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When Can Hidden Variables be Excluded in Quantum Mechanics?

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Summary. — The problem of hidden variables is examined in the axiomatic formulation of quantum mechanics based on the algebra of observables. After a brief introductory survey of the earlier investigations, we first investigate the structure of C^*-algebras which allow dispersion-free positive linear functionals. The result obtained is a direct generalization of the well-known result of von Neumann concerning the hidden variables. In the next Section, we assume, as before, that the observables form the Hermitian elements of a C^*-algebra. But we now relax the requirement on «states» and allow the so-called monotone-positive functionals (which are not necessarily linear) to represent states. It is then shown that even when such generalized states are allowed, a system admits hidden variables only if its algebra of observables is Abelian; *i.e.*, only if all observables are mutually compatible. In another Section, we investigate the question of hidden variables under the assumption that the observables, instead of forming a C^*-algebra, have a certain more general algebraic structure.

1. – Introduction.

One of the problems that have remained with us from the early days of quantum mechanics is the problem concerning the possibility of a completely deterministic description of microsystems in terms of the so-called *hidden variables*. A mathematical analysis of this problem was first given by von Neumann (1) who concluded that the existence of hidden variables would be in contradiction with the empirically verified predictions of quantum mechanics.

(1) J. von Neumann: *Mathematische Grundlagen der Quantenmechanik* (Berlin, 1932).

Von Neumann's analysis, however, has been subsequently criticized on several grounds [2].

It may be recalled here that von Neumann's conclusion is based on essentially two assumptions, one of which is about the structure of physical observables and amounts to supposing that the set of all observables has the structure of the set of all (self-adjoint) operators of a Hilbert space. The other is concerned with the mathematical formulation of the physical notion of « states » of a system. It was assumed by VON NEUMANN that all states (even the hypothetical *dispersion-free states* which appear in the formulation of the hidden-variable problem) are represented by positive linear functionals on the observables.

Both of these assumptions (especially the assumption that states are *linear* functionals) have been criticized as being unduly restrictive. It has been remarked that the so-called uncertainty relation follows immediately from these assumptions so that in postulating them, VON NEUMANN has in fact tacitly assumed the universal validity of the uncertainty principle. It is then neither surprising nor significant (so far as the hidden-variable problem is concerned) that von Neumann's assumptions exclude the existence of dispersion-free « states ». It is this line of argument which has led to the charge of « circular reasoning » against von Neumann's analysis [3].

This criticism is all the more devastating because von Neumann's assumptions are bereft of direct physical justifications. The only justification of these assumptions is the *a posteriori* one that they lead to the usual formalism of quantum mechanics. Such a justification, which is sufficient from an empirical point of view, has little compelling force in the context of the hidden-variable problem. For one is now concerned with the possibility of generalizing the usual formalism of quantum mechanics and the mere fact that a set of postulates leads to the usual formalism cannot be a sufficient recommendation for these postulates.

Once one has become critical of von Neumann's analysis, there remain two alternative approaches to the hidden-variable problem: one is to attempt at reformulating quantum mechanics so that hidden variables are allowed and a completely deterministic description of microsystems is possible in the new formulation. Such an approach has in fact been adopted and success claimed [4].

A detailed discussion and critical evaluation of the works cited in ref. [4] is beyond the scope of the present paper. It may however be mentioned that,

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[2] Instead of reviewing all criticisms that have been advanced against von Neumann's analysis, we give below only what seems to be the most pertinent criticism.

[3] *a*) L. DE BROGLIE: *La théorie de la mesure en mécanique quantique* (Paris, 1957); *b*) D. BOHM: *Causality and Chance in Modern Physics* (London, 1958).

[4] D. BOHM: *Phys. Rev.*, **85**, 166, 180 (1952); D. BOHM and J. BUB: *Rev. Mod. Phys.*, **38**, 453 (1966); see also the review article: H. FRIESTADT: *Suppl. Nuovo Cimento*, **1**, 1 (1957).

from a logical point of view, these works are somewhat obscure. For instance, the precise sense in which the all-important notion of state of a system is (presumably) extended for having hidden variables does not emerge from these works with clarity. Also, the dynamical and «kinematical» aspects of the theory seem to be mixed in a logically obscure way. Furthermore, the proposed reformulation of quantum mechanics admittedly meets conceptual difficulties when one treats multiparticle systems. Therefore, it seems to us that the works cited in ref. (4) cannot be said to have provided a definitive answer to the question of hidden variables.

The alternative left to us is to proceed axiomatically in the spirit of von Neumann. Only, one must now start with less stringent postulates than those assumed by VON NEUMANN. The aim of such an axiomatic approach is to isolate the weakest possible assumptions which must be violated for having hidden variables. Once such assumptions have been isolated, one can then decide if and how they can be altered so as to allow hidden variables. This paper is concerned with such an axiomatic approach to the problem of hidden variables and we now briefly outline the results of earlier investigations in this approach.

In a recent paper (5) GLEASON starts with essentially the same assumption regarding the observables as VON NEUMANN. More specifically, GLEASON confines his attention to a special class of observables, *viz.* those representing the so-called « yes-no experiments ». He assumes that the set of all such observables has the structure of the lattice of all projection operators of a (separable) Hilbert space H. As for the states, however, he does not identify them with positive linear functionals but represents them by « generalized probability measures » on the lattice of projections in H. The interpretation of projection operators as « yes-no experiments » and the interpretation of states as probability distributions of obtaining the answer « yes » in such experiments, almost compel us to adopt Gleason's postulates concerning the states (6). From these postulates, GLEASON deduces that states can indeed be represented by positive linear functionals on the set of operators in H and thus there are no dispersion-free states and, *a fortiori*, no hidden variables. Gleason's result may therefore be said to dispose of the criticism against the assumption that states are linear functionals. One may, however, still object that Gleason's postulates regarding the observables are too restrictive (7). One may even suspect that Gleason's

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(5) A. M. GLEASON: *Journ. Math. and Mech.*, **6**, 885 (1957).

