SPECTRAL SYNTHESIS IN REGULAR BANACH ALGEBRAS

BY

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ABSTRACT

We prove that every locally compact non-discrete abelian group G contains a compact subset E such that $A(E)$ — the restriction algebra of $A(G)$ to E — admits spectral synthesis, although it contains a closed, regular, self-adjoint subalgebra which is isomorphic to an algebra of infinitely differentiable functions on $[-1, 1]$. We also give some general results concerning the failure of spectral synthesis in regular Banach algebras.

Introduction. In 1959, P. Malliavin [5] proved that for every locally compact non-discrete abelian group G, with dual Γ, the algebra $A(G)$ of Fourier transforms of elements in $L^1(\Gamma)$ does not admit spectral synthesis. For $G = R^n$, $n \geqq 3$, L. Schwartz [8] proved this result already in 1948. In both cases the proof seems to depend in some way on the fact that $A(G)$ contains a closed subalgebra isomorphic to an algebra of differentiable functions on some interval on the real line. In Malliavin's proof, it is the algebra of functions operating on some fixed element of $A(G)$. In Schwartz's proof, it is the algebra of radial functions in $A(R^n)$, which for $n \geqq 3$, can be identified with an algebra of differentiable functions on $(0, \infty)$. Moreover, if B is a commutative semi-simple regular Banach algebra with unit in which the failure of spectral synthesis follows from the existence of an element f in B, which is not contained in the closed ideal generated by f^2, then B contains a closed subalgebra isometric-isomorphic to an algebra of functions with a bounded point derivation. We discuss this remark in Section 4 (Theorem 4.5). These observations lead to the formulation of the following general problem:

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Suppose B is a commutative, semi-simple, regular, self-adjoint Banach algebra with unit, which contains a closed subalgebra B_0 of the same type, which is isomorphic to an algebra of differentiable functions on an interval. Does it follow that spectral synthesis fails in B? We show that the answer in general is negative. In fact we prove (Theorem 3.1), that for every locally compact non discrete abelian group G there exists a compact subset $E \subset G$ such that $A(E)$ (the restriction algebra of $A(G)$ to E) admits spectral synthesis, although it contains a regular, self-adjoint, closed sub-algebra which is isomorphic to an algebra of infinitely differentiable functions on $[-1, 1]$ (and which, therefore, does not admit synthesis).

In Section 1 we recall some basic notations and definitions concerning spectral synthesis in regular Banach algebras, and discuss the concept of individual symbolic calculus. We also give a generalization of the Ditkin-Shilov theorem which we need in the sequel.

In Section 2 we consider tensor products of the form $B \hat{\otimes} C(Y)$, where B is a semi-simple, regular, self-adjoint, commutative Banach algebra with unit, and $C(Y)$ the algebra of continuous complex functions on some compact Hausdorff space Y. We show that if B satisfies a strong form of Ditkin condition (Condition (D_1) introduced in Section 1), and has a scattered maximal ideal space, then $B \hat{\otimes} C(Y)$ admits spectral synthesis. We also discuss in this section, the individual symbolic calculus of certain functions in this tensor algebra.

In Section 3 we use the results of Section 2, to prove our main result mentioned, concerning restriction algebras of group algebras.

The failure of spectral synthesis in a regular Banach algebra, which contains a closed, regular sub-algebra which does not admit synthesis, is closely related to the problem of extending certain linear functionals, from the subalgebra to the whole algebra, with some kind of preservation of support. We investigate this problem in Section 4, and give conditions for such an extension to be possible, which apply to Schwartz's example and Malliavin's theorem. We conclude with a result concerning the connection between the failure of spectral synthesis in an algebra, and the existence of a closed sub-algebra with bounded point derivations.

1. Some general results. In what follows, B denotes a semi-simple commutative, regular, self-adjoint Banach algebra with unit, represented as an algebra of functions on its maximal ideal space X. For a closed set $E \subset X$ we denote by

$$I(E) = \{f \in B;\ f^{-1}(0) \supset E\},$$

$$I_0(E) = \{f \in B;\ f^{-1}(0) \text{ is a neighborhood of } E\}.$$

For any closed ideal $I \subset B$ we denote by

$$Z(I) = \bigcap_{f \in I} f^{-1}(0).$$

It is well known that for every closed set $E \subset X$, and every closed ideal $I \subset B$ with $Z(I) = E$, the relation

$$\overline{I_0(E)} \subset I \subset I(E)$$

holds.

DEFINITIONS. A closed set $E \subset X$ is called a *set of spectral synthesis* (*an S-set* if $\overline{I_0(E)} = I(E)$.

A closed set $E \subset X$ is called a *Ditkin set* (*a D-set*) if for every $f \in I(E)$ there exists a sequence $g_n \in I_0(E) n = 1, 2, 3 \cdots$, such that $\lim_{n \to \infty} \| g_n f - f \|_B = 0$. Clearly, every D-set is an S-set. We shall say that the algebra B *admits spectral synthesis*, if every closed set $E \subset X$ is an S-set.

