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Oscillations of Scattering Amplitudes at High Energy (*).

R. J. EDEN and L. ŁUKASZUK (**)
Cavendish Laboratory - Cambridge

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Summary. — Various problems are discussed that relate to oscillations in high-energy scattering. It is shown 1) that oscillations as $s \to \infty$ of a scattering amplitude $F(s, t)$ are not restricted by a polynomial bound and analyticity in s, 2) that for sufficiently large s, the number $N(s, r)$ of zeros of $F(s, t)$ within $|t| < r < t_0$ (where t_0 is the nearest singularity) satisfies $N(s, r) < C \log s$, where C is a constant.

The nature of the singularity of a scattering amplitude at infinite energy is an important unsolved problem in the study of strong interactions of elementary particles. For some theorems and applications it is sufficient to have bounds on the modulus of the amplitude, but in general the justification of approximations and of simple models requires information about oscillations as the energy tends to infinity. One would like to know whether or when they occur, for what problems and approximations they can be neglected, and what limits can be set on their relative magnitude and their frequency per unit energy.

The purpose of this note is to discuss the present state of progress in the study of oscillations of scattering amplitudes at high energy. This is closely related to the distribution of zeros.

1. – If simplifying assumptions are made they lead immediately to severe restrictions on oscillations. For example, if the branch cut in the energy z is

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(**) On leave of absence from Institute for Nuclear Research, Warsaw.

neglected, then the amplitude $F(z)$ is regular in the entire plane and bounded by a polynomial; in this case it is a polynomial and hence it has only a finite number of zeros and is smooth (not oscillatory) at infinity. More generally it might be assumed that $F(z)$ is an exponential function of w (where $w = \log z$) in the entire w-plane. This gives a logarithmic branch cut on the real axis and the number of zeros $N(r)$, where $r = |z|$, satisfies

$$N(r) < O(\log r) . \tag{1}$$

This means that the number of oscillations per unit energy range decreases like $1/r$. An example of such a function is

$$F(z) = iz(\sin \log z + \text{const}) . \tag{2}$$

If we take $F(z)$ to be polynomial-bounded only in the z-plane, *i.e.* a strip in the w-plane, but entire for all w, the following example is allowed:

$$F(z) = iz[C + \sin(\log^2 z)] . \tag{3}$$

If $F(z)$ represents a forward scattering amplitude this example corresponds to an oscillating total cross-section.

2. – If no simplifying assumptions are made except those that have been deduced from the axioms of quantum field theory there are three observations we wish to make:

a) The upper bounds (for example the Froissart bound), that have been established by MARTIN (1), do not depend on any assumptions about oscillations of the scattering amplitude. If sufficiently strong lower bounds could be established, then the converse of theorems by KHURI and KINOSHITA (2) would lead to restrictions on the phase and hence on the allowed oscillations. However, lower bounds have been established only for an « average » amplitude (3) and even these are too weak for conclusions via the Khuri-Kinoshita theorems.

b) The axioms of quantum field theory lead to the result that $F(z)$ is regular and polynomial-bounded in the upper half z-plane (for a certain range of momentum transfer). Unless further restrictions are obtained, for example on $\operatorname{Im} F$ when z is real, this result allows $F(z)$ to have zeros on the real axis

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(1) A. MARTIN: CERN preprint No. 66/488/5-TH. 652 (1966).
(2) N. N. KHURI and T. KINOSHITA: *Phys. Rev.*, **140**, B 706 (1965).
(3) Y. S. JIN and A. MARTIN: *Phys. Rev.*, **135**, B 1369 (1964).

for any set N of measure zero. We can see this from the converse to the following theorem due to HILLE (4).

THEOREM. – If $F(z)$ is bounded and holomorphic in $\operatorname{Im} z > 0$, if $F(z)$ is not identically zero, and if

$$\lim_{y\to 0} F(x+iy) = 0\,, \qquad \text{for } x \text{ in the set } N, \tag{4}$$

then N is of measure zero. Conversely if N is a closed set of measure zero, then there exists an $F(z)$ bounded and holomorphic in $\operatorname{Im} z > 0$, such that $F(z)$ is continuous in $\operatorname{Im} z \geqslant 0$, and (4) holds but $F(z) \neq 0$ for other values of z in $\operatorname{Im} z \geqslant 0$.

c) When the set of zeros extends to infinity along the real axis there will in general be a manifold of limit points. A rather weak restriction on them has been noted by MEIMANN (5) from a theorem due to NEVANLINNA:

THEOREM. – Let $F(z)$ be regular in $\operatorname{Im} z > 0$, and let C_1 be the manifold of limit points of $F(x)$ as $x \to +\infty$ along the real axis and C_2 the manifold as $x \to -\infty$. If $F(x)$ oscillates for arbitrarily large x then either C_1 or C_2 or both will be a continuum, otherwise it would be a single point. The theorem states that if C_1 and C_2 have no points in common and if one does not surround the other, then $F(z)$ cannot be polynomial-bounded in $\operatorname{Im} z > 0$.

3. – It is evident from these theorems that one must seek additional restrictions on the scattering amplitude in order to obtain a significant simplification at high energy. The obvious sources of such restrictions are unitarity and many-variable analyticity. They have direct consequences on the nature of singularities at thresholds along the real branch cut in energy. There is some indication that these singularities are combinations of square root and logarithmic (6) so it may be that the exponential function of $\log z$ considered above is not far from the physical situation (7).

