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Rectangling a Rectangle*

C. Freiling,[1] M. Laczkovich,[2] and D. Rinne[1]

[1] California State University,
San Bernardino, CA 92407, USA
cfreilin@wiley.csusb.edu
drinne@wiley.csusb.edu

[2]Eötvös Loránd University,
Múzeum krt. 6-8, Budapest, Hungary 1088
laczk@ludens.elte.hu

Abstract. We show that the following are equivalent: (i) A rectangle of eccentricity v can be tiled using rectangles of eccentricity u. (ii) There is a rational function with rational coefficients, $Q(z)$, such that $v = Q(u)$ and Q maps each of the half-planes $\{z \mid \mathrm{Re}(z) < 0\}$ and $\{z \mid \mathrm{Re}(z) > 0\}$ into itself. (iii) There is an odd rational function with rational coefficients, $Q(z)$, such that $v = Q(u)$ and all roots of $v = Q(z)$ have a positive real part.

All rectangles in this article have sides parallel to the coordinate axes and all tilings are finite. We let $R(x, y)$ denote a rectangle with base x and height y.

In 1903 Dehn [1] proved his famous result that $R(x, y)$ can be tiled by squares if and only if y/x is a rational number. Dehn actually proved the following result. (See [4] for a generalization to tilings using triangles.)

Theorem 1 (Dehn). *If the rectangle $R(x, y)$ can be tiled by the collection of rectangles $\{R(x_n, y_n)\}$, then y/x can be written as a rational function with rational coefficients of the arguments y_n/x_n.*

It is easy to show that the rational function in Dehn's theorem has some simple properties. For example, each term in the numerator has the same degree as well as each

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term in the denominator. Furthermore, these degrees differ by exactly one. (See, for example, [2].) However, it seems that not much else is known about it. If we demand that all of the ratios y_n/x_n are equal, then by rescaling in the horizontal (or vertical) direction, the rectangles $R(x_n, y_n)$ can be made into squares. In this case Dehn's rational function reduces to a rational number and we get his result on tiling a rectangle with squares.

In this article we characterize Dehn's rational functions for the case where the ratios y_n/x_n take on two possible values. Again, by rescaling in the horizontal (or vertical) direction, it can be assumed that these two ratios are reciprocals of each other and therefore we say that the rectangle $R(x, y)$ is tiled by similar rectangles of eccentricity u.

The following partial result, for tiling a square, was proved by Szekeres and the second author [5], and later, independently, by the first and third authors [2]. Curiously, neither proof immediately generalizes to the tiling of a rectangle.

Theorem 2. *The following are equivalent:*

(i) *A square can be tiled with similar copies of $R(1, u)$.*
(ii) *u is algebraic and all conjugate roots of u have positive real part.*
(iii) *There are positive rational numbers $c_0, \ldots, c_n$ such that*

$$1 = c_0 u + \cfrac{1}{c_1 u + \cfrac{1}{\ddots + \cfrac{1}{c_n u}}}.$$

The connection between continued fraction expansions and the location of roots comes from a theorem of Wall (see [7] or [8]). We state here the version of this result for polynomials with real coefficients.

Theorem 3 (Wall). *Let $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ be a polynomial of degree one or greater with real coefficients. Let $F(z) = z^n + a_{n-2}z^{n-2} + \cdots$ and $G(z) = a_1 z^{n-1} + a_{n-3}z^{n-3} + \cdots$ be the polynomials made of the odd and the even power terms of $P(z)$. Then the following are equivalent:*

(i) *All roots of $P(z)$ have negative real part.*
(ii) *There are positive real numbers $c_0, \ldots, c_n$ so that*

$$\frac{F(z)}{G(z)} = c_0 z + \cfrac{1}{c_1 z + \cfrac{1}{\ddots + \cfrac{1}{c_n z}}}.$$

The conversion of this theorem into a result involving roots with positive real part is contained in the following lemma.

