IRREFLEXIVE BANACH SPACES ARE IMPERFECT*

BY
DINA FRIEDLAND

ABSTRACT

It is shown that irreflexive Banach spaces are imperfect: thus the "perfect" spaces are exactly the reflexive Banach spaces and "mixed" spaces do not exist.

Recall the following definitions of S. Goldberg and E. O. Thorp [1].

Definition 1. A linear operator T from X to Y, where X and Y are normed linear spaces, is called "perfectly compact" if it is compact as an operator from X to TX.

Definition 2. A B-space X is called "perfect" if all compact operators from X to every B-space Y are perfectly compact.

Definition 3. A B-space X is called "imperfect" if for any infinite dimensional B-space Y, there always is an imperfectly compact operator, $T: X \to Y$.

B-spaces which are neither perfect nor imperfect are called "mixed". Thorp [2] observed that every reflexive space is perfect, while Arteburn [3] showed that every irreflexive B-space with separable conjugate is imperfect. Arteburn's proof is modified in this paper to extend this result to all irreflexive B-spaces.

A normed linear space X will be called "semiseparable" if its conjugate X^* contains a sequence $\{f_n\}$ total over X. If X is separable, then both X and X^* are semiseparable.

Theorem 1. *Every irreflexive, semiseparable B-space is imperfect.*

Proof. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in X^*, total over X and such that $\|f_n\| = 1$. If the closed subspace Z spanned by the $\{f_n\}$ is X^*, we refer to [3]; otherwise we can choose

* This paper is part of the author's M.Sc. dissertation, performed under guidance of Dr. D. Amir at the Technion — Israel Institute of Technology. The author wishes to thank Dr. Amir for his help.

Received November 11, 1969

$$f_0 \in X^* - Z, \qquad \| f_0 \| = 1.$$

Hence, there is a $\phi \in X^{**}$ such that

$$\phi(f_0) = \| \phi \| = 1$$

$$\phi(Z) = 0.$$

Clearly $\phi \in U^{**} - JU$, where U and U^{**} are the unit balls of X and X^{**}. Since JU is dense in U^{**} [5, p. 424], there is a net $\{x_\alpha\}$ in U such that:

$$Jx_\alpha \overset{w^*}{\to} \phi$$

In particular:

$$f_n(x_\alpha) = Jx_\alpha(f_n) \to \phi(f_n) \quad \text{for } n = 0, 1, \cdots.$$

Hence, there is an α_0 such that $\alpha \geqq \alpha_0$ implies $|\phi(f_0) - f_0(x_\alpha)| < 1$.

Inductively we find α_k $(k = 1, 2, \cdots)$ such that $\alpha_k \geqq \alpha_{k-1}$, and $\alpha \geqq \alpha_k$ implies $|\phi(f_n) - f_n(x_\alpha)| < \frac{1}{2^k}$ for $n = 0, 1, \cdots, k$.

Denote $x_k = x_{\alpha_k}$, then

$$f_n(x_k) = Jx_k(f_n) \to \phi(f_n) \qquad \text{for } n = 0, 1, \cdots.$$

Now let Y be an infinite-dimensional B-space. According to [4], take in Y a basic sequence $\{y_i\}_{=0}^{\infty}$ with $\| y_i \| = 1$. Let $\{\varepsilon_i\}_{=0}^{\infty}$ be any sequence of positive numbers with $\Sigma_{i=0}^{\infty} \varepsilon_i < \infty$.

Define an operator

$$T : X \to Y$$

by

$$T = \sum_{i=0}^{\infty} \varepsilon_i f_i(\cdot) Y_i.$$

$\Sigma_{i=0}^{\infty} \varepsilon_i \| y_i \| = \Sigma_{i=0}^{\infty} \varepsilon_i < \infty$, and $\| f_i \| = 1$; therefore T is obviously a compact operator. A simple estimate shows that:

$$Tx_k \to \sum_{i=0}^{\infty} \varepsilon_i \phi(f_i) y_i.$$

Recalling that we chose ϕ such that $\phi(f_0) = 1$

$$\phi(f_i) = 0 \qquad i = 1, 2, \cdots$$

we get

$$Tx_k \to \varepsilon_0 y_0$$

Suppose T is perfectly compact, then there is an x such that

$$Tx_k \to Tx = \sum_{i=0}^{\infty} \varepsilon_i f_i(x) y.$$

and it folows that $\sum_{i=0}^{\infty} \varepsilon_i f_i(x) y_i = \varepsilon_0 y_0$.

$\{y_i\}_{i=1}^{\infty}$ is a basic sequence; therefore

$$f_0(x) = 1$$

$$f_n(x) = 0 \quad n = 1, 2, \cdots$$

which contradicts that $\{f_n\}_{n=1}^{\infty}$ is total over X.

We generalize Theorem 1 using the following characterization of reflexivity.

Theorem 2. *A B-space X is reflexive if and only if every semiseparable quotient space of X is reflexive.*

PROOF. It is known that the condition is necessary ([6, p. 56]). To see that it is sufficient, suppose X is not reflexive. Then X^* is not reflexive and it contains a separable irreflexive subspace Y ([6, p. 56]). Let $Y_\perp$ be the annihilator of O in X, and define: $Z = \frac{X}{Y_\perp}$.

According to [5, p. 72] Z^* is isometrically isomorphic to $Y_\perp{}^\perp$(the annihilator of $Y_\perp$ in X^*). $Y_\perp{}^\perp$contains the irreflexive subspace Y, hence Z^* is irreflexive. Clearly Y is total over Z.

Thus Z is an irreflexive, semiseparable quotient space of X.

Theorem 3. *Every irreflexive B-space is imperfect.*

PROOF. As observed in [3], if a quotient space of X is imperfect, so is X. Thus the proof follows directly from Theorems (1) and (2).

REFERENCES

1. S. Goldberg and E. O. Thorp, *The Range as range space for compact operators*, J. Reine Angew. Math. **211** (1952), 113–115.

2. E. O. Thorp, *The Range as range space for compact operators. II*, J. Reine Angew. Math. **217** (1964), 69–78.

3. D. R. Arteburn, *Weak perfect compactness and generalized adjoints*, J. Reine Angew. Math. **218** (1965), 121–128.

4. M. M. Day, *On the basis problem in normed spaces*, Proc. Amer. Math. Soc. **13** (1962), 655–658.

5. Dunford and Schwartz, *Linear Operators I*, Interscience, New York, 1958.

6. M. Day, *Normed Linear Spaces*, Springer, Berlin, 1958.

Department of Mathematics,
Technion—Israel Institute of Technology, Haifa