The following is well-known:

LEMMA 1.1. *Let $E \subset X$ be an S-set. If there exists a constant $c > 0$, such that for every neighborhood U of E, there exists $g \in I_0(E)$ such that $g = 1$ on $X \setminus U$, and $\| g \|_B \leqq c$, then E is a Ditkin set.*

In Section 2, we shall need the following generalization of the Ditkin-Shilov theorem:

THEOREM 1.2. *Let H be a Hausdorff space, and $\phi: X \to H$ a continuous function such that for all $y \in H$, every closed subset of $\phi^{-1}(y)$ is a Ditkin set. If E is a closed subset of X such that $\phi(\text{bdry}\, E)$ is scattered (i.e. does not contain any non-empty perfect subset), then E is an S-set.*

REMARK. If $X = H$ and ϕ is the identity map, we get the Ditkin-Shilov theorem.

PROOF. Let $I \subset B$ be a closed ideal with $Z(I) = E$, and $f \in B$ such that $E \subset f^{-1}(0)$. We shall show that $f \in I$. Let P be the set of points $y \in H$ such that f does not belong locally to I at all points of $\phi^{-1}(y)$. Since f belongs locally to I at all points of $X \setminus \text{bdry}\, E$, we have $P \subset \phi(\text{bdry}\, E)$; hence the theorem will be proved by showing that P is perfect. It is easy to see that P is closed, and we turn to show that P has no isolated points. In fact, suppose that for some $y_0 \in H$, there exists a neighborhood U such that $(U \setminus \{y_0\}) \cap P = \emptyset$. Put $\phi^{-1}(U) = V$, $\phi^{-1}(y_0) = K$, and $K \cap \mathrm{bdry}\, E = Q$. Let W be a neighborhood of Q such that $\bar{W} \subset V$, and choose $g \in B$ such that $g = 1$ on a neighborhood of Q, and $g = 0$ on $X \setminus \bar{W}$. Since Q is a D-set, there exists a sequence $g_n \in I_0(Q)$, such that $\lim_{n\to\infty} \| g_n f - f \|_B = 0$, and therefore

$$(1.1) \qquad \lim_{n\to\infty} \| g_n g f - g f \|_B = 0$$

By verifying local belonging at all points of X, we get $g_n g f \in I$, and therefore by (1.1), $gf \in I$. Hence f belongs locally to I at all points of $\phi^{-1}(y_0)$, and the proof is complete.

DEFINITIONS. We say that B *satisfies condition* (D_1) *at a point* $x_0 \in X$ if B has no non-trivial primary closed ideals at x_0 and there exists a constant $c > 0$, such that for every neighborhood U of x_0, there exists $g \in B$ such that:

a) The support of g is contained in U.

b) $g = 1$ on a neighborhood of x_0.

c) $\| g \|_B \leqq c$.

It is obvious that if B satisfies condition (D_1) at x_0 then $\{x_0\}$ is a D-set.

DEFINITION. We say that the algebra B *satisfies condition* (D_1) if B satisfies condition (D_1) at every point $x_0 \in X$. Let B^* be the dual space of B. For $\nu \in B^*$ we denote by $\Sigma(\nu)$ its support [1].

Using the Hahn-Banach theorem, we get that E is an S-set, if and only if: $\nu \in B^*$, $\Sigma(\nu) \subset E \Rightarrow \nu \in I(E)^{\perp}$. We remark that if B has no non-trivial primary closed ideals at $x_0 \in X$ then every $\nu \in B^*$ with $\Sigma(\nu) \subset \{x_0\}$ is of the form $\nu = c\delta_{x_0}$, where c is a constant and δ_{x_0} denotes the unit mass concentrated at x_0.

We finish this section with some remarks concerning individual symbolic calculus in B. Let $f \in B$; we denote by $[f]$ the set of all functions F, defined on $f(X)$, such that $F \circ f \in B$. If $F \in [f]$, we say that F operates on f; $[f]$ is called the individual symbolic calculus associated with f. With norm

$$\| F \| = \| F \circ f \|_B$$

$[f]$ forms a semi-simple, commutative Banach algebra with $f(X)$ as its maximal ideal space.

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(1) We refer the reader to [4] for the definition of support of a linear functional, and related matters concerning spectral synthesis.

We denote by

$$[[f]] = \{F \circ f;\ F \in [f]\}.$$

$[[f]]$ is a closed subalgebra of B, isometric isomorphic to $[f]$, by the correspondence $F \to F \circ f$; $F \in [f]$. It is easy to verify that $[[f]]$ consists of all $g \in B$ which respect the level lines of f, that is:

$$x_1, x_2 \in X;\ f(x_1) = f(x_2) \Rightarrow g(x_1) = g(x_2).$$

For a discussion of the relation between individual symbolic calculus and spectral synthesis we refer to [4, p. 243]. For the investigation of individual symbolic calculus in group algebras see [3] and [6].

2. The tensor product $B \hat{\otimes} C(Y)$. Throughout this section, B is a semi-simple, commutative, regular self-adjoint Banach algebra with unit, represented as an algebra of functions on its maximal ideal space X. Y will denote a compact Hausdorff space and $C(Y)$ the algebra of continuous complex functions on Y with the sup-norm.

