NORM INEQUALITIES FOR A CERTAIN CLASS OF C^∞ FUNCTIONS

BY

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For Paul Montel on his 95th birthday

ABSTRACT

In 1936 the author showed that the function $\sin(\pi(x+1)/4)$ is the entire function of least exponential type $(=\pi/4)$ among all entire functions $f(z)$ with the property that $f^{(n)}(z)$ vanishes somewhere in the real interval $[-1, 1]$ $(n=0, 1,2,\ldots)$. Now more precise results of this kind are obtained by working within the class $C^\infty[-1, 1]$.

Introduction and statement of results

In [3] the author proved long ago the following

THEOREM 1. *If $f(x)$ is an entire function of exponential type less than $\pi/4$ and is such that every one of its derivatives $f(x), f'(x), f''(x),\cdots$ vanishes somewhere in the interval $I=[-1,1]$, then $f(x)=0$ identically.*

Since the function $f(x)=\sin(\pi(x+1)/4)$ satisfies the conditions of Theorem 1 and is precisely of type $=\pi/4$, it follows that the constant $\pi/4$ can not be improved.

In 1936 Theorem 1 was a peripheral contribution towards the problem of determining the so-called Whittaker constant W (see [4, 45]). For the important more recent results of Evgrafov and Buckholtz concerning the Whittaker constant see [2], also for references. I do not pursue here this function-theoretic problem, but wish to point out that the contents of Theorem 1 can be sharpened by working within the context of functions $f(x)$ that are infinitely differentiable in the interval $I=[-1,1]$. Accordingly, the appropriate setting of our discussion is the class $\mathscr{F}$ of real or complex-valued functions defined by

$$(1)\qquad \mathscr{F}=\{f(x); f\in C^\infty(I),\ f^{(\nu)}(\eta_\nu)=0\ (\nu=0,1,\cdots),\ -1\leqq\eta_\nu\leqq 1\}.$$

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In terms of the L_{∞} (or supremum) norm in I we may state

THEOREM 2. *If $f(x) \in \mathscr{F}$ and $\|f\| > 0$, then*

$$\|f^{(n)}\| > \omega_n \|f\|, \qquad (n = 1, 2, \cdots) \tag{2}$$

where

$$\omega_n = \begin{cases} \dfrac{n!}{2^n} \dfrac{1}{|E_n|} & \text{if } n \text{ is even,} \\ \dfrac{(n+1)!}{2^{2n+1}(2^{n+1}-1)} \dfrac{1}{|B_{n+1}|} & \text{if } n \text{ is odd.} \end{cases} \tag{3}$$

Here E_n and B_{n+1} are the Euler and Bernoulli numbers, respectively. The constants ω_n in (2) are sharp for every n, i.e. none can be improved.

We find that

$$\omega_1 = \frac{1}{2}, \omega_2 = \frac{1}{2}, \omega_3 = \frac{3}{8}, \omega_4 = \frac{3}{10}, \omega_5 = \frac{15}{64}, \omega_6 = \frac{45}{244}.$$

In §4 we show that Theorem 1 is an easy consequence of Theorem 2. In §2 we derive Theorem 2 from a result to be now described.

For every natural number n we denote by $\mathscr{F}_n$ the class of functions $f(x)$ satisfying the following two conditions:

$$f(x) \in C^{n-1}(I), \text{ and } f^{(n-1)}(x) \text{ satisfies a Lipschitz condition,} \tag{4}$$

$$f^{(v)}(x) \ (v = 0, 1, \cdots, n-1) \text{ vanishes at some point of } I. \tag{5}$$

A particular element of $\mathscr{F}_n$ is the polynomial

$$s_n(x) = \int_{-1}^{x} dx_1 \int_{1}^{x_1} dx_2 \cdots \int_{(-1)^n}^{x_{n-1}} dx_n, \tag{6}$$

in view of the relations

$$s_n(-1) = 0, \ s_n'(1) = 0, \cdots, s_n^{(n-1)}((-1)^n) = 0. \tag{7}$$

In terms of the class $\mathscr{F}_n$ and the polynomial (6) we may state the following theorem.

