Bireflectional groups are frequent and play an important role among groups encountered in Geometry (see Bachmann [1]; Veblen and Young [10], e.g., §§108, 121, 122; and Coxeter [2], p. 3). It is well known that the symmetric groups are bireflectional (see, e.g., Scott [9], 10.1.17). Bireflectionality in classical groups is studied by Ellers [3].

We say that a mathematical structure T has the *bireflection property* (*b.r.p.*) if its automorphism group $\mathrm{Aut}(T)$ is bireflectional. T has *hereditarily* the b.r.p. if every substructure of T has the b.r.p. Thus, abstract sets have hereditarily the b.r.p., by the bireflectionality of the symmetric groups.

The object of this paper is the b.r.p. in trees. By a *tree* we shall mean a partial order T where the set of predecessors of each element is well-ordered. A tree T is naturally partitioned into levels T_μ labeled by ordinals, where an element a belongs to T_μ iff the set of predecessors of a has order type μ. The *length* lT of T is defined to be the first ordinal μ such that $T_\mu = \varnothing$. As an abstract set T is also a tree of length 1, we see that T has the b.r.p. whenever $lT = 1$. Micha Perles (unpublished) discovered that whenever $lT \leqq \omega$, T has the b.r.p.[†] Now, every

[†] A short proof of the b.r.p. when $lT < \omega$ runs as follows:

(a) Let $G = \Pi_{i \in I} G_i$. If G_i is bireflectional for each $i \in I$, so is G.

(b) Let $G = K[H]$ (the wreath product of H by K). If K, H are bireflectional, so is G.

(c) Let $lT < \omega$. Then $\mathrm{Aut}(T)$ is obtained from symmetric groups by finitely many applications of direct products and wreath products.

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subset of a tree of length $\leqq \omega$ forms with the inherited order again such a tree. Thus, Micha Perles' Theorem implies:

THEOREM A. *Every tree of length $\leqq \omega$ has hereditarily the b.r.p.*

We generalize it as follows (Theorem 4.13):

THEOREM B. *A tree T has hereditarily the bireflection property if and only if T has no subset order-isomorphic to $\hat{T}^0$.*

Here $\hat{T}^0$ is the tree of all zero-one sequences of length $\leqq \omega$ that contain only finitely many ones. By some obvious mutual embedding relations $\hat{T}^0$ can be replaced by an arbitrary augmented $\boldsymbol{k}$-tree (a notion to be defined shortly; see also Definitions 4.0, 4.9).

Theorem B as well as the rest of the results in this paper, were obtained in 1972, stimulated by Perles' discovery of Theorem A. The rudimentary facts we use depend on an analysis of the set of bireflections of a permutation [5], that proves useful in handling some problems in the symmetric group ([4, 6–8]). It is summarized in §2. In §3 we obtain a parallel analysis for the set of bireflections of a tree automorphism of particular type, called a bounded tree automorphism (Definition 3.5). It follows that every bounded tree automorphism has some bireflection (Theorem 3.12). Since trees of length $\leqq \omega$ have only bounded automorphisms, Theorem A follows (Corollary 3.13).

Let $\boldsymbol{k} = (k_n)_{n \in \omega}$ be a sequence of integers greater than one. A *$\boldsymbol{k}$-tree* is a tree of length ω with a unique minimal element, such that every member of the n'th level has precisely k_n successors. An *augmented $\boldsymbol{k}$-tree* $\hat{T}$ is obtained from a $\boldsymbol{k}$-tree T by selecting a countable set of paths (maximal linearly ordered subsets) of T, whose union is T, and adding a maximal element to each. It turns out that any unbounded tree automorphism witnesses a subset of the tree, isomorphic to some augmented $\boldsymbol{k}$-tree. In §4 automorphisms of $\boldsymbol{k}$-trees are studied, and it is shown that no augmented $\boldsymbol{k}$-tree has the b.r.p. (Theorem 4.12). Theorem B follows.

The set $\tilde{T}$ of all paths of a $\boldsymbol{k}$-tree T is practically the same as the cartesian product $\Pi_{n \in \omega} K_n$, where K_n has k_n elements. We endow it with the metric of first difference, the distance between two different paths being $1/n + 1$, where n is the first place they differ. With this metric, $\tilde{T}$ is a compact metric space homeomorphic to Cantor's discontinuum C. The reason for choosing this particular metric is that a T-automorphism and a $\tilde{T}$-isometry are interchangeable notions (Proposition 4.1). Several of our auxiliary results seem to have independent interest, restated topologically. We collect some in the following lemma (Lemma 4.2, Corollaries 4.3, 4.6):

LEMMA. *Let $\mathcal{A}$ be a disjointed countable family of countable dense subsets of the Cantor Discontinuum C. Endow C with the metric of first difference.* Then:

(a) *There is an isometry of C having each member of $\mathcal{A}$ for an orbit.*

(b) *Let θ be any permutation of $\mathcal{A}$. Then there is an isometry χ of C, such that χ maps A onto $\theta(A)$ for every $A \in \mathcal{A}$.*

The assumption that $\mathcal{A}$ is countable is essential. In fact, using the well ordering principle it is easy to obtain a partition of C into countable dense sets such that no homeomorphism of C has all parts for orbits. The negation of (b) is even simpler, whenever $|\mathcal{A}| = K$ and 2^K exceeds the cardinality of the continuum.

1. Notation

ω and Z denote the set $\{0, 1, 2, \cdots\}$ of natural numbers and the set of integers respectively. An ordinal number is identified with the set of smaller ordinals, and so 0 is identical with the empty set $\emptyset$. We denote integers and ordinals $\leqq \omega$ by m, n. i, j, k, p, q, r, s, t denote integers. λ, μ, ν denote arbitrary ordinals. Other greek letters denote functions.

For $m, n \in Z \cup \{\omega\}$ we write $m \mid n$ iff $m, n \in Z$ and $n = mr$ for some $r \in Z$, or if $n \in \{0, \omega\}$. Thus, $n \in Z$ and $\omega \mid n$ implies $n = 0$ and $m \mid \omega$ holds for every $m \in Z \cup \{\omega\}$.

Let θ be a function, A a subset of its domain. Then $\theta \mid A$ denotes the restriction of θ to A, and $\theta''A = \{\theta(a) : a \in A\}$. Thus θ'' is a function defined on the power set of the domain of θ.

Let α be a function defined on an ordinal ν. We think of α also as a well ordered sequence of length ν and write $l\alpha = \nu$. If ν is finite we also write $\alpha = (\alpha(0), \cdots, \alpha(\nu - 1))$. For $\mu \leqq l\alpha$ we define $\bar{\alpha}(\mu) = \alpha \mid \mu$. We write $\gamma < \alpha$ iff $\gamma = \bar{\alpha}(\mu)$ for some $\mu < l\alpha$, i.e. when γ is a proper initial segment of α.

Let A be a set. Then $|A|$ denotes the cardinality of A (identified with the smallest ordinal of that cardinality), and A^ν the set of all functions from ν to A (well ordered sequences over A of order type ν). Note that if $A = \emptyset$ or $\nu = 0$ then $A^\nu = \{\emptyset\}$. Let $A^{<\nu} = \bigcup_{\mu<\nu} A^\mu$.

A *tree* is a nonempty subset T of $A^{<\nu}$ satisfying: $\alpha \in T$ and $\mu < l\alpha$ implies $\bar{\alpha}(\mu) \in T$. Thus, $\emptyset$ is a minimal element of every tree. If T is a tree, μ an ordinal then $T_\mu = \{\alpha \in T : l\alpha = \mu\}$, and lT is the smallest μ such that $T_\mu = \emptyset$.

Let T, T' be trees, and let τ be a function defined on T. Then we let

$\tau_\mu = \tau \mid T_\mu$. Let now $\tau : T \to T'$. Then τ is *order preserving* (*o.p.*) iff for every $a, b \in T$, $a < b$ implies $\tau(a) < \tau(b)$. τ is called *order preserving onto* (*o.p.o.*) iff it is o.p. and maps T onto T'. τ is an *embedding* if it is one to one and $a < b$ if and only if $\varphi(a) < \varphi(b)$. τ is an *isomorphism* iff it is an embedding onto T, and it is an *automorphism* iff it is an isomorphism and $T = T'$.