We denote by $B \hat{\otimes} C(Y)$ the projective tensor product of B and $C(Y)$; that is, the space of all continuous functions ψ on $X \times Y$ which admit a representation of the form

$$\psi(x, y) = \sum_{j=1}^{\infty} f_j(x) g_j(y) \tag{2.1}$$

with $f_j \in B$, $g_j \in C(Y)$, and $\sum_{j=1}^{\infty} \| f_j \|_B \| g_j \|_\infty < \infty$.

We shall write $\psi = \sum_{j=1}^{\infty} f_j \otimes g_j$ for (2.1). We introduce the norm

$$\| \psi \|_{B \hat{\otimes} C(Y)} = \inf \sum_{j=1}^{\infty} \| f_j \|_B \| g_j \|_\infty$$

where the infimum is taken with respect to all possible representations of ψ in the form (2.1).

It is easy to check that with this norm $B \hat{\otimes} C(Y)$ is a regular semi-simple commutative Banach algebra with maximal ideal space $X \times Y$.

For the general theory of tensor products of Banach algebras we refer the reader to [2]. For the application of tensor products of Banach algebras to harmonic analysis see [9].

DEFINITION. Let $\nu \in B^*$, $\mu \in C^*(Y)$; we define their tensor product $\nu \otimes \mu$ as the linear functional on $B \hat{\otimes} C(Y)$, whose value at $\psi = \sum_{j=1}^{\infty} f_j \otimes g_j$ is given by

$$(2.2) \qquad \langle \psi, \nu \otimes \mu \rangle = \sum_{j=1}^{\infty} \langle f_j, \nu \rangle \langle g_j, \mu \rangle .$$

It is easy to verify that this definition does not depend on the representation of ψ.

LEMMA 2.1. *Suppose B has no non-trivial closed primary ideals at* $x_0 \in X$, *and let E be a closed subset of Y. Then every* $\omega \in (B \hat{\otimes} C(Y))^*$ *such that* $\Sigma(\omega) \subset \{x_0\} \times E$, *is of the form* $\omega = \delta_{x_0} \otimes \mu$, *where* μ *is a regular bounded Borel measure supported by E.*

PROOF. For fixed $g \in C(Y)$ let ω_g be the member of B^* defined by

$$(2.3) \qquad \langle f, \omega_g \rangle = \langle f \otimes g \ \omega \rangle, \quad f \in B.$$

Since $\Sigma(\omega) \subset \{x_0\} \times E$ it follows from (2.3) that $\Sigma(\omega_g) \subset \{x_0\}$, and since B has no non-trivial closed primary ideals at x_0, it follows from the remark in Section 1 that

$$(2.4) \qquad \omega_g = c(g)\delta_{x_0}$$

where $c(g)$ is a constant depending on g. From (2.3) we infer that $c(g) = \langle 1 \otimes g, \omega \rangle$. This shows that the map $g \to c(g)$ defines a bounded linear functional on $C(Y)$; that is, there exists a measure $\mu \in C^*(Y)$ such that

$$(2.5) \qquad c(g) = \langle g, \mu \rangle$$

for all g in $C(Y)$, and it follows by (2.3), (2.4), and (2.5) that

$$(2.6) \qquad \langle f \otimes g, \delta_{x_0} \otimes \mu \rangle = \langle f \otimes g, \omega \rangle .$$

Since the linear span of the functions $f \otimes g$, $f \in B$, $g \in C(Y)$ is dense in $B \hat{\otimes} C(Y)$, (2.6) completes the proof.

COROLLARY 2.2. *With the assumptions of Lemma* 2.1, $\{x_0\} \times E$ *is an S-set for* $B \hat{\otimes} C(Y)$.

PROOF. Lemma 2.1 shows that every functional in $(B \hat{\otimes} C(Y))^*$ which is supported by $\{x_0\} \times E$ is a measure, and therefore admits spectral synthesis.

LEMMA 2.3. *If B satisfies condition* (D_1) *at* $x_0 \in X$ *and E is a closed subset of Y, then* $\{x_0\} \times E$ *is a D-set for* $B \hat{\otimes} C(Y)$.

PROOF. By Corollary 2.2, $\{x_0\} \times E$ is an S-set; using Uryshon's lemma for $C(Y)$ and the assumption that B satisfies condition (D_1) at x_0, we infer that all of the conditions of Lemma 1.1 are satisfied.

THEOREM 2.4. *If B satisfies condition (D_1), and X is scattered, then $B \hat{\otimes} C(Y)$ admits spectral synthesis.*

PROOF. The theorem follows from Lemma 2.3 and Theorem 1.2, with ϕ as the projection map of $X \times Y$ onto X.