It is possible to obtain a bound on the number of zeros of the amplitude $F(s, t)$, for fixed s and $|t| < t_0$, from the known bounds on the amplitude for large s. The upper bound (1) is s^2 for $t_0 < 4m^2$ and the lower bound for $F(s, 0)$ is s^{-2} provided we are not too near to a zero (in general complex) of $F(s, 0)$. Because $F(s, t)$ is analytic, for fixed s, as a function of t in $|t| < t_0$, with $F(s, 0) \neq 0$,

(4) A. HILLE: *Analytic Function Theory*, vol. **2** (Theorem 19.2.4).

(5) N. N. MEIMANN: *Sov. Phys. JETP*, **16**, 1609 (1963).

(6) R. J. EDEN, P. V. LANDSHOFF, D. I. OLIVE and J. C. POLKINGHORNE: *The Analytic S-Matrix* (Cambridge, 1966).

(7) R. J. EDEN: *Journ. Math. Phys.*, (in press) (1966).

we can write Jensen's theorem in the form

$$\int_0^r \frac{1}{\tau} N(s, \tau)\, \mathrm{d}\tau = \frac{1}{2\pi}\int_0^{2\pi} \log \frac{F(s, re^{i\theta})}{F(s, 0)}\, \mathrm{d}\theta < 4 \log s\,, \tag{5}$$

where $r < t_0$ and $N(s, \tau)$ is the number of zeros in $|t| < \tau$ of $F(s, t)$.

We can obtain a bound on the number of zeros inside a circle of radius $b < r$, by taking a lower bound in the above integral to give

$$\begin{cases} N(s, b) \log \dfrac{r}{b} < 4 \log s\,, \\ N(s, b) < \dfrac{4 \log s}{\log (r/b)}\,. \end{cases} \tag{6}$$

The maximum possible number of zeros for any finite $r/b > 1$, is therefore increasing at most like $\log s$. (The same applies to zeros of $\operatorname{Im} F(s, t)$.)

It has been shown by BESSIS (8) that the first zero cannot occur nearer to $t = 0$ than $|t| \sim (\log s)^{-2}$. Our result shows that even in this case the distance between zeros must increase (on the average) so that no more than $N \sim \log s$ zeros occur within $|t| < r < t_0$ (with r/t_0 independent of s). In general, any zeros will occur for complex values of t, when s is real, so they would lead to minima in the differential cross-section $\mathrm{d}\sigma/\mathrm{d}t$ for $|t| < t_0$. In principle the zeros $t = t_k(s)$, $k = 1, 2, \ldots$, of $F(s, t)$ will determine the zeros of $F(s, 0)$ but in practice this relation may be useful only for simple models.

4. – For potential scattering it is well known (9) that a singular potential like $1/r^n$ can lead to oscillations in the total cross-section $\sigma(s)$ as $s \to \infty$. It appears that the amplitude of these oscillations becomes small compared with the cross-section as $s \to \infty$. These singular potentials give complex singularities in the angular-momentum plane and imply a number of subtractions in a dispersion relation in s that tends to infinity with t so there is no Mandelstam representation (10). However even in the case of a regular potential having a Mandelstam representation and only Regge poles in $\operatorname{Re} l > -\frac{1}{2}$, it it possible that the background integral along $\operatorname{Re} l = -\frac{1}{2}$ could lead to oscillations in the amplitude $F(s, 0)$ as $s \to \infty$. In general these oscillations would be small compared with the contribution to F from a Regge pole in $\operatorname{Re} l > -\frac{1}{2}$. For $F(s, t)$ when $t > 4m^2$, the Regge poles themselves lead to oscillations in

(8) J. D. BESSIS: CERN preprint No. TH. 653 (1966).

(9) E. VOGT and G. H. WANNIER: *Phys. Rev.*, **95**, 1190 (1954).

(10) J. CHALLIFOUR and R. J. EDEN: *Journ. Math. Phys.*, **4**, 359 (1963).

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the spectral functions. It has yet to be discovered when oscillations of $F(s, 0)$ are either absent or unimportant as $s \to \infty$.

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RIASSUNTO (*)

Si studiano vari problemi relativi alle oscillazioni nello scattering di alta energia. Si dimostra 1) che le oscillazioni, per $s \to \infty$, di un'ampiezza di scattering $F(s, t)$ non sono limitate da un legame polimoniale e dall'analiticità in s, 2) che per s sufficientemente grande, il numero $N(s, r)$ degli zeri di $F(s, t)$ entro $|t| < r < t_0$ (dove t_0 è la più vicina singolarità) soddisfa $N(s, r) < C \log s$, dove C è una costante.

(*) *Traduzione a cura della Redazione.*

Осцилляции амплитуд рассеяния при высоких энергиях.

Резюме (*). — Обсуждаются различные проблемы, которые связаны с осцилляциями в рассеянии при высоких энергиях. Показывается: 1) что при $s \to \infty$ осцилляции амплитуды рассеяния $F(s, t)$ не ограничены многочленной границей и аналитичностью по s, 2) что для достаточно больших s, $N(s, r)$-число нулей $F(s, t)$ внутри $|t| < r < t_0$ (где t_0 ближайшая сингулярность) удовлетворяет неравенству $N(s, r) < C \log s$, где C константа.

(*) *Переведено редакцией.*