Let T be a tree. A *chain* in T is a linearly ordered subset, and a *path* is a maximal chain. Define a set of functions defined on ordinals $\tilde{T}$ by $\alpha \in \tilde{T}$ iff $\bar{\alpha}(\mu) \in T$ for every $\mu < l\alpha$, but for no $b \in T$ $\alpha < b$ holds. Let $T^* = T \cup \tilde{T}$. Then T^* is the smallest tree containing T, enjoying the property that every path has a maximal element, and $\tilde{T}$ is the set of maximal elements of T^*.

Let $\tau : T \to T'$ be o.p.o. Then there exist a unique extension $\tau^* : T^* \to T'^*$ which is o.p.o. We let $\tilde{\tau} = \tau^* \mid \tilde{T}$. Thus $\tilde{\tau}$ is defined by $\overline{\tilde{\tau}(\alpha)}(n) = \tau(\bar{\alpha}(n))$. Clearly, the mapping $\theta \to \tilde{\theta}$ is a group-monomorphism of Aut(T) into $S_{\tilde{T}}$, the symmetric group on $\tilde{T}$.

We shall also need to consider trees in the broader sense mentioned in the introduction. A tree in the broad sense (*tree i.b.s.*) is a set T endowed with a partial order (transitive irreflexive relation) $<$, where for every $x \in T$ the set $(-, x) = \{y : y < x\}$ is well ordered. Most of our previously defined notions for trees have obvious counterparts for trees i.b.s., which we do not reproduce here. A tree i.b.s. T is isomorphic to a tree if and only if it satisfies the extra condition:

(*) if $(-, x) = (-, y)$ has no maximal element, then $x = y$.

Notice that (*) implies that T has a unique minimal element. Every tree i.b.s. T is naturally embeddable in a tree i.b.s. T' satisfying (*) with the same group of automorphisms (to obtain T' from T add a unique maximal element to each set $(-, x)$ with no maximal element). Hence our results for trees hold for trees i.b.s.

2. The bireflections of a permutation

We summarize here for the reader's convenience notation and results from [5]. Let A be a nonempty set and let θ be a permutation of A. For $a \in A$ let $(a)_\theta = \{\theta^m(a) : m \in Z\}$, and let $(A)^\theta = \{(a)_\theta : a \in A\}$. Thus $(a)_\theta$ is the θ-orbit containing a and $(A)^\theta$ is the partition of A into θ-orbits.

Let S_A denote the group of all permutations of A and let

$$\mathrm{BR}(\theta) = \{(\varphi, \psi) : \varphi, \psi \in S_A,\ \varphi^2 = \psi^2 = \mathbf{1},\ \theta = \varphi\psi\}.$$

Let Φ be a reflection of $(A)^\theta$, i.e. $\Phi \in S_{(A)^\theta}$ and $\Phi^2 = 1$. Define $\mathrm{BR}(\theta;\Phi) \subseteq \mathrm{BR}(\theta)$ by:

$$\mathrm{Br}(\theta;\Phi) = \{(\varphi, \psi) \in \mathrm{BR}(\theta) : \varphi'' \mid (A)^\theta = \psi'' \mid (A)^\theta = \Phi\}.$$

2.0. THEOREM. *Let* $(\varphi, \psi) \in \mathrm{BR}(\theta)$. *Then* $\varphi'' = \psi'' = \Phi$ *is a cardinality preserving reflection of* $(A)^\theta$. *Hence* $\mathrm{BR}(\theta) = \bigcup \mathrm{BR}(\theta; \Phi)$, *where* Φ *ranges on all cardinality preserving reflections of* $(A)^\theta$.

Let Φ be a cardinality preserving reflection of $(A)^\theta$. A *coupling for* Φ is a pair $\rho = (\rho_1, \rho_{-1})$ of functions defined on $((A)^\theta)^\Phi$ with values in A, satisfying for every $u \in ((A)^\theta)^\Phi$

$$u = \{(\rho_1(u))_\theta, (\rho_{-1}(u))_\theta\}.$$

$(\varphi, \psi) \in \mathrm{BR}(\theta)$ *obeys a coupling* ρ for Φ iff for every $u \in ((A)^\theta)^\Phi$, $\varphi(\rho_1(u)) = \rho_{-1}(u)$ holds.

2.1. THEOREM. *Let* Φ *be a cardinality preserving reflection of* $(A)^\theta$.

(1) *Every coupling* ρ *for* Φ *has a unique bireflection* $(\varphi, \psi) \in \mathrm{BR}(\theta; \Phi)$ *such that* (φ, ψ) *obeys* ρ.

(2) *Every* $(\varphi, \psi) \in \mathrm{BR}(\theta; \Phi)$ *obeys some coupling* ρ *for* Φ.

Let $|A| = n$, $1 \leqq n \leqq \omega$, and let θ be a cycle in S_A, i.e., $(A)^\theta = \{A\}$. When $n < \omega$ we call a pair (a, b) of members of A *an antipodal pair* (with respect to the cycle θ) if $b = \theta^s(a)$, where $s = [n/2]$ is the greatest integer not greater than $n/2$.

Let $(\varphi, \psi) \in \mathrm{BR}(\theta)$. $a_0 \in A$ is a φ*-point* (ψ*-point*) iff $\varphi(a_0) = a_0$($\psi(a_0) = a_0$). a_0 is a *reflection point* of (φ, ψ) iff it is either a φ-point or a ψ-point.

2.2. THEOREM. *Let* $\theta \in S_A$ *be a cycle,* $(\varphi, \psi) \in \mathrm{BR}(\theta)$;

(i) *if* $|A| = \omega$ *then there is exactly one reflection point,*

(ii) *if* $1 < |A| < \omega$ *then there are exactly two reflection points that form an antipodal pair* (a_0, b_0). *If* $|A|$ *is even, then* a_0, b_0 *are both* φ*-points or both* ψ*-points. If* $|A|$ *is odd, then* a_0 *is a* φ*-point and* b_0 *is a* ψ*-point.*

3. The bireflections of bounded tree automorphisms

Throughout this section T is a fixed tree and $\theta \in \mathrm{Aut}(T)$. Let $\tau^\theta : T \to (T)^\theta$ be the natural mapping, i.e., $\tau^\theta(a) = (a)_\theta$. Define a binary relation $<$ on $(T)^\theta$ by

$$(1) \qquad x < y \Leftrightarrow \exists a \in x \, \exists b \in y [a < b];$$

clearly

$$(2) \qquad x < y \Leftrightarrow \forall a \in x \, \exists b \in y [a < b] \Leftrightarrow \forall b \in y \, \exists a \in x [a < b].$$

Let T^θ denote the set $(T)^\theta$ with the relation $<$. We omit the straightforward proof of the following proposition:

3.0. PROPOSITION. *T^θ is a tree in the broad sense and $\tau^\theta : T \to T^\theta$ is order preserving onto (o.p.o.).*

3.1. COROLLARY. *τ^θ_μ maps T_μ onto T^θ_μ for every ordinal $\mu < lT$, and $lT^\theta = lT$.*

The following proposition is obvious, yet extremely useful.

3.2. PROPOSITION. *Let $a, b \in T$, $a < b$, and let $n = |(a)_\theta|$. For $m \in Z$ let $B_m = \{b' \in (b)_\theta : \theta^m(a) < b'\}$. Then $\{B_m : m \in Z\}$ is a partition of $(b)_\theta$ into n sets of cardinality k, where k is the least cardinal satisfying $n \cdot k = |(b)_\theta|$. In fact, θ^m maps B_0 onto B_m, and $B_m = B_{m'}$ iff $n \mid m - m'$.*

3.3. COROLLARY. (a) *Let $x, y \in T^\theta$, $x < y$, $|x| = m$, $|y| = n$. Then $m \mid n$.*

(b) *Let $a \in x \in T^\theta$, and define $T_a \subseteq T$, $T_x \subseteq T^\theta$ by:*

$$T_a = \{b \in T : b < a \text{ or } b = a \text{ or } a < b\},$$

$$T_x = \{y \in T^\theta : y < x \text{ or } y = x \text{ or } x < y\}.$$

If $|y| = |x|$ whenever $y \in T^\theta$ and $x < y$, then $\tau^\theta \restriction T_a$ is an isomorphism of T_a onto T_x. Hence $\tau^\theta \restriction T_x$ is an isomorphism if $|x| = \omega$.