COROLLARY 2.5. *If X is countable, then $B \hat{\otimes} C(Y)$ admits spectral synthesis.*

We turn now to some results on individual symbolic calculus in $B \hat{\otimes} C(Y)$ We begin with an observation, which uses an idea introduced in [3]. For $h \in B \hat{\otimes} C(Y)$, and $y \in Y$ we denote by h_y the function defined on X by

$$h_y(x) = h(x, y), \quad x \in X.$$

It is easy to check that the map

$$y \to h_y, \quad y \in Y$$

is a continuous map of Y into B, and

$$\| h_y \|_B \leqq \| h \|_{B \hat{\otimes} C(Y)}.$$

Let $f \in B$, $g \in C(Y)$, and $\psi = f \otimes 1 + 1 \otimes g$. The preceding remarks show that for any $F \in [\psi]$ the map

$$y \to F(f + g(y)), \quad y \in Y \tag{2.7}$$

is a continuous map of Y into B, and

$$\| F(f + g(y) \|_B \leqq \| F \circ \psi \|_{B \hat{\otimes} C(Y)} \tag{2.8}$$

for all $y \in Y$.

We denote by K the range of g; we claim that

$$t \to F(f + t); \quad t \in K \tag{2.9}$$

is a continuous map from K into B. This follows from the continuity of the map (2.7), and the continuity of g.

LEMMA 2.6. *Let $g \in C(Y)$ with range $[0, 2\pi]$, f a real function in B and $\psi = f \otimes 1 + 1 \otimes g$. If F is a function defined on the real line with period 2π which operates on ψ, then*

$$|\hat{F}(n)| \leqq c \| e^{inf} \|_B^{-1}, \qquad n = 0, \pm 1, \pm 2, \cdots,$$

where $\hat{F}(n)$ denotes the n-th Fourier coefficient of F, and c a constant.

PROOF. By (2.9) the map

$$t \to F(f + t), \quad t \in [0, 2\pi]$$

is continuous from $[0,2\pi]$ into B and (2.8) implies that

$$\sup_{0\leqq t\leqq 2\pi}\|F(f+t)\|_B \leqq \|F\circ\psi\|_{B\hat{\otimes} C(Y)} \tag{2.10}$$

Using integration of continuous B valued functions, we have

$$\hat{F}(n)e^{inf} = \frac{1}{2\pi}\int_0^{2\pi} F(f+t)e^{inf}\,dt.$$

Hence (2.10) implies that

$$\|\hat{F}(n)e^{inf}\|_B \leqq \frac{1}{2\pi}\int_0^{2\pi}\|F(f+t)\|_B dt \leqq \|F\circ\psi\|_{B\hat{\otimes} C(Y)}.$$

This proves the lemma with $c = \|F\circ\psi\|_{B\hat{\otimes} C(Y)}$.

REMARK. It follows from (2.9) that F is a continuous function on R.

LEMMA 2.7. *Let F be a function defined on* $[-1,1]$ *and* $F(\cos x)=\sum_{n=0}^{\infty} a_n\cos nx$, *for all real* x. *If*

$$\sum_{n=1}^{\infty} n^k|a_n|<\infty \qquad k=0,1,2,3,\cdots;$$

then $F\in C^\infty[-1,1]$ *(where* $C^\infty[-1,1]$ *denotes the space of infinitely differentiable functions on* $[-1,1]$.*)*

PROOF. Since $F'(\cos x)\sin x = \sum_{n=1}^{\infty} n\,a_n\sin nx$, we have

$$F'(\cos x) = \sum_{n=1}^{\infty} na_n\frac{\sin nx}{\sin x} \text{ for } x\neq k\pi \quad k=0,\pm1,\pm2,\cdots.$$

Keeping in mind that

$$\frac{\sin nx}{\sin x} = 2\sum_{j=1}^{m}\cos 2jx+1 \text{ for } n=2m+1;\ \frac{\sin nx}{\sin x} = 2\sum_{j=1}^{m}\cos(2j-1)x \text{ for } n=2m;$$

we obtain

$$F'(\cos x) = \sum_{n=0}^{\infty} b_n\cos nx \text{ with } \sum_{n=1}^{\infty} n^k|b_n|<\infty \text{ for all } k>0.$$

Thus $F\in C^1[-1,1]$, and by induction we get $F\in C^m[-1,1]$ for all $m>0$.

COROLLARY 2.8. *Let f be a real function in B such that for some* $\alpha>0$

$$\|e^{inf}\|_B\geqq e^{\alpha\sqrt{n}}, \qquad n=1,2,3,\cdots, \tag{2.11}$$

and let g be as in Lemma 2.6. If $\psi=f\otimes 1+1\otimes g$ *and* $h=\cos\psi$, *then* $[h]\subset C^\infty[-1,1]$.

PROOF. Let $F \in [h]$; define $F_1(x) = F(\cos x)$. Then $F_1 \in [\psi]$, and it follows from Lemma 2.6 and (2.11) that $F(\cos x) = \sum_{n=0}^{\infty} a_n \cos nx$ with $|a_n| \leqq e^{-\alpha\sqrt{n}}$, $n = 1,2,3,\cdots$, and by Lemma 2.7, $F \in C^{\infty}[-1,1]$.

In Section 3 we shall also need:

LEMMA 2.9. *Let f, g, ψ be as in Lemma 2.6 and $h = \cos\psi$. If*

$$\sum_{n=-\infty}^{\infty} \frac{\log \| e^{inf} \|_B}{1+n^2} < \infty \tag{2.12}$$

then $[h]$ is a regular self-adjoint Banach algebra with maximal ideal space $[-1,1]$.