THEOREM 3. *If $n \geqq 1$ and*

$$f(x) \in \mathscr{F}_n \tag{8}$$

then

$$\|f^{(n)}\| \geqq \omega_n \|f\|,$$

with the equality sign if and only if

(10) $f(x) = Cs_n(\varepsilon x)$, *where* C *is a constant and* $\varepsilon = \pm 1$.

The constant ω_n *is given by* (3) *and may also be defined as*

(11) $$\omega_n = |s_n(1)|^{-1}.$$

In §3 we show that Theorem 3 can be stated (Corollary 1) as an apparently new characteristic property of the Euler polynomial $E_n(x)$, in the interval $I^* = [0, \frac{1}{2}]$.

1. Proof of Theorem 3. We require a lemma concerning the function

(1.1) $$\Phi_n(x; \eta_0, \eta_1, \cdots, \eta_{n-1}) = \left| \int_{\eta_0}^{x} dx_1 \left| \int_{\eta_1}^{x_1} dx_2 \cdots \left| \int_{\eta_{n-1}}^{x_{n-1}} \right| dx_n \cdots \right| \right|.$$

This expression being somewhat cumbersome, we write it out completely for the case when $n = 3$:

$$\Phi_3(x; \eta_0, \eta_1, \eta_2) = \left| \int_{\eta_0}^{x} dx_1 \left| \int_{\eta_1}^{x_1} dx_2 \left| \int_{\eta_2}^{x_2} dx_3 \right| \right| \right|.$$

In [3, 14–18] the author established a lemma concerning the function (1.1) which implies readily the following

LEMMA 1. *If*

(1.2) $$-1 \leqq x \leqq 1, \ -1 \leqq \eta_\nu \leqq 1 \qquad (\nu = 0, 1, \cdots, n-1),$$

then

(1.3) $$\Phi_n(x; \eta_0, \cdots, \eta_{n-1}) \leqq \Phi_n(1; -1, 1, \cdots, (-1)^n),$$

with the equality sign if and only if

(1.4) $$x = 1, \eta_\nu = (-1)^{\nu+1}, \text{ or } x = -1, \eta_\nu = (-1)^\nu, \qquad (\nu = 0, \cdots, n-1).$$

In other words: Within the $(n+1)$ dim. cube (1.2) the function $\Phi(x; \eta)$ assumes its absolute maximum only in two opposite vertices of the cube that are described by (1.4).

Let now (8) hold and let us establish the inequality (9), where we define ω_n by (11), hence

(1.5) $$\omega_n^{-1} = |s_n(1)| = \Phi_n(1; -1, 1, \cdots, (-1)^n).$$

From the assumption (8) we know that

(1.6) $$f^{(\nu)}(\eta_\nu) = 0, \qquad (\nu = 0, \cdots, n-1),$$

for appropriate points η_ν in I. It follows that we may express $f(x)$ in terms of $f^{(n)}(x)$ by the repeated integral

$$f(x) = \int_{\eta_0}^{x} dx_1 \int_{\eta_1}^{x_1} dx_2 \cdots \int_{\eta_{n-1}}^{x_{n-1}} f^{(n)}(x_n)\, dx_n. \tag{1.7}$$

Let ξ be such that

$$|f(\xi)| = \|f\|. \tag{1.8}$$

Using Lemma 1 and (1.8) we may write the following string of equations and inequalities

$$\begin{aligned} \|f\| &= |f(\xi)| = \left| \int_{\eta_0}^{\xi} dx_1 \int_{\eta_1}^{x_1} dx_2 \cdots \int_{\eta_{n-1}}^{x_{n-1}} f^{(n)}(x_n) dx_n \right| \\ &\leqq \left| \int_{\eta_0}^{\xi} dx_1 \left| \int_{\eta_1}^{x} dx_2 \cdots \left| \int_{\eta_{n-1}}^{x_{n-1}} |f^{(n)}(x_n)| dx_n \right| \cdots \right| \right| \\ &\leqq \|f^{(n)}\| \cdot \left| \int_{\eta_0}^{\xi} dx_1 \left| \cdots \left| \int_{\eta_{n-1}}^{x_{n-1}} dx_n \right| \cdots \right| \right| \\ &= \|f^{(n)}\| \Phi_n(\xi; \eta_0, \cdots, \eta_{n-1}) \leqq \|f^{(n)}\| \Phi_n(1; -1, 1, \cdots, (-1)^n). \end{aligned} \tag{1.9}$$

But then (1.5) implies that

$$\|f\| \leqq \|f^{(n)}\| \cdot \omega_n^{-1}, \tag{1.10}$$

and the inequality (9) is established.