Define $\mathrm{BR}_<(\theta) \subseteq \mathrm{Aut}(T) \times \mathrm{Aut}(T)$, $R_\theta \subseteq \mathrm{Aut}(T^\theta)$ by:

$$\mathrm{BR}_<(\theta) = \mathrm{BR}(\theta) \cap (\mathrm{Aut}(T) \times \mathrm{Aut}(T)),$$

$$R_\theta = \{\Phi \in \mathrm{Aut}(T^\theta) : \Phi^2 = \mathbf{1} \text{ and } \forall x \in T^\theta[|\Phi(x)| = |x|]\}.$$

Note that R_θ is never empty, as the identity $\mathbf{1}$ is in R_θ.

For $\Phi \in R_\theta$ define $\mathrm{BR}_<(\theta; \Phi) \subseteq \mathrm{BR}(\theta)$ by:

$$\mathrm{BR}_<(\theta : \Phi) = \mathrm{BR}(\theta; \Phi) \cap \mathrm{BR}_<(\theta).$$

The following is a generalization of Theorem 2.0:

3.4. THEOREM. *Let $(\varphi, \psi) \in \mathrm{BR}_<(\theta)$. Then $\Phi = \varphi'' = \psi'' \in R_\theta$. Hence $\mathrm{BR}_<(\theta) = \bigcup_{\Phi \in R_\theta} \mathrm{BR}_<(\theta, \Phi)$.*

PROOF. Let $(\varphi, \psi) \in \mathrm{BR}_<(\theta)$. By Theorem 2.0 we know that $\Phi = \varphi'' = \psi''$ is a cardinality preserving permutation of T^θ of order $\leqq 2$. It is left to show that $\Phi \in \mathrm{Aut}(T^\theta)$. Let $x, y \in T^\theta$, $x < y$. Pick $a \in x$, $b \in y$ such that $a < b$. Since $\varphi \in \mathrm{Aut}(T)$, $\varphi(a) < \varphi(b)$, and so $(\varphi(a))_\theta < (\varphi(b))_\theta$. But $(\varphi(a))_\theta = \Phi(x)$, $(\varphi(b))_\theta = \Phi(y)$, and so $\Phi(x) < \Phi(y)$. Conversely, if $\Phi(x) < \Phi(y)$ then $\Phi(\Phi(x)) = x < y = \Phi(\Phi(y))$.

Since Φ is a permutation of T^θ we conclude $\Phi \in \mathrm{Aut}(T^\theta)$. □

Let $\Phi \in R_\theta$. Then $(T^\theta)^\Phi$ is a set of nonempty sets, each of which contains at most two θ-orbits of the same cardinality. In analogy with (1), define for $u, v \in (T^\theta)^\Phi$:

$$u < v \Leftrightarrow \exists x \in u \; \exists y \in v \; x < y.$$

We let $T^{\theta,\Phi}$ denote $(T^\theta)^\Phi$ with the relation $<$. Since $\Phi \in \mathrm{Aut}(T^\theta)$, Proposition 3.0 implies that $T^{\theta,\Phi}$ is a tree and the natural mapping $\tau^\Phi : T^\theta \to T^{\theta,\Phi}$ given by $\tau^\Phi(x) = (x)_\Phi$ is o.p.o. Let $\tau^{\theta,\Phi} = \tau^\Phi \tau^\theta$. Then $\tau^{\theta,\Phi} : T \to T^{\theta,\Phi}$ is o.p.o.

Let $\tau : T \to T_1$ be o.p.o. We call $\delta : T_1 \to T$ *a generalized inverse* (*g.i.*) of τ iff δ is an embedding, $\delta\tau\delta = \delta$ and $\tau\delta\tau = \tau$. It is easily checked that if $\tau_1 : T_1 \to T_2$ is also o.p.o. and δ_1 is a g.i. of τ_1 then $\delta\delta_1 : T_2 \to T$ is a g.i. of $\tau_1\tau$.

Now $\delta : T^\theta \to T$ is a g.i. of τ^θ iff δ is an o.p. choice function on T^θ. We note that τ^θ may not have a g.i. In fact, let T be the tree of all zero-one sequences of length $\leqq \omega$ and let $\theta \in \mathrm{Aut}(T)$ satisfy $T_n \in T^\theta$ for every n (we call such a θ a minimal automorphism; see Definition 4.4). Then any o.p. $\delta : T^\theta \to T$ must be constant on T^θ_ω, a set of the cardinality of the continuum, and so δ is not a choice function.

3.5. Definition. Let $\theta \in \mathrm{Aut}(T)$. We call θ a *bounded automorphism* iff for every $x \in T^\theta$ the set of cardinals $\{|y| : y < x\}$ is finite.

Note that if $lT \leqq \omega$ then every $\theta \in \mathrm{Aut}(T)$ is bounded. Also, every θ having only finite orbits or only infinite orbits is bounded.

Using Corollary 3.3(a) one easily verifies:

3.6. Proposition. *The following are the equivalent of* $\theta \in \mathrm{Aut}(T)$:

(i) *θ is bounded,*

(ii) *for every* $0 < \nu$ *and every* $a \in T_\nu$ *there is a* $\lambda < \nu$ *such that for* $\lambda \leqq \mu < \nu$, $|(\bar{a}(\mu))_\theta| = |(\bar{a}(\lambda))_\theta|$.

3.7. Proposition. *Let $\theta \in \mathrm{Aut}(T)$ be a bounded automorphism. Then τ^θ has a generalized inverse.*

Proof. Define $\delta : T^\theta \to T$ by induction. Let $\delta(\{\emptyset\}) = \emptyset$. Assume that $\delta(y)$ is already defined for $y < x \in T^\theta_\nu$, so that δ is an order preserving choice function. Let $a \in x$. For $\mu < \nu$ let $y_\mu = \tau^\theta(\bar{a}(\mu))$, $n_\mu = |y_\mu|$, $b_\mu = \delta(y_\mu)$. By Proposition 3.6 let $\lambda < \nu$, $n \leqq \omega$ satisfy $n = n_\mu$ for every $\lambda \leqq \mu < \nu$. Let $m \in Z$ satisfy $\theta^m(\bar{a}(\lambda)) = b_\lambda$. Let $\lambda < \mu < \nu$. By the induction hypothesis, $b_\lambda < b_\mu$. By Proposition 3.2, it follows from $n_\lambda = n_\mu$ that $b_\lambda < c \in T_\mu$ implies $c = b_\mu$. Now $c = \theta^m(\bar{a}(\mu))$ satisfies $c \in y_\mu$ and $b_\lambda = \theta^m(\bar{a}(\lambda)) < \theta^m(\bar{a}(\mu)) = c$. Hence $b_\mu =$

$\delta(y_\mu) = \theta^m(\bar{a}(\mu))$ for every $\lambda \leqq \mu < \nu$. Thus $\delta(y) < \theta^m(a)$ for every $y < x$, and we can make $\delta(x) = \theta^m(a)$. □

Let $\Phi \in R_\theta$, and consider $\tau^\Phi : T^\theta \to T^{\theta,\Phi}$. Since $\Phi \in \mathrm{Aut}(T^\theta)$ is bounded, it has a g.i. Let $\delta_1 : T^{\theta,\Phi} \to T^\theta$ be a g.i. of τ^Φ. Then δ_1 is an o.p. choice function on $T^{\theta,\Phi}$. Since $|u| \leqq 2$ for $u \in T^{\theta,\Phi}$, there is a unique g.i. δ_{-1} of τ^Φ satisfying $u = \{\delta_1(u), \delta_{-1}(u)\}$ for every $u \in T^{\theta,\Phi}$. We shall call (δ_1, δ_{-1}) *a complementary pair* (of g.i.s of τ^Φ).