(6) See, however, the remarks in the concluding Section of this paper.

(7) First, there is the objection that Gleason's assumption does not allow for the existence of *super-selection rules*. Existence of the so-called commutative super-selection rules can, however, be taken into account by slightly relaxing Gleason's assumption. Specifically, the lattice of « yes-no experiment » may be identified with the lattice of projections of a *discrete von Neumann algebra*. In this case, Gleason's results will still be true. It is, however, not known if Gleason's theorem holds when more general super-

success in deducing the linearity of states is due to his restrictive assumption about the observables.

Except in some special cases, such as the generators of symmetry groups like space translation, rotation, etc., there is at present little theoretical ground for deciding if a given operator of the Hilbert space represents an observable or not. The assumption of von Neumann, or its modified version which allows the existence of « commutative super-selection principles » is accepted mainly for its mathematical simplicity and for the fact that it does not seem to contradict empirical facts. But these reasons, as pointed out before, are not sufficient in the context of the hidden-variable problem.

It is thus desirable to examine the question of hidden variables in a more general mathematical setting for the observables. With this motivation, JAUCH and PIRON have recently analysed the hidden-variable problem in the lattice-theoretical formulation of quantum mechanics (8).

In this formulation, there is no need of introducing a Hilbert space. The basic object of this approach is the set of all *yes-no experiments* (called « propositions ») pertaining to the physical system and the basic assumption is that this set has the structure of a (complete) orthocomplemented, weakly modular lattice (8). This axiom is rendered physically plausible by providing appropriate physical interpretations of the lattice-theoretical operations. The assumption regarding the system of propositions is already true for *classical systems* where the propositions are in one-to-one correspondence with classes of equivalent subsets in the phase space and the lattice of propositions is therefore *Boolean*. The *Boolean* property can, in fact, be taken as the characteristic property of classical systems. For quantal systems, the Boolean property does not hold.

The essential result in ref. (8) is that a physical system admits of hidden variables only if every pair of « propositions » (*i.e.* observables corresponding to « yes-no experiments ») satisfy a special symmetrical relation which is the appropriate mathematical expression of the physical relation called « compatibility » between the « yes-no experiments ». This result leads to an easy empirical refutation of hidden variables by exhibiting physical systems with pairs of incompatible propositions. Such a system is for instance the spin of an electron and the two incompatible propositions are the polarization of the spin in two different (but not opposite) directions.

In order to obtain such a result, JAUCH and PIRON had to assume certain properties for states. Like GLEASON, they identified the states with generalized

selection rules are allowed so that the algebra of observables is not necessarily a discrete von Neumann algebra.

(8) J. M. JAUCH and C. PIRON: *Helv. Phys. Acta*, **36**, 827 (1963); a detailed account of the lattice-theoretical formulation of quantum mechanics is given in C. PIRON: *Helv. Phys. Acta*, **37**, 439 (1964).

probability measures on the lattice of all propositions. Now, a certain property for such generalized probability measures, which could be *deduced* in the restricted mathematical setting of Gleason, no longer follows from other postulates and has to be admitted as an independent axiom. This is the axiom denoted as (4)° on p. 833 of ref. (8): if $p(\cdot)$ is a state and a and b are two propositions such that $p(a) = p(b) = 1$, then $p(a \cap b) = 1$. Physically it means that if for a certain state $p(\cdot)$ the "proposition" a is true with probability 1 and the "proposition" b also is true with probability 1, then the "proposition" « a and b » is also true with probability 1. This interpretation, combined with an examination of the operational procedure of measuring propositions, makes this axiom very plausible indeed.

However, this plausibility extends only to the physically realizable states. In the problem of hidden variables, one is considering states which are not necessarily physically realizable and therefore one may defend the point of view that one may admit properties for such states which, for physically realizable states, would be inadmissible. Indeed it is easy to construct examples (9) of systems with hidden variables if one admits states which violate condition (4)°. If one adopts this point of view, then the problem of hidden variables does not exist. One simply admits states of such generality so that hidden variables are always possible and this can always be done. But, if one admits arbitrarily general states, the resulting formalism is then devoid of any useful structure so that no useful conclusion can be derived from such a formalism.

The quest for « hidden variables » becomes a meaningful scientific pursuit only if states, even physically not realizable states, are somehow restricted by physical considerations. We feel that these conditions should describe essentially the « minimal » properties which are observed on physically realizable states, since without such a restriction the problem of hidden variables dissolves into a fog of mysticism. If under such restrictions one could somehow introduce « hidden variables » consistently into the description of microsystems, then one could say that a major progress in the formal development of quantum mechanics would be accomplished. If, on the other hand, it can be shown that such a road is not possible, then such attempts at generalization of quantum mechanics cannot lead to any useful extension of the formalism. For this reason, we have found it useful to pursue the question further.

Besides the lattice-theoretical approach, there exists yet another axiomatic approach to quantum mechanics: the so-called algebraic approach. In this approach, the basic objects are not the « propositions » but rather the set of all (bounded) observables. One now postulates certain properties for this set (of observables) which furnish it with the structure of a (not necessarily asso-

(9) J. S. BELL: *Rev. Mod. Phys.*, **38**, 447 (1966).

ciative) normed algebra ([10,11]). A familiar and important example of the algebraic structure thus defined is provided by the set of Hermitian elements of a C^*-algebra ([12]). There are, of course, mathematical objects other than the (Hermitian elements of) C^*-algebras which realize the algebraic structure properties postulated for the observables ([13]). But it is doubtful whether these additional mathematical possibilities have any physical relevance. We shall therefore suppose, in the greater part of this paper, that observables of a physical system are represented by the Hermitian elements of a C^*-algebra and only in Sect. **5** shall we briefly consider the more general algebraic setting for quantum mechanics.

