ON GENERALIZED ABSOLUTELY MONOTONE FUNCTIONS

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ABSTRACT

Sufficient conditions for generalized absolutely monotone functions to possess a Taylor-type expansion in terms of the corresponding Extended Tchebycheff systems were found by Karlin and Ziegler. The question of necessary conditions, however, was left open. In this paper we solve this question by finding necessary and sufficient conditions for the validity of the expansion. The structure of the cone of generalized absolutely monotone functions and its extreme rays are also discussed.

We start by recalling briefly some definitions and basic results which will be used in the sequel. For a more detailed discussion of the results quoted here the reader is referred to the first paper on the topic [2] and to the monograph by Karlin and Studden [1].

Let $\{u_i(t)\}_{i=0}^{\infty}$ be an infinite sequence of functions belonging to $C^{\infty}[a,b]$ and such that for all n, $n = 0,1,\cdots,\{u_0,u_1,\cdots,u_n\}$ constitutes an Extended Tchebycheff system on $[a,b]$. With no loss of generality we may assume that the u_i's are of the form $u_i(t) = \phi_i(t;a)$ where

$$\phi_i(t;x) = \begin{cases} w_0(t)\displaystyle\int_a^t w_1(\xi_1)\int_a^{\xi_1} w_2(\xi_2)\cdots\int_a^{\xi_{i-1}} w_i(\xi_i)d\xi_i\cdots d\xi_1 & x \leqq t \leqq b \\ 0 & a \leqq t < x \end{cases} \tag{1}$$

for $i = 0,1,\cdots \qquad x \in [a,b]$

and $\{w_k(t)\}_{k=0}^{\infty}$ is a sequence of positive functions, each of class $C^{\infty}[a,b]$. With these functions we associate the sequence of first order differential operators

$$D_i f(t) = \frac{d}{dt}\frac{1}{w_i(t)}f(t), \qquad i = 0,1,\cdots \tag{2}$$

and the $k+1$-st order differential operators

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$$L_{-1}f(t) = f(t), \quad L_k f(t) = (D_k D_{k-1} \cdots D_0) f(t) \quad k = 0,1,\cdots$$

Definition 1. A function $\phi(t)$ defined on (a,b) is called "generalized absolutely monotone" (abbreviated G.A.M.) with respect to $\{u_i(t)\}_{i=0}^{\infty}$ provided $\phi(t)$ is of class $C^{\infty}(a,b)$ and satisfies the inequalities

$$\phi(t) \geqq 0, \qquad L_k \phi(t) \geqq 0 \quad \text{for all } t \in (a,b), \tag{3}$$

The concept of "generalized absolute monotonicity" is intimately connected with the concept of "generalized convexity cones".

Definition 2. A function $\psi(x)$ belongs to $P(u_0,\cdots,u_n)$ (and is called "convex with respect to $(u_0,\cdots,u_n)$" if for every set of points $\{x_i\}_{i=1}^{n+2}$ satisfying

$$a < x_1 < \cdots < x_{n+2} < b$$

the determinant inequality

$$\begin{vmatrix} u_0(x_1) \cdots u_0(x_{n+2}) \\ u_1(x) \cdots u_1(x_{n+2}) \\ \vdots \qquad \vdots \\ u_n(x_1) \cdots u_n(x_{n+2}) \\ \psi(x_1) \cdots \psi(x_{n+2}) \end{vmatrix} \geqq 0 \tag{4}$$

prevails.

It is proved in [2] that the cone of G.A.M. functions coincides with the intersection cone

$$P_A = P^+ \cap \left[\bigcap_{n=0}^{\infty} P(u_0,\cdots,u_n) \right]$$

where P^+ denotes the cone of continuous non-negative functions defined on (a,b). It is also shown that, if $f(x)$ is a G.A.M. function, then for all $n, n = 0,1,\cdots$ the following Taylor-type formula holds:

$$f(t) = \int_a^b \phi_n(t;x) L_n f(x) dx + \sum_{k=0}^{n} \frac{L_{k-1} f(a^+)}{w_k(a)} u_k(t). \tag{5}$$

Having all these facts at our disposal, we are prepared to state the first major theorem. We note that with no loss of generality we may take $a = 0$, $b = 1$.

Theorem 1. *If the sequence $u_i(t)$, $i = 0,1,\cdots$, generates the totality of the extreme rays of the cone P_A, then the expansion*

$$f(t) = \sum_{k=0}^{\infty} \frac{L_{k-1}f(o^+)}{w_k(o)} u_k(t), \quad t \in (0,1) \tag{6}$$

is valid for every bounded G.A.M. function.