By a *coherent coupling* for Φ we mean a coupling $\rho = (\rho_1, \rho_{-1})$ for Φ (see §2) such that ρ_i is o.p. for $i = 1, -1$. Thus a coupling $\rho = (\rho_1, \rho_{-1})$ is a coherent coupling iff ρ_1, ρ_{-1} are embeddings of $T^{\theta,\Phi}$ in T.

3.8. Proposition. *Let $\theta \in \mathrm{Aut}(T)$. Then the following are equivalent*:

(i) *τ^θ has a generalized inverse,*

(ii) *every $\Phi \in R_\theta$ has a coherent coupling.*

Proof. (i) ⇒ (ii): Let (δ_1, δ_{-1}) be a complementary pair of g.i.s for τ^Φ, and let δ be a g.i. for τ^θ. Then $\rho = (\delta\delta_1, \delta\delta_{-1})$ is easily checked to be a coherent coupling for Φ.

(ii) ⇒ (i): Let $\rho = (\rho_1, \rho_{-1})$ be a coherent coupling for $1 \in R_\theta$, the identity mapping of T^θ. Define $\delta : T^\theta \to T$ by $\delta(x) = \rho_1(\{x\})$. δ is clearly a g.i. of τ^θ. □

3.9. Corollary. *Let $\theta \in \mathrm{Aut}(T)$ be bounded. Then every $\Phi \in R_\theta$ has a coherent coupling.*

As in §2, we say that $(\varphi, \psi) \in \mathrm{BR}_<(\theta)$ obeys a coupling ρ for $\Phi \in R_\theta$ iff $\varphi(\rho_1(u)) = \rho_{-1}(u)$ for every $u \in T^{\theta,\Phi}$.

3.10. Proposition. *Let $\theta \in \mathrm{Aut}(T)$, $\Phi \in R_\theta$ and let ρ be a coherent coupling for Φ. Then there is a unique $(\varphi, \psi) \in \mathrm{BR}_<(\theta;\Phi)$ such that (φ, ψ) obeys ρ.*

Proof. By Theorem 2.1 (1) there is a unique $(\varphi, \psi) \in \mathrm{BR}(\theta;\Phi)$ such that (φ, ψ) obeys ρ. It is left to show that φ is o.p. This implies $\varphi \in \mathrm{Aut}(T)$ and $\psi = \varphi\theta \in \mathrm{Aut}(T)$, whence $(\varphi, \psi) \in \mathrm{BR}_<(\theta;\Phi)$.

Since Φ is an automorphism of T^θ and $\varphi'' = \Phi$, we conclude that φ_μ is a permutation of T_μ for every $\mu < lT$. Recall (Proposition 3.2) that for $a < b \in T$, $r \in Z$ we have $a < \theta^r(b)$ iff $n \mid r$, where $n = |(a)_\theta|$.

Let $a < b$. We shall show $\varphi(a) < \varphi(b)$. Define $x, y \in T^\theta$, $u, v \in T^{\theta,\Phi}$ and $1 \leqq n \leqq \omega$ by $x = (a)_\theta$, $y = (b)_\theta$, $u = (x)_\Phi$, $v = (y)_\Phi$, $n = |x|$. Then $x < y$, $u = \{x, \Phi(x)\} < \{y, \Phi(y)\} = v$, $|\Phi(x)| = n$. Let $\rho_i(u) = a_i$, $\rho_i(v) = b_i$, $i = 1, -1$. Since ρ_i is o.p. we have $a_i < b_i$, $i = 1, -1$. Also, by the definition of a coupling, $u = \{(a_1)_\theta, (a_{-1})_\theta\}$, $v = \{(b_1)_\theta, (b_{-1})_\theta\}$. With no loss of generality we assume $x = (a_1)_\theta$ and $y = (b_1)_\theta$. Let $m, r \in Z$ satisfy:

$$a = \theta^m(a_1), \qquad b = \theta^r(b_1).$$

By $a < b$, $a_1 < b_1$ we have $n \mid r - m$. Since (φ, ψ) obeys ρ we have

$$\varphi(a_1) = a_{-1}, \qquad \varphi(b_1) = b_{-1}.$$

By $\varphi\theta^m = \theta^{-m}\varphi$, $\varphi\theta^r = \theta^{-r}\varphi$ we have

$$\varphi(a) = \theta^{-m}(a_{-1}), \qquad \varphi(b) = \theta^{-r}(b_{-1}).$$

Let $n' = |(a_{-1})_\theta|$. By $a_{-1} < b_{-1}$ we have $\varphi(a) < \varphi(b)$ iff $n' \mid r - m$. But $(a_{-1})_\theta = \Phi(x)$. Hence by $|\Phi(x)| = n$ we have $n' = n$. Thus $\varphi(a) < \varphi(b)$. □

3.11. Proposition. *Let $\theta \in \mathrm{Aut}(T)$ and assume that τ^θ has a generalized inverse δ. Let $\Phi \in R_\theta$ and let $(\varphi, \psi) \in \mathrm{BR}_<(\theta; \Phi)$. Then (φ, ψ) obeys some coherent coupling for Φ.*

Proof. Let (δ_1, δ_{-1}) be a complementary pair of g.i.s of Φ. Define $\rho = (\rho_1, \rho_{-1})$ by $\rho_1 = \delta\delta_1$, $\rho_{-1} = \varphi\rho_1$. Then the ρ_i are obviously order preserving, and $\rho_i(u) \in \delta_i(u)$, $i = 1, -1$. □

Corollary 3.9 and Propositions 3.10, 3.11 combine in the following extension of Theorem 2.1:

3.12. Theorem. *Let $\theta \in \mathrm{Aut}(T)$ be a bounded automorphism, and let $\Phi \in R_\theta$. Then:*

(0) *There exists a coherent coupling for Φ.*

(1) *Every coherent coupling ρ for Φ has a unique $(\varphi, \psi) \in \mathrm{BR}_<(\theta; \Phi)$ such that (φ, ψ) obeys ρ.*

(2) *Every $(\varphi, \psi) \in \mathrm{BR}_<(\theta, \Phi)$ obeys some coherent coupling for Φ.*

Recalling our remark that trees of length $\leqq \omega$ have only bounded automorphisms and that R_θ is never empty we obtain:

3.13. Corollary (Micha Perles). *Let $lT \leqq \omega$. Then T has the bireflection property.*

4. k-trees and augmented k-trees

We show in this section that certain trees of length $\omega + 1$ called augmented $\boldsymbol{k}$-trees fail to have the bireflection property. Moreover, every augmented $\boldsymbol{k}$-tree is embeddable in every tree T that fails to have the b.r.p. (This follows from Theorem 4.13 and the fact that every augmented $\boldsymbol{k}$-tree can be embedded in the tree $\hat{T}^0$ defined therein.)

Let $\boldsymbol{k} = (k_n : n \in \omega)$ be a sequence of integers greater than one. Define t_n by $t_n = \prod_{i<n} k_i$. In particular, $t_0 = 1$. We fix $\boldsymbol{k}$ in the sequel.

4.0. DEFINITION. A tree T is a $\boldsymbol{k}$-tree iff $lT = \omega$ and for $n \in \omega$, $a \in T_n$ we have:

$$|\{b' : a < b', b' \in T_{n+1}\}| = k_n.$$

Recall that $T_0 = \{\emptyset\}$, and so for a $\boldsymbol{k}$-tree T we have $|T_n| = t_n$ for every $n \in \omega$.

Let T be a $\boldsymbol{k}$-tree. Define a metric d on $\tilde{T}$ by $d(\alpha,\alpha) = 0$ and $d(\alpha,\beta) = 1/(n+1)$ for $\alpha \neq \beta$, where n satisfies $\bar{\alpha}(n) = \bar{\beta}(n)$ and $\bar{\alpha}(n+1) \neq \bar{\beta}(n+1)$. For $a \in T$ let $U_a = \{\alpha \in \tilde{T} : \bar{\alpha}(la) = a\}$. Then U_a is a clopen ball of diameter $1/(la+1)$, and the family $\{U_a : a \in T\}$ is a basis to the metric topology on $\tilde{T}$.