It is not our purpose to discuss here the relative merits of the two axiomatic formulations of quantum mechanics: the C^*-algebraic and the lattice-theoretic; nor to justify on physical bases the C^*-algebraic formulation. We should, however, note that the C^*-algebraic approach cannot be subsumed under the lattice-theoretic approach nor vice versa. Consequently, the general results of JAUCH and PIRON ([8]) cannot be directly transferred to the C^*-algebraic setting of quantum mechanics. Because of this, the hidden-variable problem merits a fresh examination in the algebraic approach and the present paper has this for its task. Such a re-examination of the hidden-variable problem may have also some « topical » interest, for in recent years the C^*-algebraic formulation of quantum mechanics has proved to be useful in the problems of the so-called axiomatic field theory.

If we examine the problem of hidden variables in the algebraic approach, we are again confronted with the question concerning the properties of states, in particular, unobservable states. We shall define states, as usual, as certain positive functionals defined on the elements of the algebra R of observables. The property of positivity for a functional $\varphi(\cdot)$ means that for all positive observables A (*i.e.* for every observable A which is the square of a certain other observable B: $(A=B^2)$) $\varphi(A)$ assumes only nonnegative values. This property is so intimately connected with the interpretation of states in terms of expectation values of observables that it seems quite natural to assume it even for physically not realizable states which may appear in the formulation of the hidden variables.

All previous investigations of the problem have assumed additional properties of states which usually amount to the further assumption that states

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([10]) J. VON NEUMANN: *On an Algebraic Generalization of Quantum Mechanical Formalism*, Collected Works, vol. **3** (London, 1961).

([11]) I. E. SEGAL: *Ann. of Math.* (2), **48**, 930 (1947); *Mathematical Problems of Relativistic Physics* (New York, 1963).

([12]) For information about C^*-algebras, see J. DIXMIER: *Les C^*-algèbres et leurs représentations* (Paris, 1964).

([13]) S. SHERMANN: *Ann. of Math.*, **64**, 593 (1956).

are *linear* functionals. This assumption is however open to attack for two reasons: firstly, linearity together with positivity imply the so-called uncertainty relation so that one is again open to the charge of « circular reasoning » mentioned earlier in this Introduction. Secondly, it is hard to justify this property especially for the incompatible observables. It is therefore of interest to see to what extent *linearity* can be replaced by weaker conditions. *This is one of the main objects of this paper.*

2. – Basic notions.

We shall be dealing with a physical system of which we assume that its bounded observables are the *Hermitian elements* of an abstract C^*-algebra [12] R. We can also assume without loss of generality that R contains a unit element I. The justification for such an assumption may be found, for instance, in ref. [11]. A Hermitian element A of R is said to be *positive* (in symbols $A \geqslant 0$) if it is the square of another Hermitian element: $A \geqslant 0$ if $A = B^2$ with B Hermitian. A positive element T of R is said to be greater than a positive element S (in symbols $T \geqslant S \geqslant 0$) if $T - S$ is positive. We shall denote by R^+ the set of all positive elements in R. The bound (also called the *norm*) of any element T of R is denoted by $\|T\|$. All the topological notions which are subsequently used are always meant with respect to the topology defined by this norm in R.

A *functional* $\varphi(T)$ on R is a function from R onto the complex-number field C. We shall be dealing with bounded functionals, *i.e.* functionals $\varphi(T)$ for which there exists a finite positive number K such that

$$|\varphi(T)| \leqslant K\|T\| \qquad \text{for all } T.$$

Such functionals can be normalized to have $\varphi(I) = 1$ and this we shall henceforth do.

States of the physical system shall be represented by functionals with appropriate properties. If $\varphi(\cdot)$ represents a state and T a Hermitian element of R, then $\varphi(T)$ is interpreted as the expectation value of the observable T in the state $\varphi(\cdot)$. It is thus natural to suppose that functionals representing states assume only real values for Hermitian elements of R. Such functionals are said to be *real*. The functionals $\varphi(\cdot)$ representing states should also be *positive* in the sense that if $A \geqslant 0$, then $\varphi(A) \geqslant 0$. These two requirements on states are however as yet much too general to allow any useful conclusion to be derived. One thus usually requires further that states $\varphi(\cdot)$ are *linear*:

$$\varphi(\lambda T + \mu S) = \lambda\varphi(T) + \mu\varphi(S)\,,$$

for all T, S, in R and complex λ, μ. The last requirement is however much too stringent especially in the context of the hidden-variable problem and it is also difficult to justify it on purely physical grounds. We shall therefore not always require this property but find a substitute for it in Sect. **4**.

Of special interest in connection with the hidden-variable problem are the *dispersion-free functionals* $\varrho(\cdot)$ defined by the property

$$\varrho(A^2) = [\varrho(A)]^2 \qquad \text{for all Hermitian } A \text{ of } R.$$

Roughly speaking, the meaning of a dispersion-free functional is a «state» for which every observable has an *exact value*. Positive functionals are a convex set since for any pair of real numbers $\lambda_1 \geqslant 0$, $\lambda_2 \geqslant 0$ such that $\lambda_1 + \lambda_2 = 1$ and any pair of functionals φ_1 and φ_2, we verify that

$$\varphi(T) \equiv \lambda_1 \varphi_1(T) + \lambda_2 \varphi_2(T) \tag{1}$$

is again a positive functional. More generally, if x is a variable in a measure space M, $\mathrm{d}\mu(x)$ a positive normalized measure and $\varphi_x(T)$ a positive functional for every $x \in M$, then

$$\varphi(T) = \int_M \varphi_x(T)\, \mathrm{d}\mu(x) \tag{2}$$

is again a positive functional. We call such a state a *mixture*. A functional $\varphi(T)$ and the state which it represents is said to be *pure* if it cannot be represented by such a formula as a mixture of other functionals representing states.

