ON PERFECTLY HOMOGENEOUS BASES IN BANACH SPACES

BY

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ABSTRACT

It is proved that a Banach space is isomorphic to c_o or to l_p if and only if it has a normalized basis $\{x_i\}_{i=1}^{\infty}$ which is equivalent to every normalized block-basis with respect to $\{x_i\}_{i=1}^{\infty}$.

1. **Introduction**. F. Bohnenblust gave in [2] an axiomatic characterization of c_0 and l_p. The following proposition follows easily from his proof:

PROPOSITION 1.1. *Let $\{x_i\}_{i=1}^{\infty}$ be a normalized basis in a Banach space X. If for every normalized block-basis $\{y_i\}_{i=1}^{\infty}$ (with respect to $\{x_i\}_{i=1}^{\infty}$) and any real sequence $\{a_i\}_{i=1}^{\infty}$*

$$\left\| \sum_{i=1}^{n} a_i x_i \right\| = \left\| \sum_{i=1}^{n} a_i y_i \right\|$$

for all natural n then the basis $\{x_i\}_{i=1}^{\infty}$ is equivalent to the unit-vectors basis in c_0 or in l_p for some $p \geqq 1$. Moreover, this equivalence of the bases induces an isometric isomorphism of X onto c_0 or l_p.

The following natural question arises: If we assume only that all normalized block-bases with respect to $\{x_i\}_{i=1}^{\infty}$ are equivalent, is $\{x_i\}_{i=1}^{\infty}$ equivalent to the unit-vectors basis in c_0 or l_p?

In this paper we show that the answer is positive and the equivalence of all normalized block-bases characterizes the unit-vectors bases in c_0 and l_p.

After the preliminary lemmas of Section 2, we use the method of the proof of Lemma 4.3 of [2] to prove our main result, Theorem 3.1. A remark concerning a result of A. Pełczyński and I. Singer [6] concludes Section 3.

DEFINITIONS AND NOTATIONS. A basis $\{x_i\}_{i=1}^{\infty}$ in a Banach space is called *normalized* if $\| x_i \| = 1$ for every i. The sequence $\{y_i\}_{i=1}^{\infty}$ is called *a block-basis with respect to the basis* $\{x_i\}_{i=1}^{\infty}$ if for every i $y_i = \sum_{j=p(i)+1}^{p(i+1)} a_j x_j$, where $\{p(i)\}_{i=1}^{\infty}$

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is an increasing sequence of nonnegative integers. In this paper we discuss only one basis $\{x_i\}_{i=1}^{\infty}$ in the Banach space X; all block-bases mentioned are assumed to be block-bases with respect to the basis $\{x_i\}_{i=1}^{\infty}$. A basis $\{x_i\}_{i=1}^{\infty}$ in X is *equivalent* to a basis $\{z_i\}_{i=1}^{\infty}$ in a Banach space Z if for every real sequence $\{a_i\}_{i=1}^{\infty}$ $\sum_{i=1}^{\infty} a_i x_i$ converges if and only if $\sum_{i=1}^{\infty} a_i z_i$ converges. The closed subspace which is spanned by a sequence $\{y_i\}_{i=1}^{\infty}$ in X is denoted by $[y_i]_{i=1}^{\infty}$. A sequence $\{y_i\}_{i=1}^{\infty}$ in X is called *a basic sequence* if it forms a basis in $[y_i]_{i=1}^{\infty}$. (Every block-basis is a basic sequence in X.) Following C. Bessaga and A. Pełczyński [1] we call a basis $\{x_i\}_{i=1}^{\infty}$ *perfectly homogeneous* if it is normalized and every normalized block-basis $\{z_i\}_{i=1}^{\infty}$ with respect to $\{x_i\}_{i=1}^{\infty}$ is equivalent to the basis $\{x_i\}_{i=1}^{\infty}$.