REMARK. The convergence in (6) is uniform on every compact subset of (0,1). furthermore, if $f(t)$ is continuous at $t = 0$ and $t = 1$, the convergence is uniform over $[0,1]$. (See [2]).

Proof. Let D be the linear space $C^{\infty}(0,1)$ with the topology defined by the family of seminorms

$$\| \phi_k \|^n = \sup\{|L_p\phi(t)|;\ t \in I_k,\ p \leqq n\}$$

where $I_k = \left[\frac{1}{k}, \frac{k-1}{k}\right]$; $k = 2,3,\cdots$; $n = -1,0,1,\cdots$.

With this topology D is a complete metrizable locally convex space. We show that it is also a Montel space, i.e., every bounded set in D is relatively compact (cf. [5]). The proof is analogous to the standard proof that the space of infinitely differentiable functions on (0,1) is Montel: Let F be an infinite bounded set in D, and let $\{t_j\}_{j=1}^{\infty}$ be a dense sequence in $[0,1]$. Choose a sequence $\{f_i^{(-1,1)}\}$ from F such that $\{f_i^{(-1,1)}(t_1)\}$ converges, a subsequence $\{f_i^{(0,1)}\}$ of $\{f_i^{(-1,1)}\}$ such that $\{L_0 f_i^{(0,1)}(t_1)\}$ converges, a subsequence $\{f_i^{(-1,2)}\}$ of $\{f_i^{(0,1)}\}$ such that $\{f_i^{(-1,2}(t_2)\}$ converges, etc. A diagonal process yields a subsequence $\{f_n\}$ such that $\{L_p f_n(t_j)\}$ converges for every $p \geqq -1$ and every $j \geqq 1$. We proceed to show that $\{f_n\}$ is a Cauchy sequence.

Given p and k, choose r such that $\{t_1, t_r\}$ is η-dense in I_k, and n_0 such that

$$|L_p f_n(t_j) - L_p f_m(t_j)| < \varepsilon, \text{ for } m, n \geqq n_0 \text{ and } j \leqq r.$$

Then, for $m,n \geqq n_0$, $t \in I_k$ and $j \leqq r$ chosen so that $|t - t_j| < \eta$, we obtain

$$|L_p f_n(t) - L_p f_m(t)| < \varepsilon + 2\eta \sup \left\{ \left| \frac{d}{ds} L_p f(s) \right| ;\ s \in I_k,\ f \in F \right\} \tag{7}$$

On the other hand, for $s \in I_k$ and $f \in F$, we have

$$\| F \|_k^{p+1} \geqq \left| \frac{d}{ds} \frac{L_p f(s)}{w_{p+1}(s)} \right| = \left| \frac{w_{p+1}(s) \frac{d}{ds} L_p f - w'_{p+1}(s) L_p f}{w^2{}_{p+1}} \right|.$$

Hence, we deduce that

$$(8) \qquad \sup\left\{\left| \frac{d}{ds}L \quad (s) \right|; s \in I_k, f \in F \right\} \leqq \frac{M_{p+1}^2 \| F \|_k^{p+1} + M'_{p+1} \| F \|_k^p}{m_{p+1}}$$

where $M_p = \max\{w_p(s); 0 \leqq s \leqq 1\}$, $m_p = \min\{w_p(s); 0 \leqq s \leqq 1\}$, $M'_p = \max\{w'_p(s); 0 \leqq s \leqq 1\}$. Relations (7) and (8) imply that $\{f_n\}$ is a Cauchy sequence in D.

Let A be the set of G.A.M. functions satisfying $f(t) \leqq w_0(t)$ on $(0,1)$. Since f/w_0 is a monotone increasing function for every G.A.M. function f, $A = \{f \in C_A; z(f) \leqq 1\}$, where z is the extended real-valued functional.

$$z(f) = \lim_{t \to 1} \frac{f(t)}{w_0(t)}$$

which is finite, additive and positively-homogeneous on the cone of bounded G.A.M. functions.