Let us now assume that (9), or (1.10), holds with the equality sign. It follows that the extreme terms of (1.9) are equal. Let us now analyze the consequences of this fact. In particular we obtain that

$$\Phi_n(\xi; \eta_0, \cdots, \eta_{n-1}) = \Phi_n(1; -1, 1, \cdots, (-1)^n),$$

and by Lemma 1 we conclude that one of the two alternatives

$$\xi = 1, \eta_\nu = (-1)^{\nu+1}, \text{ or } \xi = -1, \eta_\nu = (-1)^\nu \tag{1.11}$$

must hold. Let us assume that the first one holds. We consider now the relation

$$(-1)^{[n/2]} \int_{-1}^{1} dx_1 \int_{1}^{x_1} dx_2 \cdots \int_{(-1)^n}^{x_{n-1}} F(x_n)\, dx_n = \int_{-1}^{1} K(x) F(x)\, dx \tag{1.12}$$

which transforms the repeated integral on the left into the single integral on the right hand side. This relation is to be valid for an arbitrary continuous function $F(x)$, while $[n/2]$ has its usual arithmetic meaning. A moment's reflexion will show that $K(x) \in C(I)$, in fact $K(x) \in \pi_{n-1}$ (the polynomials of degree $n-1$), and that

$$K(x) > 0 \text{ if } -1 < x < 1. \tag{1.13}$$

In terms of $K(x)$ the equality of all members of (1.9) furnishes the relation

$$\left| \int_{-1}^{1} K(x) f^{(n)}(x) dx \right| = \| f^{(n)} \| \int_{-1}^{1} K(x)\, dx,$$

which in turn implies the relation

$$\int_{-1}^{1} K(x) \{ \| f^{(n)} \| - \varepsilon f^{(n)}(x) \}\, dx = 0, \qquad (\varepsilon = 1 \text{ or } -1). \tag{1.14}$$

The integrand of (1.14) being non-negative almost everywhere we conclude that $f^{(n)}(x) = \| f^{(n)} \|$, or perhaps $f^{(n)}(x) = - \| f^{(n)} \|$, almost everywhere in I. Moreover, the first relations (1.11) and (1.6) show that

$$f(-1) = 0,\ f'(1) = 0, \cdots, f^{(n-2)}((-1)^n) = 0,$$

which imply that

$$f(x) = \pm \| f^{(n)} \| s_n(x).$$

Similarly the second alternative (1.11) will imply that $f(x) = \pm \| f^{(n)} \| s_n(-x)$. This concludes a proof of Theorem 3.

2. Proof of Theorem 2. Let $f(x) \in \mathscr{F}$ and $\| f \| > 0$. Evidently $f(x) \in \mathscr{F}_n$, for every n, and Theorem 3 shows that $\| f^{(n)} \| \geqq \omega_n \| f \|$. Since (10) can not hold because then $f^{(n)}(x)$ would not vanish anywhere in I, we conclude that the strict inequality (2) must hold, and the inequalities (2) are thereby established.

We are still to show that the constant ω_n is best possible. This requires to show the following: *If $\delta > 0$, then there exists a function $f(x) \in \mathscr{F}$ such that*

$$\| f^{(n)} \| < (\omega_n + \delta) \| f \|. \tag{2.1}$$

This we do as follows. We already know that the polynomial

$$s_n(x) = \int_{-1}^{x} dx_1 \int_{1}^{x_1} dx_2 \cdots \int_{(-1)^n}^{x_{n-1}} dx_n \tag{2.2}$$

has the properties

$$\| s_n^{(n)} \| = 1, \ \| s_n \| = \omega_n^{-1}, \tag{2.3}$$

whence

(2.4) $$\| s_n^{(n)} \| = \omega_n \| s_n \|.$$

We shall now perturb the function $s_n(x)$ slightly so that it will become an element of $\mathscr{F}$, while the relation (2.4) will be disturbed only slightly.