2. **Preliminary lemmas.** Let $\{x_i\}_{i=1}^{\infty}$ be a normalized basis in a Banach space X and let $\{f_i\}_{i=1}^{\infty}$ denote its biorthogonal sequence in X^*. In the sequel we shall consider the following property:

(a) If S and T are disjoint finite sets of positive integers and $|t| \geqq |s|$ then

$$\Big\| \sum_{i\in S} a_i x_i + t\cdot \sum_{i\in T} a_i x_i \Big\| \geqq \Big\| \sum_{i\in S} a_i x_i + s \sum_{i\in T} a_i x_i \Big\|$$

for every real $\{a_i\}$, $i \in S \cup T$.

LEMMA 2.1. *If a basis $\{x_i\}_{i=1}^{\infty}$ satisfies (a) then $\|f_i\| = 1$ for every i.*

Proof. $\|f_i\| \geqq f_i(x_i) = 1$. On the other hand

$$\|f_i\| = \sup_{\|\sum_{j=1}^{\infty} a_j x_j\| \leqq 1} \Big| f_i\Big(\sum_{j=1}^{\infty} a_j x_j\Big)\Big| = \sup_{\|\sum_{j=1}^{\infty} a_j x_j\| \leqq 1} |a_i| \leqq 1,$$

since, by (a), $|a_i| = \|a_i x_i\| \leqq \|\sum_{j=1}^{\infty} a_j x_j\| \leqq 1$.

LEMMA 2.2 *Assume that $\{x_i\}_{i=1}^{\infty}$ is a basis in a Banach space X which satisfies (a). If $|s_i| \leqq |t_i|$ for $1 \leqq i \leqq n$ then $\|\sum_{i=1}^{n} s_i x_i\| \leqq \|\sum_{i=1}^{n} t_i x_i\|$.*

Proof. Use (a) n times.

LEMMA 2.3. *Let $\{x_i\}_{i=1}^{\infty}$ be a normalized basis in a Banach space X which satisfies (a). If for some $M \geqq 1$ $\|\sum_{i=1}^{n} x_i\| \leqq M$ for every n then $\{x_i\}_{i=1}^{\infty}$ is equivalent to the unit vectors basis of c_0.*

Proof. By (a), Lemma 2.1 and Lemma 2.2

$$\max_{1\leqq i\leqq n} |a_i| \leqq \Big\| \sum_{i=1}^{n} a_i x_i \Big\| \leqq \Big(\max_{1\leqq i\leqq n} |a_i| \Big) \cdot \Big\| \sum_{i=1}^{n} x_i \Big\| \leqq M \cdot \max_{1\leqq i\leqq n} |a_i|.$$

Hence, $\sum_{i=1}^{\infty} a_i x_i$ converges if and only if $a_i \underset{i\to\infty}{\to} 0$.

Lemma 2.4. *Let $\{x_i\}_{i=1}^{\infty}$ be a perfectly homogeneous basis in a Banach space X. Then $\{x_i\}_{i=1}^{\infty}$ is an unconditional basis.*

Proof. Since for any sequence $\{a_i\}_{i=1}^{\infty}$ where $a_i = \pm 1$ the sequence $\{a_i x_i\}_{i=1}^{\infty}$ is a normalized block-basis, $\sum_{i=1}^{\infty} a_i b_i x_i$ converges if and only if $\sum_{i=1}^{\infty} b_i x_i$ converges. Hence, $\sum_{i=1}^{\infty} b_i x_i$ converges unconditionally whenever it converges. This proves Lemma 2.4.

Assume, now, that $\{x_i\}_{i=1}^{\infty}$ is a perfectly homogeneous basis in a Banach space X. By Lemma 2.4 $\{x_i\}_{i=1}^{\infty}$ is an unconditional basis. Assume, further, that $\{x_i\}_{i=1}^{\infty}$ satisfies (a). Denote by $\{\{y_i^{\alpha}\}_{i=1}^{\infty}\}_{\alpha \in I}$ the set of all normalized block-bases with respect to the basis $\{x_i\}_{i=1}^{\infty}$, I being the suitable index set. The assumed equivalence of the bases induces, for each $\alpha \in I$, an isomorphism T_α from X onto $[y_i^{\alpha}]_{i=1}^{\infty}$, defined by $T_\alpha(\sum_{i=1}^{\infty} a_i x_i) = \sum_{i=1}^{\infty} a_i y_i^{\alpha}$. (This follows from the closed graph theorem.)

