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A Content Semantics for Quantified Relevant Logics. II

Abstract. In part I, we presented an algebraic-style of semantics, which we called "content semantics", for quantified relevant logics based on the weak system *BBQ*. We showed soundness and completeness with respect to the *unreduced* semantics of *BBQ*. In part II, we proceed to show soundness and completeness for extensions of *BBQ* with respect to this type of semantics. We introduce *reduced* semantics which requires additional postulates for primeness and saturation. We then conclude by showing soundness and completeness for BB^dQ and its extensions with respect to this reduced semantics.

II. Unreduced content semantics for extensions of *BBQ*.

We first consider the axioms, and also one rule, that would be used to obtain the most commonly considered extensions of *BBQ*. For a suitable range of extensions, see [4], pp. 355–359.

Sentential axioms

S1. $(A \to B) \& (A \to C) \to . A \to B \& C.$
S2. $(A \to C) \& (B \to C) \to . A \vee B \to C.$
S3. $A \to \sim B \to . B \to \sim A.$
S4. $(A \to B) \& (B \to C) \to . A \to C.$
S5. $A \vee \sim A.$
S6. $A \to B \to . B \to C \to . A \to C.$
S7. $A \to B \to . C \to A \to . C \to B.$
S8. $A \to . A \to B \to B.$
S9. $A \to \sim A \to \sim A.$
S10. $(A \to . A \to B) \to . A \to B.$
S11. $A \to . B \to A.$
S12. $\sim A \to . A \to B.$

Quantificational axioms

Q1. $(\forall x)(A \to B) \to . A \to (\forall x)B$, where x is not free in A.
Q2. $(\forall x)(A \to B) \to . (\exists x)A \to B$, where x is not free in B.

Sentential rule

SR1. $A \Rightarrow \sim (A \to \sim A).$

The corresponding semantic postulates for these axioms and rule are presented below. To obtain content semantics for an extension of *BBQ*, just add the semantic postulates corresponding to the axioms and/or rule that are added to *BBQ*.

Semantic postulates for S1-12, Q1-2, SR1

s1. $(c \Rightarrow d) \cap (c \Rightarrow e) \leqslant c \Rightarrow (d \cap e)$.

s2. $(c \Rightarrow e) \cap (d \Rightarrow e) \leqslant (c \cup d) \Rightarrow e$.

s3. $c \Rightarrow d^* \leqslant d \Rightarrow c^*$.

s4. $(c \Rightarrow d) \cap (d \Rightarrow e) \leqslant c \Rightarrow e$.

s5. $c \cup c^* \in T$.

s6. $c \Rightarrow d \leqslant (d \Rightarrow e) \Rightarrow (c \Rightarrow e)$.

s7. $c \Rightarrow d \leqslant (e \Rightarrow c) \Rightarrow (e \Rightarrow d)$.

s8. $c \leqslant (c \Rightarrow d) \Rightarrow d$.

s9. $c \Rightarrow c^* \leqslant c^*$.

s10. $c \Rightarrow (c \Rightarrow d) \leqslant c \Rightarrow d$.

s11. $c \leqslant d \Rightarrow c$.

s12. $c^* \leqslant c \Rightarrow d$.

q1. $\bigcap \{(F^n \Rightarrow G^m) s^b/k (i_1 \ldots, i_n, j_1 \ldots, j_m) : b \in D\}$
$\leqslant F^n s (i_1 \ldots, i_n) \Rightarrow \bigcap \{G^m s^b/k (j_1 \ldots, j_m) : b \in D\}$,
where $F^n s (i_1, \ldots, i_n)$ is k-constant.

q2. $\bigcap \{(F^n \Rightarrow G^m) s^b/k (i_1 \ldots, i_n, j_1 \ldots, j_m) : b \in D\}$
$\leqslant \bigcup \{F^n s^b/k (i_1, \ldots, i_n) : b \in D\} \Rightarrow G^m s (j_1, \ldots, j_m)$,
where $G^m s (j_1, \ldots, j_m)$ is k-constant.

sr1. If $c \in T$ then $(c \Rightarrow c^*)^* \in T$.