PROOF. Since $B \hat{\otimes} C(Y)$ is self-adjoint it follows that $[h]$ is self-adjoint. Since the range of h is $[-1,1]$, the assertion about the maximal ideal space is clear.

We prove now the regularity of $[h]$. Let

$$\omega_n = \| e^{in\psi} \|_{B \hat{\otimes} C(Y)}, \qquad n = 0, \pm 1, \pm 2, \cdots.$$

Since $\| e^{in\psi} \|_{B \hat{\otimes} C(Y)} \leqq \| e^{inf} \|_B$ it follows from (2.12) that

$$\sum_{n=-\infty}^{\infty} \frac{\log \omega_n}{1+n^2} < \infty. \tag{2.13}$$

Consider the space $A(\omega_n)$ of continuous functions on R which admit a representation of the form $u(t) = \sum_{n=-\infty}^{\infty} a_n e^{int}$, $t \in R$, with $\| u \| = \sum_{n=-\infty}^{\infty} |a_n| \omega_n < \infty$. $A(\omega_n)$ is a commutative Banach algebra which is contained in $[\psi]$.

It is proved in [1] that (2.13) implies the regularity of $A(\omega_n)$. Let now E be a closed subset of $[-1,1]$ and $x_0 \in [-1,1] \setminus E$. Let $t_0 = \operatorname{arcos} x_0$, and $H = \{\operatorname{arcos} x; x \in E\}$. H is a closed subset of $[0,\pi]$. Define $-H = \{t; -t \in H\}$.

Since the algebra $A(\omega_n)$ is regular there exists $u \in A(\omega_n)$ such that $u(t) = 0$ for $t \in H \cup -H$ and $u(t_0) = u(-t_0) = 1$. Let $F_1(t) = [u(t) + u(-t)]/2$ for $t \in [-\pi,\pi]$ and $F(x) = F_1(\operatorname{arcos} x)$ for $x \in [-1,1]$. Then $F \in [h]$, $F = 0$ on E and $F(x_0) = 1$. This shows that $[h]$ is regular.

3. Group algebras. In what follows, G denotes a locally compact, non-discrete abelian group. For a closed set $E \subset G$, we denote by $A(E)$ the restriction algebra of $A(G)$ to E, which is canonically identified with $A(G)/I(E)$.

The main result of this section is the following:

THEOREM 3.1. *There exists a compact set $E \subset G$ such that $A(E)$ admits spectral synthesis, although it contains a regular, self-adjoint, closed sub-*

algebra, which is isomprhic to an algebra of infinitely differentiable functions on $[-1,1]$.

We begin with some remarks and notations. We denote by $\boldsymbol{D}$ the denumerable complete direct sum of groups of order two. We identify the elements of $\boldsymbol{D}$, as sequences (ε_n), $\varepsilon_n = 0,1$; the group operation being coordinate addition mod 2. Denote by x_m the element in $\boldsymbol{D}$, all of whose coordinates except the mth are zero. Denote by G_m the sub-group of $\boldsymbol{D}$ generated by x_m, that is $G_m = \{0, x_m\}$. Finally we denote by ξ_m the character on $\boldsymbol{D}$, defined by $\xi_m(x) = (-1)^{\varepsilon_m}$, $x = (\varepsilon_n)$.

Let $\Psi = \sum_{n=1}^{\infty} a_n \xi_n$, $\sum_{n=1}^{\infty} |a_n| < \infty$; then $\Psi \in A(\boldsymbol{D})$ and it is easy to check that for every real number u, and k positve integers $m_1, m_2, \cdots, m_k$, we have:

$$(3.1) \quad \| e^{iu\Psi} \|_{A(G_{m_1} \oplus G_{m_2}, \oplus \dots \oplus G_{m_k})} = \prod_{j=1}^{k} (|\cos a_{m_j} u| + |\sin a_{m_j} u|).$$

Let X be a compact Hausdorff space; we use the notation $V(X) = C(X) \hat{\otimes} C(X)$. For a closed set $E \subset X$, the restriction algebra of $V(X)$ to $E \times E$ is $V(E) = C(E) \hat{\otimes} C(E)$. Let H be a compact group; for any $\Psi \in A(H)$, the function Ψ^* defined on $H \times H$ by $\Psi^*(x,y) = \Psi(x+y)$, $x, y \in H$, belongs to $V(H)$, and by [9] we have:

$$(3.2) \qquad \| \Psi^* \|_{V(H)} = \| \Psi \|_{A(H)}.$$

LEMMA 3.2. *There exists a denumerable closed set* $S \subset \boldsymbol{D}$, *and a real function* $f \in V(S)$ *such that for some positive constant* α

$$(3.3) \qquad \| e^{inf} \|_{V(S)} \geqq e^{\alpha\sqrt{n}}, \qquad n = 1, 2, \cdots,$$

and

$$(3.4) \qquad \sum_{n=-\infty}^{\infty} \frac{\log \| e^{inf} \|_{V(S)}}{1+n^2} < \infty.$$