Lemma 4. *Let Q be a nonzero rational function with real coefficients and $v > 0$. Then the following are equivalent*:

(i) *Q maps each of the half-planes $\{z \mid \mathrm{Re}(z) < 0\}$ and $\{z \mid \mathrm{Re}(z) > 0\}$ into itself.*
(ii) *Q is odd and all roots of $Q(z) = v$ have positive real part.*
(iii) *There are real numbers $c_0, \ldots, c_n$ with $c_0 \geq 0$ and $c_i > 0$ for $1 \leq i \leq n$ so that*

$$Q(z) = c_0 z + \cfrac{1}{c_1 z + \cfrac{1}{\ddots + \cfrac{1}{c_n z}}}.$$

Proof. The implication (ii) $\rightarrow$ (iii) is an easy consequence of Wall's theorem. Write $Q(z)$ as $N(z)/D(z)$, a ratio of two polynomials, where N is even and D is odd, or the other way around, and apply Wall's theorem to the polynomial $N(z) - vD(z)$ using $-z$ in place of z to get roots with positive real part. Note that since multiplication of the continued fraction expansion by a positive number does not change the sign of any of the coefficients, we may take v to be one. The coefficient c_0 is zero or nonzero depending on whether the numerator or denominator of $Q(z)$ has the larger degree.

The proof that (iii) implies (i) is quickly handled by induction on n.

It remains to show that (i) implies (ii). Let $P(z) = Q(z) + Q(-z)$. We wish to show that $P(z)$ is identically zero. To begin, note that $Q(z)$ takes the imaginary axis into itself. Thus $P(iy) = Q(iy) + Q(-iy) = Q(iy) + Q(\overline{iy}) = Q(iy) + \overline{Q(iy)} = 0$. Since the numerator of $P(z)$ has an infinite number of roots, the numerator is identically zero, and hence so is $P(z)$. Finally, suppose $Q(z) = v$ has a root with real part less than or equal to zero. If this root has real part less than zero, then (i) is violated. If this root is on the imaginary axis, then by continuity we can pick z sufficiently close to the root but with negative real part so that $Q(z)$ is close enough to v so that $\mathrm{Re}(Q(z)) > 0$, also violating (i). □

To see how this leads to tilings, for each $u > 0$, let $T(u)$ denote the set of positive numbers x such that $R(1, x)$ can be tiled using similar copies of $R(1, u)$. It is immediate that:

(a) If c is a positive rational number, then $cu \in T(u)$.
(b) $x, y \in T(u) \rightarrow x + y \in T(u)$.
(c) $x \in T(u) \rightarrow 1/x \in T(u)$.

By iterating these, we get that any number in the form in (iii) above also belongs to $T(u)$. If $v = Q(u)$ where Q is an odd rational function with rational coefficients and such that all roots of $Q(z) = v$ have a positive real part, then Lemma 4 tells us that $v \in T(u)$. Furthermore, the process is constructive; that is, by expanding the rational function into a continued fraction we see exactly how to do the tiling. The tilings produced this way have a simple configuration which we call a "Wall pattern," illustrated by this example.

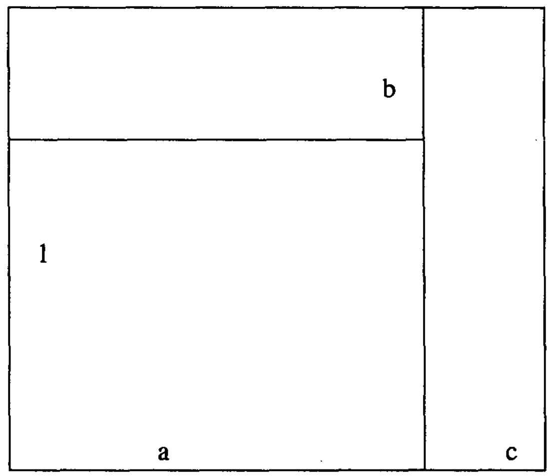

Fig. 1. Wall pattern of degree three.

We use $u = 2 - \sqrt[3]{5}$ which has minimal polynomial $z^3 - 6z^2 + 12z - 3$. Let

$$Q(z) = \frac{6z^2+3}{z^3+12z} = \cfrac{1}{\frac{1}{6}z + \cfrac{1}{\frac{12}{23}z + \cfrac{1}{\frac{23}{6}z}}}.$$

Note that Q is odd and $Q(u) = 1$. Therefore, letting $c_3 = \frac{23}{6}$, $c_2 = \frac{12}{23}$, and $c_1 = \frac{1}{6}$, we may partition a square (since a square has ratio 1) as shown in Fig. 1.