Lemma 2.5. *There exists a real $M \geqq 1$ such that for every $\alpha \in I$ both $\| T_\alpha \| \leqq M$ and $\| T_\alpha^{-1} \| \leqq M$.*

Proof. Let us first show the existence of a finite bound for the set $\{\| T_\alpha \| : \alpha \in I\}$. If $\{\| T_\alpha \| : \alpha \in I\}$ is not bounded, then by the theorem of Banach and Steinhaus there is an $x = \sum_{i=1}^{\infty} b_i x_i \in X$ such that $\| x \| = 1$ and the set $\{\| T_\alpha x \| : \alpha \in I\}$ s not bounded. Hence, we can select a sequence $\{\alpha(n)\}_{n=1}^{\infty} \subset I$ so that for every n $\| \sum_{i=1}^{\infty} b_i y_i^{\alpha(n)} \| \geqq n+1$. We construct inductively three sequences of positive integers $\{n(i)\}$, $\{p(i)\}$ and $\{q(i)\}$ in the following way: $n(1) = 1$, $p(1) = 1$ and $q(1)$ is so large that $\| \sum_{i=p(1)}^{q(1)} b_i y_i^{\alpha(n(1))} \| \geqq 1$. Suppose that $n(1), n(2), \cdots, n(k)$, $p(1), p(2), \cdots, p(k)$ and $q(1), q(2), \cdots, q(k)$ were chosen such that

(2.1) $$\left\| \sum_{i=p(j)}^{q(j)} b_i y_i^{\alpha(n(j))} \right\| \geqq 1 \quad \text{for } 1 \leqq j \leqq k$$

(2.2) $$q(j-1) < p(j) \leqq q(j) \quad \text{for } 2 \leqq j \leqq k$$

(2.3) If M_j (respectively, N_j) is the least (respectively, the largest) index of the x_is' which appear in the representations of $y_{p(j)}^{\alpha(n(j))}, y_{p(j)+1}^{\alpha(n(j))}, \cdots, y_{q(j)}^{\alpha(n(j))}$ then $N_j < M_{j+1}$ for $1 \leqq j \leqq k-1$.

Choose $n(k+1) > \max\{N_k, q(k)\} + 3$, and put $p(k+1) = \max\{N_k, q(k)\} + 1$. By (a) and Lemma 2.1, for $j \geqq 1$ $|b_j| = |f_j(\sum_{i=1}^{\infty} b_i x_i)| \leqq \| x \| = 1$, and since $\| y_j^{\alpha(n(k+1))} \| = 1$ it follows that $\| \sum_{j=1}^{p(k+1)-1} b_j y_j^{\alpha(n(k+1))} \| \leqq p(k+1)$. Therefore

$$\left\| \sum_{j=p(k+1)}^{\infty} b_j y_j^{\alpha(n(k+1))} \right\| \geqq n(k+1) - p(k+1) \geqq 2.$$

Choose $q(k+1)$ so large that $\| \sum_{j=p(k+1)}^{q(k+1)} b_j y_j^{\alpha(n(k+1))} \| \geqq 1$. Since the representation of each block $y_j^{\alpha(n(k+1))}$ contains at least one x_i, the choice of $p(k+1)$

ensures that (2.3) is satisfied for $j = k$. By (2.3) the sequence $\{y_i^{\alpha(n(k))}\}_{p(k)\leqq i\leqq q(k),\,k\geqq 1}$ forms a normalized block-basis. By (2.1) $\Sigma_{k=1}^{\infty}(\Sigma_{i=p(k)}^{q(k)} b_i y_i^{\alpha(n(k))})$ does not converge while $\Sigma_{k=1}^{\infty}(\Sigma_{i=p(k)}^{q(k)} b_i x_i)$ converges, since, by Lemma 2.3, $\{x_i\}_{i=1}^{\infty}$ is an unconditional basis. This contradicts the equivalence of the block-bases.