PROOF. We define a sequence of sets $H_k = \sum_{2^k \leqq n < 2^{k+1}} \oplus G_n$, $k = 1, 2, \cdots$, and put $S = \bigcup_{k=1}^{\infty} H_k$. S is a closed denumerable subset of $\boldsymbol{D}$. Define a sequence of positive numbers

$$(3.5) \qquad a_m = \frac{1}{2.4^k} \text{ for } 2^k \leqq m < 2^{k+1}.$$

Put $\Psi = \sum_{n=1}^{\infty} a_m \xi_m$, and let f be the function defined by $f(x,y) = \Psi(x+y)$, $x, y \in \boldsymbol{S}$. We claim that S and f satisfy (3.3) and (3.4). In fact, let n be a positive integer; consider the positive integer k such that $4^{k-1} \leqq n < 4^k$. Using (3.1), (3.2) and (3.5) we get

$$(3.6) \qquad \| e^{inf} \|_{V(S)} \geqq \| e^{inf} \|_{V(H_k)} = \| e^{in\Psi} \|_{A(H_k)} =$$
$$= \prod_{2^k \leqq m < 2^{k+1}} (|\cos na_m| + |\sin na_m|) \geqq \prod_{2^k \leqq m < 2^{k+1}} \left(1 + \frac{4}{\pi} na_m\right)^{\frac{1}{2}} > \left(\frac{9}{8}\right)^{\frac{\sqrt{n}}{2}};$$

this proves (3.3). To prove (3.4), note first that for any character γ on $\boldsymbol{D}$, and a real number u we have:

$$\| e^{iu\gamma} \|_{A(\boldsymbol{D})} = |\cos u| + |\sin u| \leqq (1 + 2|u|)^{\frac{1}{2}};$$

hence, for any integer n, we obtain:

$$(3.7) \qquad \log \| e^{in\Psi} \|_{A(\boldsymbol{D})} \leqq \frac{1}{2} \sum_{m=1}^{\infty} \log(1 + 2|n|a_m).$$

By an elementary computation we get

$$(3.8) \qquad \sum_{n=-\infty}^{\infty} \frac{\log(1 + 2a_m|n|)}{1 + n^2} \leqq ca_m \log \frac{1}{a_m} \qquad m = 1, 2, \cdots,$$

for some positive constant c. It follows from (3.5) that $\sum_{m=1}^{\infty} a_m \log(1/a_m) < \infty$; hence, (3.8) implies that

$$(3.9) \qquad \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \frac{\log(1 + 2a_m|n|)}{1 + n^2} = \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{\log(1 + 2a_m|n|)}{1 + n^2} < \infty.$$

Remembering that $\| e^{inf} \|_{V(S)} \leqq \| e^{in\Psi} \|_{A(\boldsymbol{D})}$, (3.4) follows from (3.7) and (3.9).

PROOF OF THEOREM 3.1. Take S and f which satisfy (3.3) and (3.4). Let g be a continuous function from $\boldsymbol{D}$ onto $[0, 2\pi]$. Consider the algebra $V(S) \hat{\otimes} C(\boldsymbol{D})$, and define $\phi = f \otimes 1 + 1 \otimes g$, and $h = \cos \phi$. It follows from Corollary 2.8 and Lemma 2.9 that $[h]$ is a regular, self-adjoint Banach algebra, contained in $C^\infty[-1, 1]$; hence $[[h]]$ is a regular, self-adjoint closed sub-algebra of $V(S) \hat{\otimes} C(\boldsymbol{D})$, which is isomorphic to an algebra of infinitely differentiable functions on $[-1, 1]$. Since $V(S)$ satisfies condition (D_1) it follows from Corollary 2.5 that $V(S) \hat{\otimes} C(\boldsymbol{D})$ admits spectral synthesis. Noticing that $V(S) \hat{\otimes} C(\boldsymbol{D})$ is the restriction algebra of $V(\boldsymbol{D}) \hat{\otimes} C(\boldsymbol{D})$ to $S \times S \times \boldsymbol{D}$ the theorem follows from the fact ([9], Theorem 4.2.2), that every locally compact, non-discrete, abelian group G contains a compact subset K, such that $A(K)$ is isomorphic to $V(\boldsymbol{D}) \hat{\otimes} C(\boldsymbol{D})$.

4. Extension of linear functionals. We give now some sufficient conditions for the failure of spectral synthesis in some Banach algebras. In what follows we assume that the algebra B satisfies the conditions of Section 2.

Suppose now that B contains a closed, regular sub-algebra B_0 with unit, which does not admit spectral synthesis. Denote by X_0 the maximal ideal space of B_0. There exists a continuous map $\sigma: X \to X_0$ which associates with each complex homomorphism of B its restriction to B_0; this map is onto, since B_0 being regular, any of its maximal ideals can be extended to a maximal ideal of B. Since spectral synthesis fails in B_0 there exists a functional $v_0 \in B_0{}^*$, and a function $f \in B_0$ such that $\Sigma(v_0) \subset (x \in X_0; f(x) = 0\}$ and

$$\langle f, v_0 \rangle \neq 0. \tag{4.1}$$

We can extend v_0 by the Hahn-Banach Theorem to a bounded linear functional v on B; if an extension is possible such that

$$\Sigma(v) \subset \{x \in X;\ f(x) = 0\} \tag{4.2}$$

then it follows from (4.1) that spectral synthesis fails in B. The possibility of such an extension is equivalent to the following problem: Is it possible to extend v to a bounded linear functional v on B such that

$$\Sigma(v) \subset \sigma^{-1}(\ \Sigma(v_0))\ ? \tag{4.3}$$

The results of Section 3 show that such an extension is not always possible. In the next theorem we give a condition for such an extension to be possible.