A system is said to admit ***hidden variables*** if every functional representing a physical state has the form (2) with ***dispersion-free states*** $\varphi_x(T)$.

In the next Section, we shall need the mathematical concept of an ideal $\mathscr{I}$ in an algebra R and also that of the quotient algebra $R/\mathscr{I}$. A subset $\mathscr{I}$ of the algebra R is called an ideal of R if: *a*) $\mathscr{I}$ is a linear subset of R and *b*) for every $T \in \mathscr{I}$ and any $S \in R$, both TS and ST belong to $\mathscr{I}$. Every algebra has at least two ideals, the first consists of only the zero element and the other is the entire algebra. These two ideals will be called the trivial ideals. If an algebra has no other (two-sided) ideal, it is called *simple*.

Every two-sided ideal $\mathscr{I}$ of R defines classes of equivalent elements modulo $\mathscr{I}$. Two elements T_1 and T_2 in R are said to be equivalent (modulo $\mathscr{I}$) if $T_1 - T_2 \in \mathscr{I}$. If R is a C^*-algebra and $\mathscr{I}$ a *closed* two-sided ideal of R, it can be shown that the set $R/\mathscr{I}$ whose elements are the classes of equivalent elements (modulo $\mathscr{I}$) is a C^*-algebra too.

3. – Algebras of observables which admit dispersion-free positive linear functionals.

With the preliminary matters out of the way, we can now proceed to prove an immediate generalization of the old result of von Neumann concerning hidden variables.

THEOREM 1. – A C^*-algebra R (with unit element I and containing at least one element other than multiples of unity) admits a dispersion-free positive linear functional if and only if it has a (nontrivial) closed two-sided ideal $\mathscr{I}$ such that the quotient algebra $R/\mathscr{I}$ is Abelian. For the proof of this theorem, we need the following lemmata.

Lemma 1. Let φ be a dispersion-free positive linear functional on R. Then the set $\mathscr{I}$ of all elements T in R for which $\varphi(T)=0$ is a (nontrivial) closed two-sided ideal of R.

Proof of Lemma 1. We first observe that $\mathscr{I}$ must contain elements other than the zero element. In fact, let T be any element of R which is not a multiple of the identity. Then $T-\varphi(T)I \neq 0$, but $\varphi(T-\varphi(T)I)=0$ and $T-\varphi(T)I$ belongs to $\mathscr{I}$. On the other hand, $\mathscr{I}$ cannot be the entire algebra R for the unity I cannot belong to $\mathscr{I}$.

Since φ is a linear functional, it easily follows that $\mathscr{I}$ is a linear subset. We now show that if $T\in\mathscr{I}$ and S belongs to R, then ST and TS belong to $\mathscr{I}$. For this, we first observe that if $T\in\mathscr{I}$ then $T^*\in\mathscr{I}$. In fact, for any positive linear functional φ, one has $\varphi(T^*)=[\varphi(T)]^*$ so that $\varphi(T)=0$ implies $\varphi(T^*)=0$. It follows from this remark that T belong to $\mathscr{I}$ if and only if the Hermitian elements

$$T_1=\frac{T+T^*}{2} \qquad \text{and} \qquad T_2=\frac{T-T^*}{2i}$$

also belong to $\mathscr{I}$. Since any element of R can be written as a linear combination of such Hermitian elements, it now suffices to show that for any *Hermitian* X in $\mathscr{I}$ and any S in R, both XS and SX belong to $\mathscr{I}$. But if X is Hermitian and belongs to $\mathscr{I}$, it follows from Cauchy's inequality that

$$|\varphi(SX)|^2\leqslant\varphi(SS^*)\varphi(X^2)=\varphi(SS^*)[\varphi(X)]^2=0$$

for every $S\in R$. Thus $\varphi(SX)=0$ or $SX\in\mathscr{I}$ for all $S\in R$. Similarly, one can show that XS also belongs to $\mathscr{I}$. This completes the proof that $\mathscr{I}$ is a nontrivial (two-sided) ideal.

We now verify that $\mathscr{I}$ is closed. Let T_n $(n=1, 2, \ldots)$ be a sequence of elements in $\mathscr{I}$ such that $\|T-T_n\|\to 0$ as $n\to\infty$. We have to show that $T\in\mathscr{I}$, *i.e.* $\varphi(T)=0$. Since φ is a normalized positive linear functional on a C^*-algebra, we have

$$|\varphi(T-T_n)|\leqslant\|T-T_n\|\to 0 \qquad \text{as } n\to\infty.$$

Thus

$$|\varphi(T-T_n)|=|\varphi(T)|=0\,,$$

With this, the Lemma 1 is proved in full.

Lemma 2. If φ is a dispersion-free positive linear functional on R, then

$$\varphi(ST) = \varphi(S)\varphi(T) \qquad \text{for all } S,\ T \in R.$$

Proof of Lemma 2. Since $T - \varphi(T)I \in \mathscr{I}$ for all T, we have $ST - \varphi(T)S \in \mathscr{I}$, hence $\varphi(ST) - \varphi(S)\varphi(T) = 0$.

Proof of Theorem 1. Suppose φ is a dispersion-free positive linear functional on R. Then $\mathscr{I}_\varphi \equiv \{T|T \in R,\ \varphi(T) = 0\}$ is a nontrivial closed two-sided ideal of R (Lemma 1). Let T be any element and $\{T\}$ the class of all elements T_1 such that $T - T_1 \in \mathscr{I}_\varphi$. This class is one of the elements of the residue class algebra $R/\mathscr{I}_\varphi$. If $S \in \{T\}$ then $\varphi(S) = \varphi(T)$. Thus $\{T\} \to \varphi(T)$ defines a mapping from $R/\mathscr{I}_\varphi$ onto the field of complex numbers. It follows from the definition of $\mathscr{I}_\varphi$ that it is one-to-one, and the linearity of φ implies that it is linear. Furthermore $\varphi(TS) = \varphi(T)\varphi(S)$ (Lemma 2), thus $\{T\}\cdot\{S\} = \{TS\}$ is mapped into $\varphi(TS) = \varphi(T)\varphi(S)$. We have thus verified that the mapping $\{T\} \to \varphi(T)$ is an algebraic isomorphism of $R/\mathscr{I}_\varphi$ onto the complex numbers C. Thus $R/\mathscr{I}_\varphi$ is Abelian. This proves the « only if » part of Theorem 1. The « if » part follows from the well-known fact that every Abelian C^*-algebra admits a dispersion-free positive linear functional. The theorem is thus proved in full.