We let $a = c_3u = \frac{23}{6}u$, $b = c_2ua = 2u^2$, and $c = c_1u(1+b) = \frac{1}{6}u + \frac{1}{3}u^3$. This rectangle is then $(a+c)\times(1+b) = (4u+\frac{1}{3}u^3)\times(1+2u^2)$. Since $u^3-6u^2+12u-3 = 0$, $(4u + \frac{1}{3}u^3) = (1 + 2u^2)$, so this is indeed a square.

The pattern visible in Fig. 1 is as follows. The lower left $1 \times a$ rectangle has a ratio which is a rational multiple of $1/u$. Adjoined to the top of it is a rectangle whose ratio is a rational multiple of u. Adjoined to the side is then a rectangle whose ratio is a rational multiple of $1/u$. Each of these rectangles can easily be tiled using rectangles similar to $R(1, u)$. Repeatedly adjoining rectangles whose ratios are alternately rational multiples of u and $1/u$ creates the Wall pattern.

The theorem proved below characterizes Dehn's rational functions for tiling a rectangle with similar rectangles. Again, the geometric interpretation is that when such a tiling is possible, one can be constructed using a Wall pattern. The shortcoming is that if the eccentricity of the tiles, u, happens to be an algebraic number, then the process is no longer algorithmic. In that case there may be infinitely many odd rational functions such that $v = Q(u)$. Furthermore, one choice for Q may have its continued fraction in the desired form while others do not. So trying to determine whether a tiling is possible may require a search through all possible such rational functions. In tiling a square this problem does not occur since in that case we can take $Q(z)$ to be either $(P(-z)+P(z))/(P(-z)-P(z))$ or its reciprocal, where $P(z)$ is the minimal polynomial for u. Any other odd rational function with $Q(u) = 1$ must reduce to one of these.

Perhaps the difficulty in passing from tilings of a square to tilings of a rectangle can be best understood as follows. If we wish to tile a square with rectangles of eccentricity u, and u is an algebraic number of degree n, then the complexity of the Wall pattern is also n. However, in tiling a rectangle, no such bound on the complexity of the "Wall pattern" is known.

We also present a technique for establishing, in the algebraic case, necessary conditions for the existence of such a tiling. In the case where u is of degree two this will lead to a complete solution of the problem.

A simple characterization for Dehn's rational function when more than two ratios y_n/x_n are used is also unknown.

Theorem 5. *For any pair of positive real numbers u and v, the following are equivalent:*

(i) *The rectangle $R(1, v)$ can be tiled with similar copies of $R(1, u)$.*
(ii) *There is a rational function with rational coefficients, $Q(z)$ such that $v = Q(u)$, and Q maps each of the half-planes $\{z \mid \mathrm{Re}(z) < 0\}$ and $\{z \mid \mathrm{Re}(z) > 0\}$ into itself.*
(iii) *There is an odd rational function with rational coefficients, $Q(z)$ such that $v = Q(u)$, and all the roots of $v = Q(z)$ have positive real part.*
(iv) *There are rational numbers $c_0, \dots, c_n$ with $c_0 \geq 0$ and $c_i > 0$ for $1 \leq i \leq n$ so that*

$$v = c_0 u + \cfrac{1}{c_1 u + \cfrac{1}{\ddots + \cfrac{1}{c_n u}}}.$$

Proof. The equivalence of parts (ii)–(iv) is obvious from the statement of Lemma 4. That (iv) implies (i) follows from our discussion above of the Wall pattern. It remains to show that (i) implies (ii). We distinguish two cases.

Case 1: The number u is transcendental. Let $\mathbf{Q}^*$ denote the collection of rational functions with rational coefficients, $\mathbf{Q}(u)$ the field $\{Q(u) \mid Q \in \mathbf{Q}^*\}$, and V the vector space of $\mathbf{R}$ over $\mathbf{Q}(u)$. Let $B = (1, b_1, b_2, \dots)$ be a (Hamel) basis for this space. For any real number x, there is a function $Q_x \in \mathbf{Q}^*$ such that $Q_x(u)$ is the coefficient of one in the expansion of x over B. Since u is transcendental, the function Q_x is unique up to reduction to lowest terms.

Let r be a fixed transcendental complex number with nonzero real part so that r is in the domain of every function in $\mathbf{Q}^*$. Define the "Hamel area" (or just "area") of $R(x, y)$ to be $A(x, y) = \mathrm{Re}(Q_x(r) \cdot \overline{Q_y(r)})$. Since $Q_x + Q_y = Q_{x+y}$, the area function is additive in both coordinates: $A(x, y) + A(x, z) = A(x, y+z)$ and $A(x, y) + A(z, y) = A(x+z, y)$.