Assume that the set $\{\|T_\alpha^{-1}\| : \alpha \in I\}$ is not bounded; so there exist sequences $\{\alpha(n)\}_{n=1}^{\infty} \subset I$ and $\{z_n\}_{n=1}^{\infty} \subset X$ such that if $z_n = \Sigma_{j=1}^{\infty} b_j^n x_j$ then for every n $\|\Sigma_{j=1}^{\infty} b_j^n y_j^{\alpha(n)}\| \leqq 2^{-n}$ and $\|\Sigma_{j=1}^{\infty} b_j^n x_j\| \geqq n+1$.
We choose, again, sequences $\{n(j)\}$, $\{p(j)\}$ and $\{q(j)\}$ such that (2.2) and (2.3) are satisfied in addition to the following

$$\left\| \sum_{j=p(k)}^{q(k)} b_j^{n(k)} x_j \right\| \geqq 1 \text{ for } k \geqq 1. \tag{2.4}$$

Put $n(1) = 1$, $p(1) = 1$ and choose $q(1)$ so large, that $\|\Sigma_{j=p(1)}^{q(1)} b_j^{n(j)} x_j\| \geqq 1$. $n(k+1)$ and $p(k+1)$ are also chosen as in the first part, and $q(k+1)$ is so large that $\|\Sigma_{j=p(k+1)}^{q(k+1)} b_j^{n(k+1)} x_j\| \geqq 1$. (This construction is possible since $\{y_i^{\alpha(n)}\}_{i=1}^{\infty}$ is a basis in $[y_i^{\alpha(n)}]_{i=1}^{\infty}$ and satisfies (a). By Lemma 2.1, if $\{g_i^n\}_{i=1}^{\infty}$ denotes the sequence of biorthogonal functionals of $\{y_i^{\alpha(n)}\}_{i=1}^{\infty}$, then $\|g_i^n\| = 1$. It follows that for all natural i and n $|b_i^n| = |g_i^n(\Sigma b_j^n y_j^{\alpha(n)})| \leqq 2^{-n} < 1$.) By (a) and Lemma 2.2

$$\left\| \sum_{j=p(k)}^{q(k)} b_j^{n(k)} y_j^{\alpha(n(k))} \right\| \leqq \left\| \sum_{j=1}^{\infty} b_j^{n(k)} y_j^{\alpha(n(k))} \right\| \leqq 2^{-n(k)}.$$

It follows that $\Sigma_{i=1}^{\infty}(\Sigma_{j=p(i)}^{q(i)} b_j^{n(i)} y_j^{\alpha(n(i))})$ converges while $\Sigma_{i=1}^{\infty}(\Sigma_{j=p(i)}^{q(i)} b_j^{n(i)} x_j)$ certainly does not, by (2.4). But by (2.3) the sequence $\{y_j^{\alpha(n(i))}\}_{p(i)\leqq j\leqq q(i),\,i\geqq 1}$, forms a normalized block-basis; it follows that the last block-basis is not equivalent to the basis $\{x_i\}_{i=1}^{\infty}$ — a contradiction. This completes the proof of Lemma 2.5.

3. The main theorem.

THEOREM 3.1. *Let $\{x_i\}_{i=1}^{\infty}$ be a normalized basis in a Banach space X. Then $\{x_i\}_{i=1}^{\infty}$ is perfectly homogeneous if and only if it is equivalent to the unit-vectors basis of c_0 or of l_p for some $p \geqq 1$.*

Proof. The "if" part is obvious, since the unit-vectors bases in c_0 and l_p are perfectly homogeneous. Let us prove the other part. By Lemma 2.4 $\{x_i\}_{i=1}^{\infty}$ is an unconditional basis. By [3] p. 73 Theorem 1(v) we may assume that $\{x_i\}_{i=1}^{\infty}$ satisfies (a), hence, by Lemma 2.5 it satisfies the following property:

(b) There exists a real $M \geqq 1$ such that for every normalized block basis $\{z_i\}_{i=1}^{\infty}$, $n \geqq 1$ and real $a_1, a_2, \cdots, a_n$

$$M \cdot \left\| \sum_{i=1}^{n} a_i z_i \right\| \geqq \left\| \sum_{i=1}^{n} a_i x_i \right\| \geqq M^{-1} \cdot \left\| \sum_{i=1}^{n} a_i z_i \right\|.$$

Define for $k \geqq 1$

$$\lambda_k = \left\| \sum_{i=1}^{k} x_i \right\|. \tag{3.1}$$

It follows from (a) that for $k \geqq 1$

$$\lambda_{k+1} \geqq \lambda_k. \tag{3.2}$$

By (b), for every increasing sequence $\{p(i)\}_{i=1}^{n}$ of positive integers

$$M^{-1} \leqq \left\| \sum_{j=1}^{n} x_j \right\| \cdot \left\| \sum_{j=1}^{n} x_{p(j)} \right\|^{-1} \leqq M. \tag{3.3}$$

It follows that

$$M^2 \cdot \left\| \sum_{i=1}^{n} x_i \right\| \geqq \left\| \sum_{i=1}^{n^k} x_i \right\| \cdot \left\| \sum_{i=1}^{n^{k-1}} x_i \right\|^{-1}. \tag{3.4}$$

(We substitute for z_i in the right side inequality of (b) the normalized block $\left\| \sum_{j=1}^{n^{k-1}} x_{j+(i-1)n^{k-1}} \right\|^{-1} \cdot \left(\sum_{j=1}^{n^{k-1}} x_{j+(i-1)n^{k-1}} \right)$ and use (3.3).)

Using 3.4 we can prove by induction that for every natural n and k

$$M^{2k} \cdot \left\| \sum_{i=1}^{n} x_i \right\|^k \geqq \left\| \sum_{i=1}^{n^k} x_i \right\|. \tag{3.5}$$

On the other hand, by the left-side inequality of (b) and by (3.3)

$$\left\| \sum_{i=1}^{n} x_i \right\| \leqq M^2 \cdot \left\| \sum_{i=1}^{n^k} x_j \right\| \cdot \left\| \sum_{j=1}^{n^{k-1}} x_j \right\|^{-1}.$$

Again it follows by induction that for every natural n and k

$$\left\| \sum_{i=1}^{n} x_i \right\|^k \leqq M^{2k} \cdot \left\| \sum_{i=1}^{n^k} x_i \right\|. \tag{3.6}$$

(3.1), (3.5) and (3.6) yield

$$M^{-2k} \cdot \lambda_{n^k} \leqq \lambda_n^k \leqq M^{2k} \cdot \lambda_{n^k} \text{ for every } n \text{ and } k. \tag{3.7}$$

For any natural N, n and k let $h = h(N, n, k)$ be the non-negative integer for which $N^h \leqq n^k < N^{h+1}$.

By (3.2) and (3.7)

$$h \cdot \log \lambda_N \leqq \log(M^{2h} \cdot \lambda_{N^h}) = 2h \cdot \log M + \log \lambda_{N^h} \leqq 2h \cdot \log M + \log \lambda_{n^k} \leqq$$
$$\leqq 2h \cdot \log M + \log(M^{2k} \cdot \lambda_n^k) = 2h \cdot \log M + 2k \cdot \log M + k \cdot \log \lambda_n.$$

Since $h \leqq k \cdot \log n \cdot (\log N)^{-1} \leqq h + 1$, we have
$(k \cdot \log n \cdot (\log N)^{-1} - 1) \cdot \log \lambda_N \leqq 2k \cdot \log n \cdot \log M (\log N)^{-1} + 2k \cdot \log M + k \cdot \log \lambda_n$.

Dividing by $k \log n$ and passing to the limit as $k \to \infty$ we get

(3.8) $(\log \lambda_N)\cdot(\log N)^{-1} \leqq (2\log M)\cdot((\log N)^{-1} + (\log n)^{-1}) + (\log \lambda_n)\cdot(\log\ n)^{-1}$.