To do this we use the auxiliary function

(2.5) $$h(x) = \int_0^x e^{-t^{-1}(1-t)^{-1}}\, dt \Big/ \int_0^1 e^{-t^{-1}(1-t)^{-1}}\, dt, \qquad (0 \leqq x \leqq 1),$$

having the properties

(2.6) $$h^{(\nu)}(0) = 0 \qquad (\nu = 0, 1, 2, \cdots)$$

(2.7) $$h(1) = 1, \qquad h^{(\nu)}(1) = 0 \qquad (\nu = 1, 2, \cdots),$$

(2.8) $h(x)$ increases from 0 to 1 in $[0, 1]$.

Let α be a small positive number and let us define the even function

(2.9) $$g_\alpha(x) = \begin{cases} 1 & \text{if } \quad \alpha \leqq x \leqq 1, \\ h(x/\alpha) & \text{if } \quad 0 \leqq x \leqq \alpha, \\ g_\alpha(-x) & \text{if } \quad -1 \leqq x \leqq 0. \end{cases}$$

Evidently

$$g_\alpha(x) \in \mathscr{F}.$$

It follows that an n-fold integral $f_\alpha(x)$ of $g_\alpha(x)$ will also be in $\mathscr{F}$, provided that each of the functions

$$f_\alpha(x), f_\alpha'(x), \cdots, f_\alpha^{(n-1)}(x),$$

vanishes somewhere in I. An n-fold integral satisfying these conditions is evidently

(2.10) $$f_\alpha(x) = \int_{-1}^x dx_1 \int_1^{x_1} dx_2 \cdots \int_{(-1)^n}^{x_{n-1}} g_\alpha(x_n)\, dx_n.$$

From $f_\alpha^{(n)}(x) = g_\alpha(x)$, (2.8) and (2.9), it is clear that

(2.11) $$\| f_\alpha^{(n)} \| = 1.$$

From (2.9) we see that if $x \neq 0$ then $g_\alpha(x) \to 1$ as $\alpha \to +0$. From (2.2) and (2.10) it follows that

$$\| f_\alpha \| \to \| s_n \| = \omega_n^{-1} \text{ as } \alpha \to +0,$$

and therefore

(2.12) $$\|f_\alpha\| > \frac{1}{\omega_n + \delta} \text{ provided that } \alpha \text{ is sufficiently small.}$$

For such small α we therefore have that

$$\|f_\alpha^{(n)}\| = 1 < (\omega_n + \delta)\|f_\alpha\|.$$

Since $f_\alpha(x) \in \mathscr{F}$, the inequality (2.1) is established.

3. Determination of the constants ω_n and an extremum property of the Euler polinomials. The constants ω_n were defined by (11) and (6). Let us sketch the derivation of the explicit expressions (3). In (6) we reverse the order of the lower limits of integration and define the new polynomials.

(3.1) $$r_n(x) = \int_{(-1)^n}^{x} dx_1 \int_{(-1)^{n-1}}^{x_1} dx_2 \cdots \int_{-1}^{x_{n-1}} dx_n, \qquad (r_0(x) = 1).$$

These form an Appell sequence of polynomials having the generating function

(3.2) $$\frac{2e^t}{e^{2t} + e^{-2t}} e^{xt} = \sum_0^\infty r_n(x) t^n$$

(See [3, 19]). They are related to the $s_n(x)$ by

(3.3) $$s_n(x) = \pm\, r_n((-1)^{n+1} x).$$

Now (11) and (3.3) show that

(3.4) $$(\omega_n)^{-1} = |s_n(1)| = \begin{cases} |r_n(-1)| & \text{if } n \text{ is even,} \\ |r_n(1)| & \text{if } n \text{ is odd.} \end{cases}$$

From these results and the generating function

(3.5) $$\frac{2}{e^t + e^{-t}} = \sum_0^\infty \frac{E_n}{n!} t^n$$

of the Euler numbers E_n, we easily obtain the expressions (3) for even values of n. Details may be omitted.