Suppose $R(x, y)$ is tiled by the rectangles $R(x_n, y_n)$. By extending each horizontal line segment across the entire rectangle $R(x, y)$ we form a refinement into rectangles $R(w_n, z_n)$. It now follows that $A(x, y) = \sum A(x_n, y_n)$ since both sides are equal to $\sum A(w_n, z_n)$. It is also readily seen that $A(x, y) = A(y, x)$. If $y = ux$, then $Q_y(r) = r Q_x(r)$, and hence $A(x, y) = \mathrm{Re}(Q_x(r) \cdot \overline{Q_{ux}(r)}) = \mathrm{Re}(Q_x(r) \cdot \overline{Q_x(r)} \cdot \overline{r}) = a \cdot \mathrm{Re}(r)$ for some $a \geq 0$.

We actually prove something slightly stronger than what is required. Suppose we have a finite collection of tiles $R(p_n, q_n)$ such that, for each n, either $p_n/q_n = u$ or $q_n/p_n = u$. Suppose further that we make vertical and horizontal cuts in these rectangles to form a new collection of tiles $R(x_n, y_n)$ and that these are rearranged (without any waste) to tile the rectangle $R(1, v)$. Then $A(1, v) = \sum A(x_n, y_n) = \sum A(p_n, q_n)$, which is either zero or has the same sign as $\mathrm{Re}(r)$. On the other hand, $A(1, v) = \mathrm{Re}(Q_1(r) \cdot \overline{Q_v(r)}) = \mathrm{Re}(1 \cdot \overline{Q_v(r)}) = \mathrm{Re}(Q_v(r))$. It must be then that $\mathrm{Re}(r)$ and $\mathrm{Re}(Q_v(r))$ are not on opposite sides of the imaginary axis. Although we assumed r to be transcendental, by continuity this is true for all complex r in the domain of Q_v. That is, Q_v maps each of the left and right closed half-planes into itself. It follows from the Open Mapping Theorem that Q_v is either identically zero or maps each of the left and right open half-planes into itself. The proof will then be finished if we show $Q_v(u) = v \neq 0$. For this it suffices to show that $v \in \mathbf{Q}(u)$. If $v \notin \mathbf{Q}(u)$ let $b_1 = v + 1$ in the basis selection. Then $v = (-1)(1) + (1)(v + 1)$ so $Q_v(u) = -1$, contradicting the fact that u and $Q_v(u)$ are not on opposite sides of the imaginary axis.

Case 2: *The number u is algebraic.* Suppose $R(1, v) = [0, 1] \times [0, v]$ is tiled by the rectangles $R(x_j, y_j)$ $(j = 1, \dots, n)$ such that, for each j, either $x_j/y_j = u$ or $y_j/x_j = u$. Let $z_1, \dots, z_m$ denote the widths of the horizontal strips (ordered top to bottom) formed when the horizontal line segments of the tiling are extended across the entire rectangle $R(1, v)$.

We use a technique given by Gale (see [3]) to develop a system of equations representing such a tiling. Let $A = \{a_{i,j}\}$ be the $m \times n$ incidence matrix where $a_{i,j} = 1$ if $R(x_j, y_j)$ meets the ith horizontal (open) strip and $a_{i,j} = 0$ otherwise. Note that for each such strip there is some rectangle which meets it but does not meet any horizontal strip below it. Therefore, for each row of A there is a column which has a one in that row and no ones below that row. The matrix A therefore, has rank m. Using that the length of each horizontal strip is one, we immediately get that $AX = E$ where X is the column vector $(x_1, \dots, x_n)^{\mathrm{T}}$ and E is the column vector $(1, \dots, 1)^{\mathrm{T}}$. We now let B represent the $n \times m$ matrix $\{b_{i,j}\}$ where $b_{i,j} = x_i/y_i$ if $R(x_i, y_i)$ meets the interior of the jth horizontal strip and zero otherwise. In other words, B is formed by taking the transpose of A and replacing the ones in each row by the ratio x_i/y_i of the rectangle corresponding to that row. The system of equations $BZ = X$ where Z is the column vector $(z_1, \dots, z_m)$ merely states that each tile has the correct ratio. The elements of B are either zero or u or $1/u$. If each nonzero entry were replaced by one we would just have the transpose of A. Since the rank of A is m, $\det(AA^{\mathrm{T}}) \neq 0$ (see, for example, [6]). Similarly, $\det(AB) \neq 0$. To see this, note that $B = CA^{\mathrm{T}}$ where C is a diagonal matrix with diagonal entries u or $1/u$. Let $\sqrt{C}$ be formed in the obvious way using diagonal entries $\sqrt{u}$ or $1/\sqrt{u}$. It is then easy to calculate that $AB = A\sqrt{C}\sqrt{C}A^{\mathrm{T}} = A\sqrt{C}(A\sqrt{C})^{\mathrm{T}}$ and as before $A\sqrt{C}$ is of rank m so $\det(AB) \neq 0$. We can then deduce from the system of equations $AX = E$ and $BZ = X$, that $Z = (AB)^{-1}E$ and that $X = BZ$.