A is closed and convex in D. We show next that it is bounded, hence compact. By definition we have, for every $f \in A$ and $k \geqq 2$:

$$\| f \|_k^{-1} \leqq A_k^{-1} = M_0 .$$

The other bounds are found by induction. Suppose that we already have

$$\| f \|_k^p \leqq A_k^p , \quad k = 2, 3, \cdots$$

The relationship

$$\frac{d}{dt} \frac{L_{p+1}f}{w_{p+2}} = L_{p+2}f \geqq 0$$

implies that for any $s \in I_k$, we have

$$(9) \qquad \frac{L_{p+1}f(s)}{w_{p+2}(s)} \leqq \frac{L_{p+1}f(t)}{w_{p+2}(t)} \quad \frac{k-1}{k} \leqq t \leqq \frac{2k-1}{2k}.$$

By transposing sides and recalling the definitions, we have

$$(10) \qquad \frac{d}{dt} \frac{L_p f(t)}{w_{p+1}(t)} = L_{p+1}f(t) \geqq \frac{m_{p+2}}{M_{p+2}} L_{p+1}f(s), \quad \frac{k-1}{k} \leqq t \leqq \frac{2k-1}{2k}$$

Since $L_p f / w_{p+1}$ is a positive function, (10) implies

$$(11) \qquad \frac{m_{p+2} L_{p+1} f(s)}{M_{p+2}} \leqq 2k \frac{L_p f\left(\dfrac{2k-1}{2k}\right)}{w_{p+1}\left(\dfrac{2k-1}{2k}\right)}.$$

Thus, we finally obtain the new bound

$$\| f \|_k^{p+1} \leqq \frac{2kM_{p+2}A_{2k}}{m_{p+1}m_{p+2}} = A_k^{p+1}. \tag{12}$$

We turn next to the question of extreme points of A. We first show that all extreme points have to be of the form $au_j(t)$, $j = 0, 1, \cdots$. Indeed if $f \in A$ is not of the form au_j it does not lie, by our assumption, on an extreme ray of P_A. Hence f possesses a representation

$$f = f_1 + f_2 \quad f_1, f_2 \in P_A; \ f_1, f_2 \text{ are linearly independent.} \tag{13}$$

Observing that $z(f_1) + z(f_2) = z(f) \leqq 1$ since $f \in A$ and that $z(f_i) > 0$, $i = 1,2$, we deduce that (13) can be rewritten in the form

$$f = \frac{z(f_1)}{(zf)} \frac{z(f)f_1}{z(f_1)} + \frac{z(f_1)}{z(f)} \frac{z(f)f_2}{z(f_2)}. \tag{14}$$

Since $z(f)f/z(f_i) \in A$, $(i = 1,2)$, relation (14) shows that f is not an extreme point of A.

Since the functions $au_j(t)$, $j = 0\ 1, \cdots$ are obviously extreme points of A if and only if, $a = 0$ or $a = 1/z(u_j)$, we conclude that the extreme points of A are exactly the function 0 and $u_j/z(u_j)$, $j = 0, 1, \cdots$.

By the well known theorem of Choquet (cf. e.g., [4]), there exists, for every $f \in A$, a probability measure μ on A which represents f and is supported by the set $\varepsilon(A)$ of extreme points of A. Since $\varepsilon(A)$ is countable, such a probability measure is discrete, i.e.

$$\mu\left(\frac{u_j}{z(\mu_j)}\right) = \mu_j \geqq 0, \quad j = 0, 1, \cdots.$$

The fact that μ represents f means that, for every continuous linear functional h on D,

$$h(f) = \int h d\mu = \sum_{j=0}^{\infty} \mu_j h(u_j/z(u_j)) = \sum_{j=0}^{\infty} \frac{\mu_j}{z(u_j)} h(u_j). \tag{15}$$

In particular, for the evaluation functionals $h(f) = f(t)$, $t \in (0,1)$ we obtain

$$f(t) = \sum_{j=0}^{\infty} a_j u_j(t) \qquad \left(a_j = \frac{\mu_j}{z(u_j)} \geqq 0\right). \tag{16}$$

Since, in a Montel space, weak and strong sequential convergence are equivalent, we conclude that the generalized derivatives $L_p f$ converge uniformly on each I_k. This validates the term by term generalized differentiation of (16), and thus proves the validity of the expansion (6) for every $f \in A$, hence for every bounded G.A.M. function. This completes the proof of the theorem.

We turn now to the main result of the paper, expounding the necessary and sufficient conditions for the validity of the expansion (6).