Similarly, the generating function

(3.6) $$\frac{2e^{xt}}{1 + e^t} = \sum_0^\infty \frac{E_n(x)}{n!} t^n$$

of the Euler polynomials $E_n(x)$, and the known relations

(3.7) $$E_n(0) = -\frac{2}{n+1}(2^{n+1} - 1)B_{n+1}$$

easily furnish the relations (3) for odd values of n.

We also mention that (3.2) and (3.6) show that

$$r_n(x) = (-1)^n \frac{4^n}{n!} E_n\left(\frac{1-x}{4}\right). \tag{3.8}$$

These relations show that our Theorem 3 may be interpreted as describing a new characteristic extremum property of the Euler polynomial $E_n(x)$ in the interval $[0, \frac{1}{2}]$. We state it as

COROLLARY 1. *Let $\mathscr{F}_n^*$ denote the class of functions $F(x)$ defined in $I^* = [0, \frac{1}{2}]$ and satisfying the three conditions*

(i) *$F(x) \in C^{n-1}(I^*)$, and $F^{(n-1)}(x)$ satisfies a Lipschitz condition,*

(ii) *$F^{(\nu)}(x)$ $(\nu = 0, \cdots, n-1)$ vanishes at some point of I^*,*

(iii) *$\| F^{(n)} \| = n!$.*

If $F(x) \in \mathscr{F}_n^*$, then the inequality

$$\| F \| \leqq \| E_n(x) \|$$

holds, with the equality sign if and only if

$$F(x) = \pm E_n(x) \text{ or } F(x) = \pm E_n(\tfrac{1}{2} - x).$$

Finally, we shall need below the following

LEMMA 2. *The constants ω_n satisfy the relation*

$$\lim_{n \to \infty} (\omega_n)^{1/n} = \frac{\pi}{4}. \tag{3.9}$$

A proof follows immediately from the estimate of $r_n(x)$ given in [3, formula (29)].

4. Proof of Theorem 1. Let $f(x)$ be an entire function of exponential type $= \gamma$. It is known (see [1, 11]] that

$$\overline{\lim_{n \to \infty}} |f^{(n)}(0)|^{1/n} = \gamma. \tag{4.1}$$

We first establish

LEMMA 3. *If*

$$\| f^{(n)} \| = \max |f^{(n)}(x)| \text{ in the interval } I = [-1, 1], \tag{4.2}$$

then

$$\overline{\lim_{n \to \infty}} \| f^{(n)} \|^{1/n} = \gamma. \tag{4.3}$$

PROOF. If $\gamma_1 > \gamma$, then (4.1) shows that

$$|f^{(n)}(0)| < \gamma_1^n \text{ if } n > N.$$

The expansion of $f^{(n)}(x)$ in powers of x shows that, if $n > N$, then

$$\|f^{(n)}\|^{1/n} \leqq (|f^{(n)}(0)| + \frac{1}{1!}|f^{(n+1)}(0)| + \frac{1}{2!}|f^{(n+2)}(0)| + \cdots)^{1/n}$$

$$\leqq \left(\gamma_1^n + \frac{1}{1!}\gamma_1^{n+1} + \cdots\right)^{1/n} = \gamma_1 (1 + \frac{1}{1!}\gamma_1 + \cdots)^{1/n} = \gamma_1 e^{\gamma_1/n},$$

whence

$$\overline{\lim} \|f^{(n)}\|^{1/n} \leqq \gamma_1.$$

Letting $\gamma_1 \to \gamma$ we obtain (4.3).

A proof of Theorem 1 now follows immediately. Indeed, let $f(x)$ be of exponential type γ, and let $f, f', \cdots$ have the required zeros in $I = [-1, 1]$. By Theorem 2

$$\|f^{(n)}\| > \omega_n \|f\|$$

whence

$$\|f^{(n)}\|^{1/n} > (\omega_n)^{1/n} \|f\|^{1/n}.$$

Lemma 2 and 3 show that if $n \to \infty$ we obtain that

$$\gamma \geqq \pi/4.$$

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