A rather typical example which illustrates this theorem is constructed as follows. For the algebra R we take the subset of 3×3 matrices which are of the form

$$T \equiv \begin{pmatrix} \alpha & \beta & 0 \\ \gamma & \delta & 0 \\ 0 & 0 & z \end{pmatrix}.$$

The dispersion-free linear functional is then given by $\varphi(T) = z$. The two-sided ideal $\mathscr{I}_\varphi$ consists of all the matrices of the given form with $z = 0$. The equivalence classes $R/\mathscr{I}_\varphi$ consist of all matrices T for which z has the same value. The lemmata and the theorem are easily verified on this example. An immediate consequence of Theorem 1 is

Corollary 1. A simple C^*-algebra with unit element has no nontrivial dispersion-free positive linear functional.

We also have

Corollary 2. A *factor* in a separable Hilbert space has no dispersion-free positive linear functionals.

The proof of Corollary 2 follows from the following observations: the factors of type $\mathrm{I}_n(n < \infty)$, II_1 and III_∞ are known to be simple, so that Corollary 1 is applicable in these cases. As for the remaining two types I_∞ and II_∞ their closed two-sided ideals are known and one can verify that the residue class algebras defined by their ideals are not Abelian.

For factors of type I, we recover thus easily the theorem of von Neumann concerning hidden variables in quantum mechanics.

4. – Monotone-positive functionals.

In this Section, we shall investigate the question of hidden variables without assuming that states are *linear* functionals. Before doing this, it will be well to comment briefly on the meaning of hidden variables.

The important question, with which the hidden-variable problem is concerned, is whether there are dispersion-free functionals *with appropriate properties* so that they may be usefully considered as « states », albeit unobservable « states », of the system and whether the *physical states* of the system can be obtained from such dispersion-free « states » by suitable averaging procedures. If the answer to this question is yes, one may then say that the physical system in question admits hidden variables. Somewhat more precisely, we have

Definition 1. A physical system specified by the C^*-algebra R of observables and the set Σ of physically realizable states is said to admit hidden variables if every state $\varphi \in \Sigma$ is of the form

$$\varphi(T) = \int_M \varrho_x(T)\, d\mu_\varphi(x)\,, \tag{3}$$

for all Hermitian T of R.

Here, x denotes the variable element of a set M, μ_φ a positive measure defined on (a suitable class of subsets of) M and $\varrho_x(\cdot)$ (for each x in M) is a dispersion-free functional (on R) with *appropriate properties*.

Definition 1 is still imprecise; for we have not yet specified the properties that are required of the dispersion-free functionals ϱ_x occurring in relation (3). Admittedly, these functionals need not represent physically realizable states and we are therefore not allowed to assume for them all the properties common to functionals in Σ. Yet, we cannot allow these functionals to be completely arbitrary; for nothing useful will be gained by admitting such arbitrary objects in the theory. The problem of hidden variables may therefore be said to revolve around the central question: what properties should one require of the dispersion-free states?

Evidently, it is quite reasonable to suppose that the functionals are *positive*. But the condition of positivity alone still leaves us with too wide a class of objects and there is no hope that a useful theory can be constructed with them. Therefore, the condition of positivity must be supplemented by further restrictive conditions.

In the last Section, we took the supplementary condition to be linearity and obtained, as was to be expected, a generalization of the earlier result of von Neumann. But the condition of linearity, as mentioned earlier, is too stringent a condition in the context of the hidden-variable problem. Therefore, we search now for a weaker condition which could replace linearity. One such condition, which suggests itself naturally, is a strengthened form of positivity, the so-called *monotone-positivity*.

Definition 2. A functional $\varphi(T)$ on a C^*-algebra R is called *monotone-positive* if the conditions $T, S \in R^+$ and $T \geqslant S$ imply that $\varphi(T) \geqslant \varphi(S) \geqslant 0$.

It is obvious that every positive linear functional is monotone-positive whereas the converse is not true. It also seems to be a minimal condition which one should impose on functionals so that they can be usefully considered as « states » of the system. The physical meaning of monotone-positivity is the following: we recall that the value of a functional φ for a Hermitian element T of R is the expectation value, or mean value of the observable T in the state φ. If $T \geqslant S$ then there exists another *positive* observable A such that $T - S = A$. If T and S were compatible observables (that is if they commuted), then one would have $\varphi(T) \geqslant \varphi(S)$ for every state as required by the condition of monotone-positivity since then T and S could be measured in principle by the same apparatus and every value of T is necessarily greater than a simultaneously measured value of S. We extend this requirement to observables which are not necessarily compatible.

We shall then suppose in this Section that the dispersion-free « states » $\varrho_x(T)$, in terms of which the notion of hidden variables is formulated, are *monotone-positive* functionals on R. With this stipulation, Definition 1 acquires a precise meaning and we may now ask the basic question: when does a physical system admit hidden variables in the sense of Definition 1? To answer this question, we need an assumption about the set Σ of all *physically realizable states*.

Assumption. The set Σ of physically realizable states is a *full* set of functionals in the sense that if T and S are any two observables and

$$\varphi(T) \geqslant \varphi(S) \geqslant 0 \qquad \text{for all } \varphi \in \Sigma,$$

then $T \geqslant S \geqslant 0$.