THEOREM 4.1. *A necessary and sufficient condition that $v_0 \in B_0^*$ admits an extension $v \in B^*$ such that* (4.13) *holds is that for some positive constant c*

$$|v_0(f)| \leqq c \, \|f + h\|_B \tag{4.4}$$

holds, for all $f \in B_0$ and $h \in I_0(E)$, with $E = \sigma^{-1}(\ \Sigma(v_0))$.

PROOF. It is obvious that condition (4.4) is necessary and we therefore prove only the sufficiency. Let $v_0 \in B_0^*$ such that (4.4) holds. We first extend v_0 to a functional v_1 on $B_0 + I_0(E)$, which is defined by:

$$v_1(f+h) = v_0(f), \quad f \in B_0, \ h \in I_0(E). \tag{4.5}$$

One checks easily that v_1 is well defined and (4.4) implies that it is bounded by c. Let $v \in B^*$ an extension of v_1 to B, which exists by the Hahn-Banach Theorem. It follows from (4.5) that $\Sigma(v) \subset E$ and the theorem is proved.

We consider now, in a general setting, a situation which occurs in the disproofs of spectral synthesis by Schwartz and Malliavin.

Let B, B_0, X, X_0 and σ, be as in the preceding discussion. For $f \in B_0$, we denote by $\hat{f}$ its Gelfand transform. Denote by $\hat{B}_0$, the algebra of Gelfand transforms, of elements of B_0.

Let B_1 be a semi-simple, commutative, regular, Banach algebra with unit, represented as an algrbra of functions on its maximal ideal space E, which is a closed subset of X_0.

In what follows, we suppose that there exists a bounded linear transformation $T: B \to B_1$ such that:

$$T(fg) = \hat{f}_{|E} Tg, \; f \in B_0, \; g \in B. \tag{4.6}$$

$$T1 = 1 \tag{4.7}$$

It follows that the restriction algebra of $\hat{B}_0$ to E, is contained in B_1, and therefore to every functional $\omega \in B_1^*$ is associiated a functional $v_0 \in B_0^*$ defined by

$$\langle f, v_0 \rangle = \langle \hat{f}_{|E}, \omega \rangle, \quad f \in B_0.$$

Keeping the same notations we have:

THEOREM 4.2. *The functional $v = T^*\omega$ is an extension of v_0 to B, which satisfies*

$$\Sigma(v) \subset \sigma^{-1}(\Sigma(\omega)) \tag{4.8}$$

PROOF. It follows from (4.7) that v is an extension of v_0. To prove (4.8), consider a function $g \in B$ which vanishes on a neighborhood U of σ^{-1} $(\Sigma(\omega))$. Let V be a neighborhood of $\Sigma(\omega)$ in X_0, such that $\sigma^{-1}(V) \subset H$. Since B_0 is regular, there exists $h \in B_0$ such that $\hat{h} = 0$ on a neighborhood of $\Sigma(\omega)$, and $\hat{h} = 1$ on $X_0 \setminus V$; hence $h = 1$ on $X \setminus U$, and therefore $g = hg$.

Hence, using (4.6), and keeping in mind that the support of $\hat{h}$ is disjoint from $\Sigma(\omega)$, we obtain

$$\langle g, v \rangle = \langle \hat{h}_{|E} Tg, \omega \rangle = 0,$$

and the proof is complete.

COROLLARY 4.3. *If there exist $\omega \in B_1^*$ and $f \in B_0$, such that*

$$\Sigma(\omega) \subset \hat{f}^{-1}(0); \; \langle \hat{f}_{|E}, \omega \rangle \neq 0$$

then spectral synthesis fails in B.

COROLLARY 4.4. *If* $\hat{B}_0 = B_1$, *then every* $v_0 \in B_0$ *has an extension* $v \in B^*$, *such that* (4.3) *holds.*

SCHWARTZ'S EXAMPLE. Consider the algebra $A(\boldsymbol{R}^n)$ $n \geqq 3$, and its closed subalgebra $\mathscr{R}_n$, of radial functions. Let

$$K = \left\{ X = (x_1, x_2, \cdots, x_n) \in \boldsymbol{R}^n ;\ 1 \leqq \left(\sum_{j=1}^{n} x_j^2 \right)^{\frac{1}{2}} \leqq 2 \right\}$$

and denote by $A(K)$ and $\mathscr{R}(K)$ the restriction algebras of $A(\boldsymbol{R}^n)$ and $\mathscr{R}_n$ to K respectively. Let M be the radial mean operator on $A(\boldsymbol{R}^n)$. The conditions of Theorem 4.2 are satisfied for $B = A(K)$, with $B_0 = \mathscr{R}(K)$, $B_1 = C^1[1,2]$, and the operator T defined by $Tf_{|K} = Mf_{|[1,2]}$, $f \in A(\boldsymbol{R}^r)$.