4.1. PROPOSITION. *Let T be a $\boldsymbol{k}$-tree. Then χ is an isometry of $\tilde{T}$ if and only if $\chi = \tilde{\theta}$ for some $\theta \in \mathrm{Aut}(T)$.*

The straightforward proof is left to the reader.

The abundance of isomorphisms between two $\boldsymbol{k}$-trees is displayed in the following lemma.

4.2. ISOMORPHISM LEMMA. *Let T^0, T^1 be $\boldsymbol{k}$-trees. For $n \in \omega$, $i = 0, 1$ let A^i_n be a countable dense subset of $\tilde{T}^i$ so that $n \neq m$ implies $A^i_n \cap A^i_m = \emptyset$. Then there is an isomorphism $\tau : T^0 \to T^1$ such that for every $n \in \omega$, $\tilde{\tau}'' A^0_n = A^1_n$.*

PROOF. Let $A^i_n = \{\alpha^i_{nm} : m \in \omega\}$ be an enumeration of A^i_n. Let $\{(p_s, q_s) : s \in \omega\}$ be any fixed enumeration of $\omega \times \omega$. Thus for any $n \in \omega$, the sets $\{s : p_s = n, q_s$ is even$\}$ and $\{s : p_s = n, q_s$ is odd$\}$ are infinite disjointed subsets of ω.

We shall define for $s \in \omega$, $i = 0, 1$, B^i_s, τ^s so that:

(1) B^i_s is a finite subset of $\tilde{T}^i$ and $B^i_s \subset B^i_{s+1}$.

(2) Let $C^i_s = \{\bar{\beta}(r) : \beta \in B^i_s,\ r \in \omega\}$. Then $T^i_s \subset C^i_s$.

(3) $\tau^s : C^0_s \to C^1_s$ is an isomorphism, and $\tau^s = \tau^{s+1} \mid C^0_s$.

(4) B^i_{s+1} contains the first member of $A^i_{p_s}$ not in B^i_s, where $i = 0$ or $i = 1$ according as q_s is even or odd.

(5) $\alpha \in A^0_n \cap B^0_s$ if and only if $\tilde{\tau}^s(\alpha) \in A^1_n \cap B^1_s$.

We show first that defining $\tau : T^0 \to T^1$ by $\tau \mid C^0_s = \tau^s$ we obtain an isomorphism as required. Indeed, by (2) and (3) τ is well defined, and is an isomorphism of T^0 onto T^1. Let $n \in \omega$. We show that $\tilde{\tau}'' A^0_n = A^1_n$. Indeed let $\alpha = \alpha^0_{nm} \in A^0_n$. Suppose s satisfies: There are distinct $s_0, \cdots, s_{m-1}$ smaller than s such that $p_{s_j} = n$ and q_{s_j} is even, $j = 0, \cdots, m-1$. Then by (4) and (1), $\alpha^0_{nj} \in B^0_s$ for $j < m$. If in addition $p_s = n$ and q_s is even, then $\alpha^0_{nm} \in B^0_{s+1}$ by (4), and so $\tilde{\tau}(\alpha^0_{nm}) = \tilde{\tau}^{s+1}(\alpha^0_{nm}) \in A^1_n$ by (5). Thus $\tilde{\tau}'' A^0_n \subseteq A^1_n$. The proof of the inclusion $A^1_n \subseteq \tilde{\tau}'' A^0_n$ is similar.

We now define B^i_s, τ^s satisfying (1)–(5) by induction. Let $B^i_0 = \{\alpha^i_{00}\}$ and $\tau^0(\bar{\alpha}^0_{00}(n)) = \overline{\alpha^1_{00}}(n)$.

Induction Step. Assume that B_s, τ^s are already defined so that (1)–(5) hold.

I. Let $\alpha^i \in A^i_{p_s}$ be first not in B^i_s, with $i = 0$ or $i = 1$ according as q_s is even or odd. Let t be smallest such that $B^i_s \cap U_{\bar{\alpha}^i(t)} = \emptyset$. By (2), $s < t$. Also, $a = \bar{\alpha}^i(t-1) \in C^i_s$. Let $b = \tau^s(a)$. Then by (1), (3), $b \in T^{1-i}_{t-1}$, and we may pick $b' \in T^{1-i}_t$ satisfying $b < b'$ and $b' \notin C^{1-i}_s$. Since $A^{1-i}_{p_s}$ is dense in $\tilde{T}^{1-i}$, we may pick $\alpha^{1-i} \in A^{1-i}_{p_s} \cap U_{b'}$. Let $B^i = B^i_s \cup \{\alpha^i\}$, $C^i = \{\bar{\beta}(r) : \beta \in B^i, r \in \omega\}$ and define $\tau' : C^0 \to C^1$ by $\tau' \mid B^0_s = \tau^s$, and $\tau'(\bar{\alpha}^0(r)) = \bar{\alpha}^1(r)$, $r \in \omega$. Clearly τ' is an isomorphism of C^0 onto C^1, and $\bar{\tau}'(\alpha^0) = \alpha^1$.

II. For $a \in T^i_s$ let $D_a = \{b \in T^i_{s+1} : a < b,\ b \notin C^i\}$. Then $|D_a| = |D_{\tau'(a)}|$ for every $a \in T^0_s$, so we fix a bijection $\tau_a : D_a \to D_{\tau'(a)}$. For every $a \in T^i_s$ and $b \in D_a$ pick $\alpha^i_b \in A^i_0 \cap U_b$. Let:

$$B^i_{s+1} = B^i \cup \bigcup_{a \in T^i_s} \{\alpha^i_b : b \in D_a\}.$$

Define $\tau^{s+1} : C^0_{s+1} \to C^1_{s+1}$ by $\tau^{s+1} \mid C^0 = \tau'$, and $\tau^{s+1}(\bar{\alpha}^0_b(r)) = \bar{\alpha}^1_{\tau_a(b)}(r)$ for $a \in T^0_s$, $b \in D_a$ and $r \in \omega$. It is readily checked that (1)–(5) hold. □

4.3. Corollary. *Let T be a $\mathbf{k}$-tree and let $(A_n : n \in \omega)$ be a sequence of mutually disjoint countable dense subsets of $\tilde{T}$. Let σ be a permutation of ω. Then there is an isometry χ of $\tilde{T}$ such that χ maps A_n onto $A_{\sigma(n)}$ for every $n \in \omega$.*

4.4. Definition. Let T be a $\mathbf{k}$-tree. Then $\theta \in \mathrm{Aut}(T)$ is called a *minimal automorphism* iff $T^\theta = \{T_n : n \in \omega\}$.

The term minimal is borrowed from topological dynamics, where a homeomorphism is called minimal when every orbit is dense. Indeed, the reader will easily verify:

4.5. Proposition. *Let T be a $\mathbf{k}$-tree, $\theta \in \mathrm{Aut}(\theta)$. Then the following are equivalent:*

(i) *θ is minimal.*

(ii) *$(\alpha)_{\tilde{\theta}}$ is dense for some $\alpha \in \tilde{T}$.*

(iii) *$(\alpha)_{\tilde{\theta}}$ is dense for every $\alpha \in \tilde{T}$.*

Perhaps the most natural example of a minimal automorphism of a $\mathbf{k}$-tree is the following one. Let T be the $\mathbf{k}$-tree of all finite sequences $a = (a_0, \cdots, a_{n-1}) \in \omega^{<\omega}$ satisfying $0 \leqq a_i < k_i$. With $a \in T_n$ associate a natural number m_a by:

$$m_a = \sum_{i=0}^{n-1} a_i t_i$$

where, as usual, $t_i = \Pi_{j<i} k_j$. Thus, if $k_n = 10$ for all n, then $a_{n-1}a_{n-2}\cdots a_0$ is the decimal representation of m_a, possibly with some zeros in front of it.