Roughly speaking, this assumption guarantees that the set Σ of *physical states* is not too small. It is thus a reasonable property to require of the set Σ.

We now prove the

THEOREM 2. – A physical system characterized by the C^*-algebra R of observables and a *full* set Σ of physically realizable states admits hidden variables only if the algebra R is Abelian.

Proof of Theorem 2. Since the system admits hidden variables, every physical state $\varphi(T)$ is of the form

$$\varphi(T) = \int_M \varrho_x(T)\, \mathrm{d}\mu_\varphi(x) ,$$

where $\varrho_x(T)$, $\mathrm{d}\mu_\varphi(x)$ and M have the meaning mentioned in Definition 1. Now let T and S be two elements of R such that $T \geqslant S \geqslant 0$. For every $\varphi \in \Sigma$, we have

$$\varphi(T^2) = \int_M \varrho_x(T^2)\, \mathrm{d}\mu_\varphi(x) = \int_M [\varrho_x(T)]^2\, \mathrm{d}\mu_\varphi(x)$$

and

$$\varphi(S^2) = \int_M \varrho_x(S^2)\, \mathrm{d}\mu_\varphi(x) = \int_M [\varrho_x(S)]^2\, \mathrm{d}\mu_\varphi(x) . \tag{4}$$

Since $\varrho_x(T)$ are assumed to be monotone-positive, we have $\varrho_x(T) \geqslant \varrho_x(S) \geqslant 0$ and therefore

$$[\varrho_x(T)]^2 \geqslant [\varrho_x(S)]^2 \qquad \text{for all } x \in M. \tag{5}$$

It results from (4) and (5) that

$$\varphi(T^2) \geqslant \varphi(S^2) \geqslant 0 \qquad \text{for all } \varphi \in \Sigma.$$

Since Σ is a *full* set, we obtain $T^2 \geqslant S^2 \geqslant 0$. We have thus proved that if the physical system specified by the C^*-algebra R of observables admits hidden variables, then the conditions $T, S \in R^+$ and $T \geqslant S$ imply that $T^2 \geqslant S^2$. Theorem 2 can now be easily proved by applying the following mathematical

THEOREM 3. – Let R be a C^*-algebra. If, for *every* pair T, S of elements in R with the property $T \geqslant S \geqslant 0$, we also have $T^2 \geqslant S^2$, then the algebra R is Abelian ([14]).

Theorem 2, then, excludes hidden variables whenever the physical system has noncompatible observables. One could, of course, plead for dropping the

([14]) T. OGASAWARA: *Journ. Sc. Hiroshima Univ.*, **18**, 179 (1954).

requirement of *monotone-positivity* on dispersion-free states. There is, however, no weaker postulate in sight which would allow hidden variables and yet introduce sufficient structure into the set of dispersion-free states, so as to give an useful and «nontrivial» theory.

In the earlier investigation (8), the meaning assigned to hidden variables is essentially the same as that given by our Definition 1. Hidden variables may however be given a somewhat wider meaning in the sense of the following

Definition 3. A physical system with C^*-algebra R of observables and the set Σ of physical states is said to admit «generalized hidden variables» if, for every φ in Σ and any Hermitian T of R, there exists a positive measure μ_φ^T on a measure space M, such that

$$\varphi(T) = \int_M \varrho_x(T)\,\mathrm{d}\mu_\varphi^T(x)\,.$$

Here, x denotes the variable element of M and $\varrho_x(T)$ denotes dispersion-free functionals with appropriate properties.

The «generalized hidden variables» are more general than the hidden variables of Definition 1 in that the averaging process (mathematically symbolized by the measure μ_φ^T) by which the physical state φ is obtained from the dispersion-free «states», is now allowed to depend not only on the physical state φ, but also on the observable T whose expectation value is being measured. It is such generalized hidden variables which seem to be alluded to in several remarks of BOHM (15).

Now, neither the results of this Section nor the «proof» (16) of ref. (8), as it stands, can exclude such generalized hidden variables. It is however interesting that the proof of ref. (8) can be modified so as to exclude even this possibility in the lattice-theoretical setting of quantum mechanics (16). But

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(15) Cf. D. BOHM: l. c.

(16) If the physical system admits of no super-selection rules and if a strengthened form of the property (4)° (*viz.* that denoted as (4) in ref. (8)) is required of the states, then Corollary 1 to Theorem 1 of Jauch and Piron is, of course, sufficient to exclude the «generalized hidden variables». The question is if «generalized hidden variables» can be also excluded without resorting to the stringent assumption (4) about the states.

Now, Theorem 2 of Jauch and Piron does not depend on axiom (4) for its proof and excludes hidden variables in the sense of Definition 1 of this paper. The proof of Theorem 2, however, needs to be modified before it can exclude «generalized hidden variables». The required modification follows from adopting a definition of «compatibility» which is different from, though equivalent to, that adopted in ref. (8). This is: the propositions a and b are compatible if and only if $(a \cap b) \cup (a' \cap b) \cup (a \cap b') \cup (a' \cap b') = I$. The equivalence of this definition with that in ref. (8) is shown in the already cited work of PIRON. The details of the proof of the slightly generalized version of Theorem 2 of ref. (8) which excludes «generalized hidden variables» is left to the reader.

no such result is available in the C^*-algebraic frame unless one assumes the too stringent condition of linearity.

5. – Hidden variables in the general frame of quantum mechanics.

In this Section, we shall briefly discuss the hidden-variable problem in a more general algebraic setting of quantum mechanics than the C^*-algebraic one [10,11]. The set of postulates outlined by SEGAL [11] will be our starting point. We shall not describe this general algebraic approach in full, but only mention the relevant assumptions. For a fuller discussion of the postulates and their physical motivations, the reader may see the cited references.