MALLIAVIN'S THEOREM. The principle of Malliavin's disproof of spectral synthesis in group algebras, can be stated in the general setting of regular Banach algebras, as follows [4, p. 231]:

If there exists a real function $f \in B$, and a non-trivial bounded Borel measure μ on X such that

$$\int_{-\infty}^{\infty} |u| \; \| e^{iuf}\mu \|_{B^*} d\mu < \infty \tag{4.9}$$

then spectral synthesis fails in B. By replacing if necessary μ by $\Psi\mu$ for some $\Psi \in B$, and f by $f + \alpha$, for some real constant α, (4.9) is not altered, and we may therefore assume (see [4], p. 232) that

$$\int_{-\infty}^{\infty} \langle 1, e^{iuf}\mu \rangle du \neq 0. \tag{4.10}$$

Consider the continuous function on R

$$v(t) = \int^{\infty} \langle 1, e^{iuf}\mu \rangle e^{-iut} du, \quad t \in R;$$

By (4.10), there exists a positive number r, such that $v(t) \neq 0$ for $t \in [-r, r]$. Suppose now, in addition, that the algebra $[f]$ is regular (this is the case in Malliavin's construction). The conditions of Theorem 4.2 are then satisfied for B with $B_0 = [[f]]$, $B_1 = C^1[-r, r]$, and the transformation T, defined by

$$Tg(t) = \frac{1}{v(t)} \int_{-\infty}^{\infty} \langle g, e^{iuf}\mu \rangle e^{-iut} du, \quad t \in [-r, r]$$

for $g \in B$.

Another example, in which the same principle holds, is in the disproof of spectral synthesis for the algebra $V(G) = C(G) \hat{\otimes} C(G)$, where G is a compact infinite group [9].

Let V_0 be the algebra of functions $f \in V(G)$ such that $f(x,y) = f(x+y,0)$ for all $x, y \in G$. The conditions of Theorem 4.2 (in fact of Corollary 4.4) are satisfied for $V(G)$ with $B_0 = V_0$, $B_1 = A(G)$ and the transformation T (introduced in [9]) defined by

$$Tf(x) = \int_G f(x-y,y)dy, \quad x \in G, \; f \in V(G).$$

We finish this section with a result in the opposite direction. Recall that a bounded point derivation on B at a point $x_0 \in X$ is a linear functional $D \in B^*$ such that:

$$D(fg) = f(x_0)Df + g(x_0)Dg,$$

for all f and g in B.

THEOREM 4.5. *Suppose that there exists a function $f \in B$ such that the closed ideal generated by f^2 in B does not contain f. Then B has a closed subalgebra B_0 which is isomorphic isometric to an algebra of functions B_1 which has a non-trivial bounded point derivation.*

PROOF. Let B_0 be the closed algebra generated by f in B; denote by K the range of f; K is a compact set in the plane. To each polynomial $P(z) = \sum_{k=0}^{n} a_k z^k$ on K we associate the function $P(f) = \sum_{k=0}^{n} a_k f^k$ in B_0; we define $\| P \| = \| P(f) \|_B$; and denote by B_1 the completion of the algebra of polynomials on K with respect to this norm. B_1 is an algebra, isomorphic isometric to B_0. Let (f^2) be the closed ideal generated by f in B. Since $f \notin (f^2)$ there exists a functional $v \in B^*$ which annihilates (f^2) and $\langle f, v \rangle = 1$. By replacing v by $v - \langle 1, v \rangle \delta_{x_0}$, for some $x_0 \in f^{-1}(0)$ we may assume that $\langle 1, v \rangle = 0$. Let D be the functional in $B_1{}^*$ which corresponds to v by the canonical map of B^* into B_1^*. D annihilates the ideal generated by z^2 in B_1, $\langle z, D \rangle = 1$, and $\langle 1, D \rangle = 0$; hence for any polynomial $P \in B_1$ we have $\langle P, D \rangle = P'(0)$; since the polynomials are dense in B_1, D is a non-trivial bounded point derivation on B_1, at $z = 0$.

REMARK. Theorem 4.5 shows that whenever spectral synthesis fails in B for principal ideals (that is, there exists an $f \in B$ such that $(f) \neq (f^2)$) there exists a functional $v \in B^*$ which does not admit spectral synthesis, which is an extension, of a bounded point derivation on a closed subalgebra of B. One can show that if for all real $f \in B$ we have $\| e^{inf} \| = O(|n|^k)$, $(|n| \to \infty)$, where k is a positive integer depending on f, then a necessary condition, for spectral synthesis to fail in B is, that it fails for principal ideals.

In all known cases in which spectral synthesis fails for a regular Banach algebra B, it fails already for principal ideals. The following problem, therefore, arises:

Suppose that for all $f \in B$, we have $f \in (f^2)$. Does it follow that B admits spectral synthesis?

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