It is easily checked that the mapping $\tau_n(a) = m_a$ is a bijection of T_n onto $\{0,\cdots,t_n - 1\}$, and so for $0 \leqq m < t_n$ let $a_{n,m}$ be the unique $a \in T_n$ with $m_a = m$. Define θ_n by:

$$\theta_n(a_{n,m}) = a_{n,m+1}, \qquad m + 1 < t_n$$

and

$$\theta_n(a_{n,t_n-1}) = a_{n,0}.$$

Then obviously $T_n = (a_{nm})_{\theta_n}$ for any $0 \leqq m < r_n$. It is easily checked that θ defined by $\theta \mid T_n = \theta_n$ is indeed an automorphism of T, and so θ is a minimal automorphism.

Lemma 4.2 implies:

4.6. Corollary. *Let T be a $\mathbf{k}$-tree. For $n \in \omega$ let $A_n \subseteq \tilde{T}$ be a countable dense set, such that $A_n \cap A_m = \varnothing$ for $n \neq m$. Then there is an isometry χ of $\tilde{T}$ such that A_n is a χ-orbit for every $n \in \omega$.*

Proof. Let T^0 be an arbitrary $\mathbf{k}$-tree and let θ be a minimal automorphism of T^0. For $n \in \omega$ let A_n^0 be a $\tilde{\theta}$-orbit so that A_n^0, A_m^0 are different (hence disjoint) if $n \neq m$. By Proposition 4.5 each A_n^0 is dense. By Lemma 4.2 let $\tilde{\tau} : \tilde{T}^0 \to \tilde{T}$ be an isometry mapping A_n^0 onto A_n. Let $\chi = \tilde{\tau}\tilde{\theta}\tilde{\tau}^{-1}$. □

We discuss next the bireflections of a minimal automorphism of a $\mathbf{k}$-tree. The familiarity with the fixed points of a possible bireflection (§2) turns out to be useful.

For an arbitrary permutation φ of a set A, let F^φ denote the set of points fixed by φ, i.e.:

$$F^\varphi = \{a \in A : \varphi(a) = a\}.$$

Clearly, if T is a tree, $\varphi \in \mathrm{Aut}(T)$, then F^φ is also a tree.

Now let T be a $\mathbf{k}$-tree, let $\theta \in \mathrm{Aut}(T)$ be a minimal automorphism, and let $(\varphi, \psi) \in \mathrm{BR}_<(\theta)$. We wish to determine $F^{\tilde{\varphi}}$, $F^{\tilde{\psi}}$ from F^φ, F^ψ. Let $F = F_\varphi \cup F_\psi$. Then F is a tree. Since $T_n \in T^\theta$ is φ-invariant and ψ-invariant, we have by Theorem 2.2: $F_n = F \cap T_n$ is a two-element set for every $n > 0$. We say that F *splits at n* if for some $a \in F_n$ we have $a < b$ for every $b \in F_{n+1}$.

4.7. Proposition. *Let $n > 0$. F splits at n if and only if k_n is even.*

Proof. By Theorem 2.2 there are $b, b' \in T_{n+1}$ such that $b' = \theta^s(b)$, where $s = [\frac{1}{2}t_{n+1}]$, and $F_{n+1} = \{b, b'\}$. Let $a = \bar{b}(n)$. Then $a < b'$ if and only if $\theta^s(a) = a$, that is, if and only if $t_n \mid s$.

Case 0. k_n is even, say $k_n = 2r$. Then $s = [\frac{1}{2}t_{n+1}] = [\frac{1}{2}k_n t_n] = [\frac{1}{2}\cdot 2r \cdot t_n] = [r \cdot t_n] = r \cdot t_n$, and so $t_n \mid s$, so $a < b'$.

Case 1. k_n is odd, say $k_n = 2r+1$. Then $s = [\frac{1}{2}(2r+1)t_n] = [(r+\frac{1}{2})t_n] = rt_n + [\frac{1}{2}t_n]$. Since $n > 0$, we have $2 \leqq k_{n-1} \leqq t_n$ and so $0 < [\frac{1}{2}t_n] < t_n$. Hence $t_n \nmid s$ and so $a < b'$ does not hold. □

Clearly $F^{\tilde{\varphi}} = \widetilde{F^{\varphi}}$, $F^{\tilde{\psi}} = \widetilde{F^{\psi}}$ and $F^{\tilde{\varphi}} \cup F^{\tilde{\psi}} = \tilde{F}$.

We first determine $|\tilde{F}|$. Since F is an infinite tree but F_n is finite for all n, $\tilde{F}$ is nonempty, by König's Lemma. Thus, $|\tilde{F}| \geqq 1$.

Case A. k_n is odd for all but finitely many values of n. Then there is an n_0 such that F does not split at n for $n \geqq n_0$. Since $|F_n| = 2$ for all n, we conclude that $|\tilde{F}| = 2$.

Case B. k_n is even for infinitely many values of n. Then F splits at n infinitely often, and since $|F_n| = 2$ for all n, $\tilde{F}$ has at most one member. By $|\tilde{F}| \geqq 1$ we conclude that $|\tilde{F}| = 1$.

We next discuss $F^{\tilde{\varphi}}$ and $F^{\tilde{\psi}}$ separately. There are three cases, two of which are dual.

Case I. k_n is an odd integer for every $n \in \omega$. Then by Theorem 2.2 (b), $|F_n^{\varphi}| = |F_n^{\psi}| = 1$ for every $n \in \omega$, and so $|F^{\tilde{\varphi}}| = |F^{\tilde{\psi}}| = 1$; that is, there exist one $\tilde{\varphi}$-point and one $\tilde{\psi}$-point.

Case II. There is a smallest n_0 such that k_{n_0} is even. By Theorem 2.2 (b), $F_{n_0} = F_{n_0}^{\varphi}$ or $F_{n_0} = F_{n_0}^{\psi}$. Case II splits accordingly, and we have:

Case II_φ (*Case* II_ψ). $F_{n_0} = F_{n_0}^{\varphi}$ ($F_{n_0} = F_{n_0}^{\psi}$). Then $F_n^{\psi}(F_n^{\varphi})$ is empty for $n \geqq n_0$, since $F^{\psi}(F^{\varphi})$ is a tree. Thus $\tilde{F} = F^{\tilde{\varphi}}$ ($\tilde{F} = F^{\tilde{\psi}}$).

We summarize this discussion in:

4.8. Theorem. *Let T be a* **k**-*tree and θ a minimal automorphism of T. Let $(\varphi, \psi) \in \mathrm{BR}_<(\theta)$. Then $(\tilde{\varphi}, \tilde{\psi})$ is a bireflection of $\tilde{\theta}$ and the following holds:*

(1) *If k_n is odd for all but finitely many n's then $(\tilde{\varphi}, \tilde{\psi})$ has precisely two reflection points.*

(2) *If k_n is even for infinitely many values of n then $(\tilde{\varphi}, \tilde{\psi})$ has precisely one reflection point.*

(3) *If k_n is odd for all n then there are precisely one $\tilde{\varphi}$-point and one $\tilde{\psi}$-point.*

(4) *If n_0 is smallest such that k_{n_0} is even and the two reflection points of the bireflection $(\varphi_{n_0+1}, \psi_{n_0+1})$ of θ_{n_0+1} are both φ-points (ψ-points) then all reflection points of $(\tilde{\varphi}, \tilde{\psi})$ are $\tilde{\varphi}$-points ($\tilde{\psi}$-points).*

(5) *Let $x \in (\tilde{T})^{\tilde{\theta}}$. Then $\tilde{\varphi}''x = x \Leftrightarrow \tilde{\psi}''x = x \Leftrightarrow x$ contains a reflection point of $(\tilde{\varphi}, \tilde{\psi})$.* □

Proof. (1)–(4) are discussed above. (5) follows from Theorems 2.0 and 2.2 (i).

4.9. Definition. An *augmented **k**-tree* is a tree $\hat{T}$ of length $\omega+1$, satisfying $\hat{T}=T\cup A$, where T is a $\mathbf{k}$-tree and A is a countable dense subset of $\tilde{T}$. An automorphism $\hat{\theta}\in \mathrm{Aut}(\hat{T})$ is called *minimal* if $\hat{\theta}\mid T$ is a minimal automorphism of T.