The value of v is not mentioned in these equations, but can easily be calculated to be the sum of the elements of Z which is the same as the sum of the elements in $(AB)^{-1}$ Thus v as well as all the entries in X and Z are positive numbers which are expressed as rational functions of u. In particular, $v = Q(u)$ for some $Q \in \mathbf{Q}^*$. (This fact also follows from Theorem 2 of [4].) We need to show that Q is odd and that the roots of $Q(z) = 1$ are all in the right half of the complex plane.

To see this, we form a new matrix B' by replacing all values of u in the matrix B with a positive transcendental number u'. The determinant of AB' is still nonzero and we therefore define $Z' = (AB')^{-1}E$, $X' = B'Z'$, and let v' be the sum of the entries in Z' which is also the sum of the entries in $(AB')^{-1} = Q(u')$. The value of u' may be chosen so close to u that all of the entries in Z' and X' are still positive. Then since v' is the sum of the entries in Z', the vector Z' tells us how to cut up $R(1, v')$ into horizontal strips, while the equation $AX' = E$ tells us how to cut up each of these strips into rectangles whose lengths are entries of the matrix X', and finally the equation $B'Z' = X'$ tells us how to combine the rectangles from different strips, using each rectangle exactly once, to form tiles of eccentricity u'. Note that these equations do not quite tell us how to tile $R(1, v')$ with rectangles of eccentricity u', but they do tell us how to cut tiles of eccentricity u' into pieces and rearrange them (without any waste) to form a tiling of $R(1, v')$. It therefore follows from the transcendental case that $v' = Q'(u')$ for some odd $Q' \in \mathbf{Q}^*$ and that the roots of $Q'(z) = 1$ are all in the right half of the complex plane. However, since u' is transcendental and $Q'(u') = Q(u')$, we have $Q' = Q$. Therefore Q is odd and all the roots of $Q(z) = 1$ are in the right half-plane. □

We now present a technique which will generate necessary conditions for the existence of such a tiling in the algebraic case. Let u be an algebraic number with minimal polynomial $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. Then the companion matrix of u is defined to be

$$U = \begin{bmatrix} 0 & \cdots & & 0 & -a_0 \\ 1 & 0 & & & -a_1 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & & \ddots & 0 & \\ 0 & \cdots & 0 & 1 & -a_{n-1} \end{bmatrix}.$$

Let v be in the field generated by u, so that $v = P(u)$ for some rational polynomial P. Then we call $V = P(U)$ a *companion matrix of v with respect to u*.

The purpose of the companion matrices is to effect multiplication of polynomials in u by the numbers u and v. More precisely, if x is in the field generated by u, then x can be written in the form $x = x_{n-1}u^{n-1} + \cdots + x_1 u + x_0$. We associate x with the vector $\bar{x} = (x_0, x_1, \ldots, x_{n-1})$. The vector $(U\bar{x}^{\mathrm{T}})^{\mathrm{T}}$ is the associated vector for the number ux and $(V\bar{x}^{\mathrm{T}})^{\mathrm{T}}$ is the associated vector for the number vx.