THEOREM 2. *Let* $\{u_i\}_{i=0}^{\infty}$ *be an infinite Extended Tchebycheff system on* $[0,1]$. *Then, a necessary and sufficient condition for each* G.A.M. *function to possess the representation* (6) *is that for each* t, $0 < t < 1$, *there exist an* s, $t < s < 1$, *for which*

$$\lim_{n\to\infty} \frac{u_n(t)}{u_n(s)} = 0. \tag{17}$$

Proof. Let $\phi(t)$ be an arbitrary G.A.M. function. Then relation (5) holds for all n. Furthermore, since $u_i(t)L_{i-1}\phi(0^+)/w_i(o)$ is nonnegative for all k and so is the integral, we deduce that

$$s_n(t) = \sum_{i=0}^{n} \frac{L_{i-1}\phi(0^+)}{w_i(o)} u_i(t) \tag{18}$$

is a nondecreasing sequence bounded above by $\phi(t)$. Hence, it converges for all $t \in [0,1]$ to a function $s(t) < \infty$. Moreover, since $s_n(t)$ is a G.A.M. function and the cones $P(u_0, \cdots, u_k)$ are closed with respect to pointwise convergence, we conclude that $s(t)$ is a G.A.M. function and thus belongs to $C^\infty(0,1)$. This implies, by Dini's theorem, that the convergence is uniform on compact subsets of $[0,1)$. This argument was adopted from [2].

We note next that the same reasoning applies to the sequences $\{L_k s_n(t),\, n \geqq k\}$

$$L_k s_n(t) = \sum_{i=k+1}^{n} \frac{L_{i-1}\phi(0^+)}{w_i(o)} L_k u_i(t), \quad k = 0,1,\cdots, \tag{19}$$

albeit with respect to the "$k+1$-st reduced Extended Tchebycheff system" (see [1], p. 394). It follows that

$$L_k s(t) = \sum_{i=k+1}^{\infty} \frac{L_{i-1}\phi(0^+)}{w_i(o)} L_k u_i(t), \quad k = 0,1,\cdots, t \in [0,1). \tag{20}$$

Define now the function

$$g(t) = \phi(t) - s(t) = \lim_{n\to\infty} \int_0^1 \phi_n(t;x) L_n\phi(x)\, dx. \tag{21}$$

We note that $\phi_n(t;x)$ is non-negative and belongs to $P(u_0,\cdots,u_k)$ for all $n \geqq k$ (see [3]). Since we also have $L_n\phi(x) \geqq 0$ for all $n \geqq -1$, it follows that $g(t)$ is a G.A.M. function. We further note that (18) and (20) imply that

$$g(0^+) = 0; \; L_k g(0^+) = 0, \qquad k = 0,1,\cdots. \tag{22}$$

We deduce that in order to prove that the expansion (6) is valid it suffices to show that if $g(t)$ is a G.A.M. function for which (22) holds then it must vanish identically. Conversely, for the expansion (6) to be valid it is clearly necessary that each G.A.M. function satisfying (22) vanish identically.

Thus, the assertion of Theorem 2 is equivalent to the assertion of the following Lemma, which deserves a separate statement.

LEMMA. *A necessary and sufficient condition for all G.A.M. functions* $g(t)$ *satisfying* (22) *to vanish identically in* (0, 1) *is that* (17) *hold.*

Proof. a) *Sufficiency.* We start by assuming that (17) holds. Let $g(t)$ be a G.A.M. function satisfying (22). Let x be an arbitrary point in (0,1), and let y, $x < y < 1$, be the point for which

$$\lim_{n\to\infty} \frac{u_n(x)}{u_n(y)} = 0. \tag{23}$$

Since $g(t)$ is a G.A.M. function, the system

$$\{u_0(t), u_1(t), \cdots, u_n(t), g(t)\}$$

is a Weak Tchebycheff system for all n, $n = 0,1,\cdots,$. Hence, by making n points coalesce with 0, choosing x as the $n+1$-st point and y as the $n+2$-nd point, we have the following determinant inequality:

$$\begin{vmatrix} w_0(0) & 0 & \cdots & 0 & u_0(x) & u_0(y) \\ 0 & w_1(0) & 0 & \cdots & u_1(x) & u_1(y) \\ . & 0 & \cdots & \cdots & . & . \\ . & . & . & \cdots & . & . \\ . & . & 0 & w_{n-1}(0) & . & . \\ . & . & . & 0 & u_n(x) & u_n(y) \\ 0 & 0 & . & 0 & g(x) & g(y) \end{vmatrix} \geqq 0, \tag{24}$$

which is valid for all n. (Note that the zeros in the last row are a consequence of (22)). Inequality (24) is equivalent to

$$g(x) \leqq g(y)\frac{u_n(x)}{u_n(y)} \quad 0 < x < y < 1, \quad n = 0, 1, \cdots. \tag{25}$$

Hence, relation (23) implies that $g(x) = 0$, and since x was arbitray, $g(t) \equiv 0$, $0 < t < 1$.