By the Isomorphism Lemma 4.2, every two augmented $\mathbf{k}$-trees are isomorphic.

Let (p,k) denote the greatest common divisor of the integers p and k.

4.10. Theorem. *Assume that for some $1<p<\omega$, $(p,k_n)=1$ for all $n\in\omega$. Then every augmented **k**-tree has a minimal automorphism $\tilde{\theta}$ admitting no bireflection.*

Proof. By the Isomorphism Lemma 4.2 it is enough to show that there exists an augmented $\mathbf{k}$-tree which has a minimal automorphism θ admitting no bireflection.

Let T be an arbitrary $\mathbf{k}$-tree, and let $\eta\in \mathrm{Aut}(T)$ be minimal. Let $\alpha_0\in\tilde{T}$, and for $m\in Z$ let $\alpha_m=\tilde{\eta}^m(\alpha_0)$. Let $q\in\omega$ satisfy $5<q$ and $(q,k_n)=1$ for every $n\in\omega$. Define $\theta\in\mathrm{Aut}(T)$ by $\theta=\eta^q$. By $(q,k_n)=1$ we have for $a\in T_n$:

$$(a)_\theta=\{\theta^r(a):r\in Z\}=\{\eta^{qr}(a):r\in Z\}=\{\eta^r(a):r\in Z\}=(a)_\eta=T_n$$

and so θ is also minimal.

Set $x_m=(\alpha_m)_{\tilde{\theta}}$. Clearly $x_m=x_{m'}$ iff $q\mid m-m'$.

Let $B=\{x_{-2},x_0,x_1\}$, $A=x_{-2}\cup x_0\cup x_1$.

Then A is a countable dense subset of $\tilde{T}$ (Proposition 4.5). Let $\hat{T}=T\cup A$. Thus, $\hat{T}$ is an augmented $\mathbf{k}$-tree. Define $\hat{\theta}\in\mathrm{Aut}(\hat{T})$ by $\hat{\theta}\mid T=\theta$, $\hat{\theta}\mid A=\tilde{\theta}\mid A$. Then $\hat{\theta}$ is a minimal automorphism of $\hat{T}$. We shall show that $\mathrm{BR}_<(\hat{\theta})=\varnothing$.

In fact, suppose not and let $(\hat{\varphi},\hat{\psi})\in\mathrm{BR}_<(\hat{\theta})$. Let $\varphi=\hat{\varphi}\mid T$, $\psi=\hat{\psi}\mid T$. Then $(\varphi,\psi)\in\mathrm{BR}_<(\theta)$.

Claim 1. *$\varphi\eta^j=\eta^{-j}\varphi$ for every $j\in Z$.*

Proof. Let $a\in T_n$ and let $t_n=|T_n|$. Since $(q,k_n)=1$ for all n, $t_n=\Pi_{i<n}k_i$, we have $(q,t_n)=1$ and so we can find $r\in Z$ so that $t_n\mid qr-j$. Thus $\eta^{qr}(b)=\eta^j(b)$, and $\eta^{-qr}(b)=\eta^{-j}(b)$ for every $b\in T_n$. Now, by $\varphi\theta^r=\theta^{-r}\varphi$ and by a, $\varphi(a)\in T_n$:

$$\varphi\eta^j(a)=\varphi\eta^{qr}(a)=\varphi\theta^r(a)=\theta^{-r}\varphi(a)=\eta^{-qr}(\varphi(a))=\eta^{-j}\varphi(a). \qquad \square$$

Note that Claim 1 implies that $(\varphi,\varphi\eta)\in\mathrm{BR}_<(\eta)$.

Claim 2. *Let $\tilde{\Phi}=\tilde{\varphi}''$. If $\tilde{\Phi}(x_i)=x_i$ then for every $j\in Z$, $\tilde{\Phi}(x_{i+j})=x_{i-j}$.*

PROOF. Let $\tilde{\Phi}(x_i) = x_i$. Then $\tilde{\varphi}(\alpha_i) \in (\alpha_i)_\theta$, and so for some $r \in Z$, $\tilde{\varphi}(\alpha_i) = \tilde{\theta}^r(\alpha_i) = \tilde{\eta}^{qr}(\alpha_i)$. By Claim 1, $\tilde{\varphi}\tilde{\eta}^j = \tilde{\eta}^{-j}\tilde{\varphi}$ and so:

$$\tilde{\varphi}(\alpha_{i+j}) = \tilde{\varphi}\tilde{\eta}^j(\alpha_i) = \tilde{\eta}^{-j}\tilde{\varphi}(\alpha_i) = \tilde{\eta}^{-j}\tilde{\eta}^{qr}(\alpha_i) = \tilde{\eta}^{qr}(\alpha_{i-j}) = \tilde{\theta}^r(\alpha_{i-j}).$$

Thus $\tilde{\varphi}(\alpha_{i+j}) \in (\alpha_{i-j})_{\tilde{\theta}}$, and so $\tilde{\varphi}''(\alpha_{i+j})_{\tilde{\theta}} \cap (\alpha_{i-j})_{\tilde{\theta}} \neq \emptyset$. Hence by Theorem 2.0, $\tilde{\varphi}''(\alpha_{i+j})_{\tilde{\theta}} = (\alpha_{i-j})_{\tilde{\theta}}$, that is, $\tilde{\Phi}(x_{i+j}) = x_{i-j}$. □

Consider now the set $\hat{T}^{\hat{\theta}}_\omega = B = \{x_{-2}, x_0, x_1\}$. Since $(\hat{\varphi}, \hat{\psi}) \in \mathrm{BR}_<(\hat{\theta})$, B must be invariant under $\tilde{\Phi}$. Since every orbit of $\tilde{\Phi}$ contains at most two elements, B contains at least one fixed point of $\tilde{\Phi}$. But by Theorem 4.8 $\tilde{\Phi}$ has at most two fixed points, so $\hat{T}^{\hat{\theta}}_\omega$ contains precisely one fixed point of $\tilde{\Phi}$. Now by Claim 2:

If $\tilde{\Phi}(x_{-2}) = x_{-2}$ then $\tilde{\Phi}(x_0) = x_{-4}$.

If $\tilde{\Phi}(x_0) = x_0$ then $\tilde{\Phi}(x_1) = x_{-1}$.

If $\tilde{\Phi}(x_1) = x_1$ then $\tilde{\Phi}(x_0) = x_2$.

But since $q > 5$ and $x_m = x_{m'}$ iff $q \mid m - m'$ we have $\{x_{-2}, x_0, x_1\} \cap \{x_{-4}, x_{-1}, x_2\} = \emptyset$, and so B cannot be an invariant set of $\tilde{\Phi}$. □

4.11. THEOREM. *Assume that for some $1 < p < \omega$, $(p, k_n) = 1$ for all $n \in \omega$. Let T be a $\mathbf{k}$-tree and θ a minimal automorphism of T. Then there exist an augmented $\mathbf{k}$-tree $\hat{T} = T \cup A$ such that $\hat{\theta}$ defined by $\hat{\theta} \mid T = \theta$, $\hat{\theta} \mid A = \tilde{\theta} \mid A$ is a minimal automorphism of $\hat{T}$ admitting no bireflection.*

PROOF. Let q be any natural number satisfying $(q, k_n) = 1$ for every $n \in \omega$. By the proof of Theorem 4.10, Theorem 4.11 will follow if we find a minimal automorphism η such that $\theta = \eta^q$. For every $n \in \omega$ let p_n be a natural number satisfying $t_n \mid qp_n - 1$, where $t_n = \Pi_{i<n} k_i$. The existence of p_n is guaranteed by $(t_n, q) = 1$. Now define a permutation η_n of T_n by $\eta_n = \theta^{p_n}_n$. Then we have for $a \in T_n$:

$$\eta^q_n(a) = \theta^{p_n q}(a) = \theta(a)$$

and so η defined by $\eta \mid T_n = \eta_n$ satisfies $\eta^q = \theta$. It is left to show that $\eta \in \mathrm{Aut}(T)$. Since η_n is a permutation of T_n it is enough to show that: $a < b$ implies $\eta(a) < \eta(b)$. Indeed, let $a < b$, $a \in T_n$, $b \in T_m$. Then:

$$\eta(a) = \eta_n(a) = \theta^{p_n}(a), \qquad \eta(b) = \eta_m(b) = \theta^{p_m}(b)$$

so

$$\eta(a) < \eta(b) \qquad \text{iff } t_n \mid p_m - p_n.$$

Now $t_n \mid qp_n - 1$, $t_m \mid qp_m - 1$ and $t_n \mid t_m$, so $t_n \mid qp_m - 1$ hence $t_n \mid q(p_m - p_n)$, and since $(t_n, q) = 1$ we have $t_n \mid p_m - p_n$. □

4.12. THEOREM. *No augmented $\mathbf{k}$-tree has the bireflection property.*

PROOF. By the Isomorphism Lemma 4.2 it is enough to show that there exists an augmented $\boldsymbol{k}$-tree having an automorphism $\hat{\theta}$ with $\mathrm{BR}_<(\hat{\theta}) = \emptyset$.