The basic object of this approach is again the set O of all (bounded) observables of the system. However, we no longer suppose that O forms the set of Hermitian elements of a C^*-algebra. We shall rather postulate:

1) The set O is a real linear space, so that if A and B are any two observables and α a real number, then the sum $A+B$ and the multiples αA are defined.

2) O contains a unit element I and to every A in O, there is an element A^2 in O: the square of A.

With the help of squaring operation, one can introduce a *quasi-product* $A \circ B$ in O by setting $A \circ B = \frac{1}{4}\,[(A+B)^2 - (A-B)^2]$. The quasi-product thus defined is now required to have the following property:

3) $$(\alpha A)^r = \alpha^r A^r \qquad \text{and} \qquad A^{r+s} = A^r \circ A^s,$$

where A^r is defined recursively by the following relations:

$$A^0 = I \qquad \text{and} \qquad A^r = A \circ A^{r-1} \qquad\qquad (r, s = 0, 1, 2, \ldots).$$

It should be noticed that the quasi-product is not necessarily associative *i.e.* $A \circ (B \circ C)$ is not necessarily identical with $(A \circ B) \circ C$. Nor does the quasi-product necessarily satisfy a distributive law with respect to the operation of taking sums of observables.

The following postulate now introduces a topological structure in O:

4) A *bound* $\|A\|$ is assigned to every A in O such that:

a) $\|A\| \geqslant 0$, and $\|A\| = 0$ if and only if $A = 0$.

b) $\|\alpha A\| = |\alpha|\,\|A\|$ and $\|A+B\| \leqslant \|A\| + \|B\|$.

c) If A_n is a sequence of elements in O such that $\|A_n - A_m\| \to 0$ as $n, m \to \infty$, then there exists an element of O, say A, such that $\|A_n - A\| \to 0$ as $n \to \infty$.

d) $\|A^2\| = \|A\|^2$ and $\|A^2 - B^2\| \leqslant \max[\|A\|^2, \|B\|^2]$.

e) A^2 is a continuous function of A in the topology defined by the bound.

5) For every observable A in O, there exists a pair of *positive* elements A_+ and A_- such that $A = A_+ - A_-$ and $A^2 = A_+^2 + A_-^2$. In this connection, an observable is called positive, as before, if it is the square of another observable.

It is easy to verify that postulates 1) through 5) are satisfied by the set of Hermitian elements of a C^*-algebra. These postulates define, however, a more general algebraic structure than that of the (set of Hermitian elements in) C^*-algebra.

Postulate 5) was not explicitly assumed by SEGAL. It seems however to be a property which one may reasonably require of the observables. It says that every observable can be thought of as linear combination of two compatible positive observables: the so-called positive and negative « parts » of the given observable.

The mathematical structure of the set O of observables is, then, defined by postulates 1) through 5). As to the states, we shall suppose that they (whether physically realizable or not) are all represented by (normalized) positive linear functionals on R. We shall, of course, not suppose that every positive linear functional is a physically realizable state. But we shall suppose that the set Σ of *physical states* is sufficiently large so that if $\varphi(A) = 0$ for all $\varphi \in \Sigma$, then $A = 0$.

We now ask the question: when is it possible to represent *every* physical state $\varphi \in \Sigma$ in the form

$$\varphi(T) = \int_M \varrho_x(T)\, d\mu_\varphi^T(x) , \tag{6}$$

where ϱ_x are dispersion-free positive linear functionals on O?

In order to answer this question, we need the concept of a « *derivation* » on O.

Definition. A mapping $T \to D(T)$ from O into O is called a derivation if:

1) $$D(\alpha T) = \alpha D(T) , \qquad D(T_1 + T_2) = D(T_1) + D(T_2) ,$$

and

2) $$D(T^2) = 2T \circ D(T) .$$

The answer to the previous question is contained in the following

THEOREM 4. – Every *physical state* $\varphi(T)$ of the system is of the form (6) only if O has no (nontrivial) derivation.

Proof of Theorem 4. We first show that if $\varrho(T)$ is a dispersion-free positive linear functional on O and $\mathrm{D}(T)$ a derivation, then $\varrho(\mathrm{D}(T))=0$.

Now, it can be easily shown that $\mathrm{D}(I)=0$. It will hence suffice to show that $\varrho(\mathrm{D}(S))=0$ where $S=T-\varrho(T)I$.

Let S_+ and S_- denote the positive and negative parts of S (see postulate 5)). We then have $S=S_+-S_-$ and $S^2=S_+^2+S_-^2$. Since ϱ is dispersion-free and $\varrho(S)=0$, we have

$$\varrho(S_+^2)+\varrho(S_-^2)=\varrho(S^2)=[\varrho(S)]^2=0\,.$$

Further, S_+^2 and S_-^2 are positive so that $\varrho(S_+^2)\geqslant 0$ and $\varrho(S_-^2)\geqslant 0$. Thus, the last equation implies that $\varrho(S_+^2)=\varrho(S_-^2)=0$ from which it also follows that $\varrho(S_+)=\varrho(S_-)=0$.

S_+ being positive is the square of some observable, say B; $S_+=B^2$. Since ϱ is dispersion-free and $\varrho(S_+)=0$, it follows that $\varrho(B)=0$. Now $\mathrm{D}(S_+)=\mathrm{D}(B^2)=2B\circ\mathrm{D}(B)$. Therefore,

$$\varrho(\mathrm{D}(S_+))=2\varrho(B\circ\mathrm{D}(B))=\tfrac{1}{2}\varrho\big((B+\mathrm{D}(B))^2-(B-\mathrm{D}(B))^2\big)=$$
$$=\tfrac{1}{2}\{[\varrho(B+\mathrm{D}(B))]^2-[\varrho(B-\mathrm{D}(B))]^2\}=2\varrho(B)\,\varrho(\mathrm{D}(B))=0\,.$$

Similarly, it can be proved that $\varrho(\mathrm{D}(S_-))=0$. Hence

$$\varrho(\mathrm{D}(S))=\varrho(\mathrm{D}(S_+))-\varrho(\mathrm{D}(S_-))=0\,.$$

The proof of Theorem 4 can be now easily completed. If every physical state φ can be expressed by relation (6) in terms of dispersion-free positive linear functionals, then

$$\varphi(\mathrm{D}(T))=0 \qquad \text{for } \textit{all} \text{ physical states.}$$

But this implies that $\mathrm{D}(T)=0$. Hence there is only the trivial derivation on O which maps every T onto the zero element.