Theorem 6. *If u is algebraic and $v = P(u)$ and $R(1, v)$ can be tiled with similar copies of $R(1, u)$, then, for every symmetric matrix M, if MU is positive semidefinite, then so is MV.*

Proof. Let M be a symmetric matrix with MU positive semidefinite. Let B be a basis for the vector space of $\mathbf{R}$ over $\mathbf{Q}(u)$ such that B contains the number one. Then, for any real number x, the coefficient of one in the unique expansion of x using this basis is a rational polynomial of u. We denote this polynomial by $x_n u^n + \cdots + x_0$, and we let $\bar{x}$ be the vector $(x_0, \ldots, x_{n-1})$. We now define the "area" of a rectangle $R(x, y)$ to be $\bar{x}M\bar{y}^{\mathrm{T}}$. As mentioned above, if $y = ux$, then $\bar{y} = (U\bar{x}^{\mathrm{T}})^{\mathrm{T}}$ and if $y = vx$, then $\bar{y} = (V\bar{x}^{\mathrm{T}})^{\mathrm{T}}$. Therefore, if $y = ux$, then the "area" of $R(x, y)$ is $\bar{x}MU\bar{x}^{\mathrm{T}}$ which is nonnegative since MU is positive semidefinite. Since M is symmetric, the "area" of $R(x, y)$ and $R(y, x)$ are equal so any rectangle of eccentricity u has nonnegative area. Also, our "area" function is additive so that if a rectangle R is tiled by rectangles $\{R_n\}$, then the "area" of R is the sum of the "areas" of the R_n. If $R(1, v)$ can be tiled by rectangles of eccentricity u, then any rectangle $R(x, y)$ with $y = vx$ can be so tiled. Therefore, the "area" of any such rectangle must be nonnegative. Given any n-vector $\bar{z}$ of rational numbers there is always a positive number x such that $\bar{x} = \bar{z}$. Therefore, for any such $\bar{z}$, $\bar{z}MV\bar{z}^{\mathrm{T}}$ must be nonnegative. However, if this holds for rational vectors, it must also hold for real vectors and hence MV is positive semidefinite. □

In the case where u has degree two, this is enough to solve the problem of tilings:

Theorem 7. *Let u be a positive algebraic number of degree two so that $u^2 = au - c$ for some rational numbers a and c, $c \neq 0$. Let u' be its conjugate root. Let v be in the field generated by u so that $v = du + b$ for some rationals d and b. Then $R(1, v)$ can be tiled by rectangles similar to $R(1, u)$ if and only if either $b = 0$ and $d > 0$ or else $bc/a > 0$ and $(ad + b)/a \geq 0$.*

Proof. Suppose first that $R(1, v)$ can be so tiled. If $b = 0$, then since $v > 0$ we have $d > 0$ and we are done. So assume $b \neq 0$. If $a \neq 0$ let M be the 2×2 diagonal matrix $\begin{bmatrix} c/a & 0 \\ 0 & c^2/a \end{bmatrix}$. Let U be the companion matrix for u, that is the 2×2 matrix $\begin{bmatrix} 0 & -c \\ 1 & a \end{bmatrix}$. Then MU is the 2×2 matrix $\begin{bmatrix} 0 & -c^2/a \\ c^2/a & c^2 \end{bmatrix}$ which is positive semidefinite. The companion matrix for v is $V = dU + bI$ where I is the 2×2 identity matrix. However, then MV is the 2×2 matrix $\begin{bmatrix} bc/a & -dc^2/a \\ dc^2/a & c^2(ad + b)/a \end{bmatrix}$. If either of the diagonal entries bc/a or $c^2(ad+b)/a$ is less than zero, then this matrix will not be positive semidefinite. Therefore $bc/a > 0$ and $(ad + b)/a \geq 0$. If $a = 0$ we let $M = \begin{bmatrix} c & 0 \\ 0 & c^2 \end{bmatrix}$ and the same proof shows that $b \geq 0$. Similarly letting $M = \begin{bmatrix} -c & 0 \\ 0 & -c^2 \end{bmatrix}$ proves that $b \leq 0$. So if $a = 0$, then $b = 0$ contradicting our assumption that $b \neq 0$. Now let $v = du + b$ and first suppose that $b = 0$ and $d > 0$. Then v is a positive rational multiple of u and can be trivially tiled. Finally suppose that $bc/a > 0$ and $(ad + b)/a \geq 0$. Then using $v = c_0u + 1/c_1u = (c_0c_1u^2 + 1)/c_1u$ where $c_0 = (ad + b)/a$ and $c_1 = a/bc$ we find that $c_0 \geq 0$ and $c_1 > 0$. By Theorem 5 we are done. □

The previous theorem might raise hopes that a similar theorem could be proved for algebraic numbers of higher degree. However, even for degree three, the conditions start to get complicated.

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