Let T be the $\boldsymbol{k}$-tree of all finite sequences a satisfying $0 \leqq a(i) < k_i$ for $0 \leqq i < la$. Let $T^0 \subseteq T$ be the tree of all finite sequences satisfying $0 \leqq a(i) < 2$, and let θ^0 be a minimal automorphism of T^0. We extend θ^0 to an automorphism θ of T as follows:

For $a \in T$ let $r_a = \min\{i : 2 \leqq a(i) \text{ or } i = la\}$, and let $a^0 = \bar{a}(r_a)$. Then $a^0 \in T^0$ for every $a \in T$ and $a^0 = a$ iff $a \in T^0$. Define $a^1 \in T$ by the equality $a = a^0 \cdot a^1$, where $\cdot$ denotes concatenation. Finally, define $\theta(a)$ by:

$$\theta(a) = \theta^0(a^0) \cdot a^1.$$

Clearly, $\theta \mid T^0 = \theta^0$. Also, $(a)_\theta = \{\theta^{0m}(a^0) \cdot a^1 : m \in Z\} = (a^0)_{\theta^0} \cdot a^1$, so $|(a)_\theta| = 2^{r_a}$ for every $a \in T$. Now for $a \in T_n$, $r_a = n$ iff $a \in T^0_n$. Thus $\{x \in T^\theta_n : |x| = 2^n\} = (T^0_n)^\theta$. Hence $(T^0_n)^\theta$ is invariant under every $\Phi \in R_\theta$. It follows from Theorem 3.4 that if $(\varphi, \psi) \in \mathrm{BR}_<(\theta)$ and $\varphi^0 = \varphi \mid T^0$, $\psi^0 = \psi \mid T^0$ then (φ^0, ψ^0) is a bireflection of θ^0. Hence $\tilde{T}^0$ is $\tilde{\varphi}$-invariant and $\tilde{\psi}$-invariant.

We now make T into an augmented $\boldsymbol{k}$-tree $\hat{T}$ as follows. By Theorem 4.11 pick a countable $A^0 \subseteq \tilde{T}^0$, dense in $\tilde{T}^0$ and invariant under $\tilde{\theta}^0$ such that $\hat{\theta}^0$ defined by $\hat{\theta}^0 \mid T^0 = \theta^0$, $\hat{\theta}^0 \mid A^0 = \tilde{\theta}^0 \mid A^0$ is a minimal automorphism of $\hat{T}^0 = T^0 \cup A^0$ with $\mathrm{BR}_<(\hat{\theta}^0) = \emptyset$.

Let $A^1 = \{\alpha \in \tilde{T} : \exists i[\alpha(i) \geqq 2 \wedge \forall j > i[\alpha(j) = 0]]\}$. Then A^1 is clearly $\tilde{\theta}$-invariant, $A^0 \cap A^1 = \emptyset$ and $A^0 \cup A^1$ is a countable dense subset of $\tilde{T}$.

Let $A = A^0 \cup A^1$ and $\hat{T} = T \cup A$. Define $\hat{\theta} \in \mathrm{Aut}(\hat{T})$ by $\hat{\theta} \mid T = \theta$, $\hat{\theta} \mid A = \tilde{\theta} \mid A$. We show that $\hat{\theta}$ admits no bireflection in $\mathrm{Aut}(\hat{T})$.

Indeed, let $(\hat{\varphi}, \hat{\psi}) \in \mathrm{BR}_<(\hat{\theta})$. Let $\varphi = \hat{\varphi} \mid T$, $\psi = \hat{\psi} \mid T$, $\hat{\varphi}^0 = \hat{\varphi} \mid \hat{T}^0$, $\hat{\psi}^0 = \hat{\psi} \mid \hat{T}^0$. Now $(\varphi, \psi) \in \mathrm{BR}_<(\theta)$, and so by the previous remarks $\tilde{T}^0$ is $\tilde{\varphi}$-invariant and $\tilde{\psi}$-invariant. Hence $(\hat{\varphi}^0, \hat{\psi}^0) \in \mathrm{BR}_<(\hat{\theta}^0)$, a contradiction. □

Let us say that a tree T has *hereditarily the bireflection property* iff every subset of T, considered as a tree in the broad sense, the partial order being $<$, has the bireflection property. We conclude with the following characterization of the class of trees with this property.

4.13. THEOREM. *Let T be a tree. The following are equivalent*:

(i) *T has hereditarily the bireflection property.*

(ii) *$\hat{T}^0$ is not embeddable in T, where $\hat{T}^0$ is the tree of all zero-one sequences of length $\leqq \omega$ with only finitely many ones.*

PROOF. (i) $\Rightarrow$ (ii). Indeed, if (ii) fails, and τ is an embedding of $\hat{T}^0$ in τ, then $\tau''\hat{T}^0 \subseteq T$ is a tree in the broad sense failing to have the bireflection property so (i) fails.

(ii) $\Rightarrow$ (i). Assume T fails to have hereditarily the bireflection property. Let $T_* \subseteq T$ testify to it. Let $\theta \in \mathrm{Aut}(T_*)$ fail to have the bireflection property. By Theorem 3.12, θ cannot be a bounded automorphism. Let $x \in T_*^\theta$ satisfy $\{|y| : |y| < x\}$ is infinite. By discarding a subset of T_* if necessary we may assume that $\{y : y < x\} = \{y_n : n \in \omega\}$ where $|y_n| < |y_{n+1}|$. Let $t_n = |y_n|$. By Proposition 3.2 and Corollary 3.3, $k_n = t_{n+1}/t_n$ is an integer greater than one, for every $a \in y_n$ the set $\{b \in y_{n+1} : a < b\}$ has precisely k_n elements, and $\{b \in x : a < b\}$ is nonempty. Thus $x \cup \bigcup_{n \in \omega} y_n$ is isomorphic to an augmented $\boldsymbol{k}$-tree. □

References

1. F. Bachmann, *Aufbau der Geometrie aus dem Spiegelungsbegriff*, 2nd edition, Springer, New York–Heidelberg–Berlin, 1973.

2. H. S. M. Coxeter, *Regular Complex Polytopes*, Cambridge University Press, 1974.

3. E. W. Ellers, *Bireflectionality in classical groups*, Can. J. Math. **29** (1977), 1157–1162.

4. G. Moran, *Permutations as products of k conjugate involutions*, J. Combinatorial Theory Ser. A, **19** (1975), 240–242.

5. G. Moran, *The bireflections of a permutation*, Discrete Math. **15** (1976), 55–62.

6. G. Moran, *The product of two reflection classes of the symmetric group*, Discrete Math. **15** (1976), 63–77.

7. G. Moran, *Reflection classes whose cubes cover the alternating group*, J. Combinatorial Theory Ser. A, **21** (1976), 1–19.

8. G. Moran, *Some coefficients in the center of the group algebra of the symmetric group*, Discrete Math. **21** (1978), 75–81.

9. W. R. Scott, *Group Theory*, Prentice Hall, Englewood Cliffs, N.J., 1964.

10. O. Veblen and J. W. Young, *Projective Geometry*, Vol. II, Blaisdell, New York, 1946.

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