By interchanging the rôles of n and N we get

(3.9) $(\log \lambda_n)(\log n)^{-1} \leqq (2\log M)\cdot((\log N)^{-1} + (\log n)^{-1}) + (\log \lambda_N)(\log N)^{-1}$.

By (3.8) and (3.9)

$$\left|(\log \lambda_n)(\log n)^{-1} - (\log \lambda_N)(\log N)^{-1}\right| \leqq (2\log M)[(\log n)^{-1} + (\log N)^{-1}]$$

therefore the sequence $\{(\log \lambda_n)(\log n)^{-1}\}_{n=1}^{\infty}$ converges to a limit c, and since $1 \leqq \lambda_n \leqq n$, we get that $0 \leqq c \leqq 1$. Passing to the limits as $N \to \infty$ in (3.8) and (3.9) we get:

$$c \log n \leqq 2\log M + \log \lambda_n = \log(M^2\lambda_n)$$

and $\log(M^{-2}\cdot\lambda_n) \leqq c\log n$, hence, for every n

(3.10) $$M^{-2}\cdot n^c \leqq \lambda_n \leqq M^2\cdot n^c.$$

If $c = 0$ then $\{\lambda_n\}_{n=1}^{\infty}$ is a bounded sequence, therefore by Lemma 2.3 X is isomorphic to c_0. If $1 \geqq c > 0$, put $c = 1/p$. We have

(3.11) $$M^{-2}\cdot n^{1/p} \leqq \left\| \sum_{i=1}^{n} x_i \right\| \leqq M^2\cdot n^{1/p}.$$

Let r_i be any positive rational number for $1 \leqq i \leqq n$ and assume that $r_i = m^{-1}\cdot k_i$, where m and k_i are positive integers. It follows from (a), (b), (3.11) and (3.3) that

(3.12) $$\left\| \sum_{i=1}^{n} r_i^{1/p} x_i \right\| =$$

$$= m^{-1/p}\cdot\left\| \sum_{i=1}^{n} k_i^{1/p} x_i \right\| \geqq M^{-2}\cdot m^{-1/p}\cdot\left\| \sum_{i=1}^{n} \left\| \sum_{j=1}^{k_i} x_j \right\| \cdot x_i \right\|$$

$$\geqq M^{-3}\cdot m^{-1/p}\left\| \sum_{i=1}^{n} \left[\left\| \sum_{j=1}^{k_i} x_j \right\| \cdot \left\| \sum_{j=1}^{k_i} x_{j+\sum_{m=1}^{i-1} k_m} \right\|^{-1} \cdot \left(\sum_{j=1}^{k_i} x_{j+\sum_{m=1}^{i-1} k_m} \right) \right] \right\|.$$

(We substitute in (b) for z_i the normalized block

$$\left\| \sum_{j=1}^{k_i} x_{j+\sum_{m=1}^{i-1} k} \right\|^{-1} \cdot \left(\sum_{j=1}^{k_i} x_{j+\sum_{m=1}^{i-1} k_m} \right)).$$

By (3.3), (3.12) yields

(3.13) $$\left\| \sum_{i=1}^{n} r_i^{1/p} x_i \right\| \geqq M^{-4}\cdot m^{-1/p}\cdot\left\| \sum_{i=1}^{k} x_i \right\|$$

where $k = \sum_{i=1}^{n} k_i$. By (3.11)

$$\left\| \sum_{i=1}^{n} r_i^{1/p} x_i \right\| \geqq M^{-6} \cdot \left(\sum_{i=1}^{n} k_i \right)^{1/p} \cdot m^{-1/p} \geqq M^{-6} \left(\sum_{i=1}^{n} r_i \right)^{1/p}.$$

Hence, using (a), it is easily proved that for any real $a_1, a_2, \cdots, a_n$

$$\left\| \sum_{i=1}^{n} a_i x_i \right\| \geqq M^{-6} \cdot \left(\sum_{i=1}^{n} |a_i|^p \right)^{1/p}. \tag{3.14}$$