Necessity. We will show that if (17) does not hold then there exists a G.A.M. function which satisfies (22) but is not identically zero.

We note first that by substituting in (25) the G.A.M. function $g(t) = u_{n+1}(t)$, we find that

$$\{u_n(x)/u_n(y)\}_{n=0}^{\infty} \text{ is a monotone decreasing sequence if } x < y. \tag{26}$$

Assume now that (17) does not hold. Then there exists a point t_0, such that for all s, $t_0 < s < 1$, the sequence $\{u_n(t_0)/u_n(s)\}$ does not tend to 0 as $n \to \infty$. Since it is monotone decreasing by (26), it is bounded from below.

Consider the sequence of functions

$$\left(\frac{u_n(t)}{u_n(t_0)}\right), \qquad 0 < t < 1.$$

By (26), and by the boundedness of the sequence for each fixed t, which is a consequence of (17), we find that this sequence converges pointwise, for all $0 < t < 1$.

Let the limit function be denoted by $v(t)$. We can easily deduce that $v(t)$ is a G.A.M. function, which does not vanish identically. We now show that it satisfies (22).

Let s and t be two points such that $0 < t < s < 1$ and let k be an arbitrary integer such that $k \geqq -1$. Using the construction of Theorem 1 and observing that for some l the points s and t belong to I_l, we deduce the existence of a subsequence of natural numbers $\{m\}_{m \in N}$ such that $L_j[u_m(x)/u_m(t_0)]$ converges uniformly in I_l for all j. Hence, it converges to $L_j v(x)$. In particular,

$$\lim_{m\to\infty} L_k\left[\frac{u_m(t)}{u_m(t_0)}\right] = L_k v(t), \ \lim_{m\to\infty} L_k\left[\frac{u_m(s)}{u_m(t_0)}\right] = L_k v(s).$$

Noting that $L_k u_n(t)$ is a "reduced" ECT-system, we deduce, like in (25), the inequality

$$L_k u_m(t) \leqq L_k u_m(s) \frac{L_k u_{k+2}(t)}{L_k u_{k+2}(s)}, \text{ for } m \geqq k+2.$$

Dividing throughout by $u_m(t_0)$ and then letting $m \to \infty$, we obtain the inequalities

$$0 \leqq L_k v(t) \leqq L_k v(s) \frac{L_k u_{k+2}(t)}{L_k u_{k+2}(s)}, \qquad 0 < t < s < 1.$$

Keeping s fixed and letting $t \to 0^+$, we find that the expression on the right-hand side tends to 0, and thus also $L_k v(0^+) = 0$. This is true for all k so that $v(t)$ indeed satisfies (22) while not being identically zero.

This completes the proof of the Lemma and of Theorem 2.

The present, simplified proof of the necessity part of the Lemma was suggested to us by Professor Karlin.

The following theorem, concerning the extreme-rays structure, is an easy consequence of the Lemma, and will therefore be stated without proof.

THEOREM 3. *If* (17) *holds, then the sequence of functions* $u_i(t)$, $i = 0, 1, \cdots$, *generates the totality of the extreme rays of the cone* P_A *of G.A.M. functions.*

REMARK. If we restrict ourselves to the cone $P_A \cap B$ of bounded G. A. M. functions, then condition (17) should be replaced by

$$(17') \qquad \lim_{n \to \infty} \frac{u_n(t)}{u_n(1)} = 0, \text{ for all } t, \qquad 0 < t < 1,$$

and Theorems 2 and 3 as well as the Lemma will still hold. The proof of the Lemma follows similar lines and is, in fact, slightly simpler. Thus, the following statements are equivalent:

(i) Each function of $P_A \cap B$ has a representation (6).

(ii) $\{u_i(t)\}$ satisfies condition (17′).

(iii) The extreme rays of $P_A \cap B$ are exactly those generated by $\{u_i\}_{i=0}^{\infty}$.

Making use of Theorem 2, we immediately obtain a characterization of the cone dual to P_A — the cone of G.A.M. functions (see [2]).

THEOREM 4. *Let* $\{u_i\}_0^\infty$ *be an Extended Tchebycheff system such that* (17) *holds. A signed measure* $d\mu$ *belongs to the dual cone of* P_A *if and only if*

$$(22) \qquad \int_0^1 u_i d\mu \geqq 0, \quad i = 0, 1, \cdots.$$

We conclude this paper by observing that the sufficient conditions given in [2] have to imply that (17) holds. This implication is by no means easy to verify, even in the simple cases, and may be therefore considered as corollary of our present results.

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