In order to understand the physical meaning of Theorem 4, we should remember that, when we specialize to the case of C^*-algebra, the typical examples of derivations are the mappings which assign to elements T of the C^*-algebra their *commutator* with some given element of the algebra. The absence of (non-trivial) derivations on O may thus be taken as an appropriate mathematical characterization of the physical situation that all observables of the system are mutually compatible.

Theorem 4, then, excludes hidden variables in the general algebraic setting of quantum mechanics if one assumes that the dispersion-free «states» have the properties of linearity and positivity. Unfortunately, we have now no such result if linearity is replaced by a weaker condition, for instance, monotone-positivity.

6. – Concluding remarks.

We conclude this paper with a brief remark about the notion of « state ».

Underlying the accustomed representations of states as functionals (or probability measures) defined on *all* (bounded) observables (or propositions), there are two tacit assumptions. These are:

1) Any of the (bounded) observables of the system can be measured irrespective of the state in which the system is. In other words, the possibility that there may be states of the system such that certain of the observables cannot be measured on them is excluded by this assumption. Yet it is not difficult to imagine states of physical systems such that measurement of certain observables would be beyond the reach of available experimental technique. In all such cases, it is assumed that one can at least think of an ideal « thought experiment ».

If this assumption is not made, then one cannot represent states by functionals defined on *all* bounded observables and one would have to face the task of finding criteria that would single out the observables which can be measured on a given state.

2) The second assumption lies somewhat deeper and its denial would call for a radical revision of our conception of « states » and « observables ». According to our present conception, an observable represents, in general, not one but a whole class of experimental arrangements. What is more, the mathematical representation of observables as operators of a Hilbert space (or elements of a lattice, etc.) permits the possibility that the class of experimental arrangements corresponding to a given observable may contain mutually incompatible or exclusive experimental arrangements. For instance, if we suppose that every self-adjoint operator of a Hilbert space represents an observable, then it is not difficult to find observables A, B and C such that A and B do not commute and $C(\neq \alpha I)$ is a *function* of A *alone* and also of B *alone*. The observable C (being a function of A alone and also of B alone) can now be measured either by an experimental arrangement corresponding to A or that of B. Since A and B are incompatible, the class of experimental arrangements corresponding to C would thus contain mutually incompatible elements. Yet, in spite of this possibility, it is generally assumed (and this is the second tacit assumption) that the measured value of a given observable does not depend on the choice of experimental arrangement. Without some such assumption, « expectation value of an observable » would have no unambiguous meaning and the accustomed mathematical description of states in terms of « expectation value » of observables (or « propositions ») would not be possible.

Therefore, if one questions these two assumptions, all the « impossibility proofs » of hidden variables would thereby be put in question ([17]). But this line of thought can be pursued profitably only when an alternative mathematical description of states, which does not suffer from the limitation of the above tacit assumptions, has been outlined.

This, however, has not yet been seriously attempted.

Finally, we should recall that there exists also the possibility of « approximate hidden variable » ([18]) as already mentioned by JAUCH and PIRON ([8]). None of the known « impossibility proofs » are strong enough to exclude such possibilities.

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([17]) BELL ref. ([9]) has in fact taken recourse to similar arguments for refuting the « impossibility proof » flowing from the Gleason's theorem.

([18]) A system characterized by the algebra of observables R is said to admit of « approximate hidden variables » if, for any given $\varepsilon \gtrneqq 0$ (no matter how small), every physical state φ can be represented in the form $\varphi(T) = \int_M \varrho_x(T)\, \mathrm{d}\mu_\varphi(x)$, where $\varrho_x(T)$ are functions on R (with appropriate properties so as to be usefully considered as « states ») such that $|\varrho_x(T^2) - [\varrho_x(T)]^2| \leqslant \varepsilon$ for all T with $\|T\| \leqslant 1$. In other words, if a system admits « approximate hidden variables », then every physical state of the system is a mixture of « states » with arbitrarily small (although nonzero) « dispersion ».

RIASSUNTO (*)

Si esamina il problema delle variabili nascoste nella formulazione assiomatica della meccanica quantistica basata sull'algebra degli osservabili. Dopo una breve rassegna introduttiva degli studi precedenti, si analizza dapprima la struttura delle algebre C^* che consentono funzionali lineari positivi privi di dispersione. Il risultato ottenuto è una diretta generalizzazione del ben noto risultato di von Neumann riguardante le variabili nascoste. Nella successiva Sezione si suppone, come prima, che gli osservabili formino gli elementi hermitiani di un'algebra C^*. Ma ora si indeboliscono le condizioni sugli « stati » e si lascia che i cosiddetti funzionali monotoni positivi (che non sono necessariamente lineari) rappresentino gli stati. Si dimostra allora che, anche quando sono ammessi questi stati generalizzati, un sistema ammette variabili nascoste solo se la sua algebra degli osservabili è abeliana, cioè solo se tutti gli osservabili sono mutuamente compatibili. In un'altra Sezione si studia la questione delle variabili nascoste nell'ipotesi che gli osservabili, invece di formare un'algebra C^*, abbiano una certa struttura algebrica più generale.

(*) *Traduzione a cura della Redazione.*

Резюме не получено.