Similar arguments yield the following

$$\left\| \sum_{i=1}^{n} r_i^{1/p} x_i \right\| = m^{-1/p} \left\| \sum_{i=1}^{n} k_i^{1/p} x_i \right\| \tag{3.15}$$

$$\leqq M^2 . m^{-1/p} . \left\| \sum_{i=1}^{n} \left\| \sum_{j=1}^{k_i} x_j \right\| \cdot x_i \right\| \leqq$$

$$\leqq M^3 \cdot m^{-1/p} \cdot \left\| \sum_{i=1}^{n} \left[\left\| \sum_{j=1}^{k_i} x_j \right\| \cdot \left\| \sum_{j=1}^{k_i} x_{j+\sum_{m=1}^{i-1} k_m} \right\|^{-1} \cdot \left(\sum_{j=1}^{k_i} x_{j+\sum_{m=1}^{i-1} k_m} \right) \right] \right\|$$

$$\leqq M^4 \cdot m^{-1/p} \cdot \left\| \sum_{i=1}^{k} x_i \right\| \leqq M^6 \cdot m^{-1/p} \cdot \left(\sum_{i=1}^{n} k_i \right)^{1/p} = M^6 \cdot \left(\sum_{i=1}^{n} r_i \right)^{1/p}.$$

(The notations in (3.15) are the same as in (3.12).) Again, by (a), for any real $a_1, a_1, \cdots, a_n$

$$\left\| \sum_{i=1}^{n} a_i x_i \right\| \leqq M^6 . \left(\sum_{i=1}^{n} |a_i|^p \right)^{1/p} \tag{3.16}$$

(3.14) and (3.16) show that $\{x_i\}_{i=1}^{\infty}$ is equivalent to the unit-vector basis in l_p. This completes the proof of Theorem 3.1.

REMARK. Using the deep result of A. Dvoretzky [4] A. Pełczyński and I. Singer proved in [6] the following

PROPOSITION 3.2. *Let E be an infinite-dimensional Banach space with an unconditional basis in which all normalized unconditional basic sequences are equivalent. Then E is isomorphic to l_2.*

Proposition 3.2 has the following alternative proof: By Theorem 3.1 E is isomorphic either to c_0 or to l_p for some $p \geqq 1$. For $2 \neq p > 1$ one can construct in l_p a subspace isomorphic to the space $(E_1 \oplus E_2 \oplus \cdots)_p$, where E_k denotes the k-dimensional euclidean space. This can be done without using [4] (see e.g. [5].) The space $Y = (E_1 \oplus E_2 \oplus \cdots)_p$ has an unconditional basis non-equivalent to the unit-vectors basis in l_p, $1 < p \neq 2$. In fact, if $\{x_i\}_{i=\frac{1}{2}(n-1)n+1}^{\frac{1}{2}n(n+1)}$ plays the rôle of the unit vectors basis in $E_n \subset Y$, $n = 1, 2, \cdots$, the sequence $\{x_i\}_{i=1}^{\infty}$ forms an unconditional normalized basis in Y. If it were equivalent to the unit vectors basis $\{e_i\}_{i=1}^{\infty}$ in l_p we would get that, for some fixed $M \geqq 1$, every natural n and any real $a_1, a_2, \cdots, a_n$

$M^{-1} \cdot \| \sum_{i=1}^n a_i e_i \| \leqq \| \sum_{i=1}^n a_i x_{i+\frac{1}{2}(n-1)n} \| \leqq M \cdot \| \sum_{i=1}^n a_i e_i \|$, which is known to be false, since $\| \sum_{i=1}^n a_i e_i \| = (\sum_{i=1}^n | a_i |^p)^{1/p}$ while $\| \sum_{i=1}^n a_i x_{i+\frac{1}{2}(n-1)n} \|$ is equal to $(\sum_{i=1}^n | a_i |^2)^{\frac{1}{2}}$. It follows that there exist normalized unconditional basic sequences in l_p which are not equivalent to the unit vectors basis. Similar basic sequences can be easily constructed in c_0 and in l_1. It follows that E is isomorphic to l_2.

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