As should be clear, the semantic postulates sn, qm and sr1 correspond to the axioms Sn, Qm and the Rule SR1, for $n = 1, \ldots, 12$ and $m = 1,2$.

Instead of proving soundness and completeness for each of these axioms and rule with respect to their correspondents, we can generalize the situation by constructing semantic postulates corresponding to arbitrary formula--schemes and rules, and prove soundness and completeness at this level of generality. This means that a content semantics can be provided for *any* extension of *BBQ*.

However, we do need to specify the conditions on formula-schemes that we are going to treat. If we consider the quantificational axioms and rules of *BBQ* and the above extensions, we see conditions of the form 'x_k does not occur free' and we see a formula-scheme of the form 'A^{x_l}/x_k, where x_l is free for x_k in A'. We restrict consideration to these two expressions as they seem to cover all the extensions of *BBQ* that one is likely to meet, and any extensions of *BBQ* containing other expressions can most likely be dealt with in a similar manner to these. Certainly, the following procedure can deal with any formula-scheme, built up using the formation rules and so we can still justifiably say that *all* extensions of *BBQ* can be provided with a content semantics. We will need, however, to transform the quantificational semantic postulates for the specific extensions of *BBQ*, given above, into the forms that emerge from the general treatment about to be given.

The construction of semantic postulates for axioms and rules is as follows:

The procedure extends that used by Lavers in [15] for the sentential fragment.

(i) For axiom-schemes and rules with no quantifiers nor expressions of either of the forms 'x_k is not free in A' or 'A^{x_l}/x_k, where x_l is free for x_k in A',
   (a) replace formula-schemes $A, B, C, \ldots$ by $c, d, e, \ldots$ (distinct elements of C),
   (b) replace '$\sim$' by '*' to the right of the element of C (can be a composite element such as $(c \cup d)^* \cap e$),
   (c) replace '&' by '$\cap$' between (composite) elements of C,
   (d) replace '$\vee$' by '$\cup$' between (composite) elements of C,
and (e) replace '$\rightarrow$' by '$\Rightarrow$' between (composite) elements of C, provided '$\rightarrow$' was not a main connective of the original formula-scheme.
   [Take '$\leftrightarrow$' as defined: $A \leftrightarrow B =_{df} (A \rightarrow B) \& (B \rightarrow A)$.]

(ii) For axiom-schemes and rules with at least one quantifier or expression of one of the forms 'x_k is not free in A' or 'A^{x_l}/x_k, where x_l is free for x_k in A',
   (a) replace 'A', 'B', 'C', ... by '$F^n s(i_i, \ldots, i_n)$', '$G^m s(j_1, \ldots, j_m)$', '$H^l s(k_1, \ldots, k_l)$', ... (i.e. a function from $\mathscr{F}$, an element s of S (same s in each case), a sequence of positive integers the number of which is the same as the number of arguments of the function from $\mathscr{F}$), and replace 'A^{x_l}/x_k, where x_l is free for x_k in A', by '$F^n s(i_1, \ldots, i_n)^l/k$', where $F^n s(i_1, \ldots, i_n)$ is what is used to replace A and $(i_1, \ldots, i_n)^l/k$ is $(i_1, \ldots, i_n)$ with each occurrence of k replaced by l, provided that no replacement step (f) or (g) eliminates such occurrences,
   (b) replace '$\sim$' by '*' to the right of the element of $\mathscr{F}$ (also can be a composite element such as $(F^n \cup G^m)^* \cap H^l$),
   (c) replace '&' by '$\cap$' between (composite) elements of $\mathscr{F}$, and follow this by 's' and the sequence $(i_1, \ldots, i_n, j_1, \ldots, j_m)$, where $(i_1, \ldots, i_n)$ is the sequence of positive integers associated with the left conjunct and $(j_1, \ldots, j_m)$ is the sequence of positive integers associated with the right conjunct,
   (d) replace '$\vee$' by '$\cup$' between (composite) elements of $\mathscr{F}$, followed by 's' and the sequence described in (c) for '&',
   (e) replace '$\rightarrow$' by '$\Rightarrow$' between (composite) elements of $\mathscr{F}$, followed by 's' and the sequence described in (c) for '&', provided '$\rightarrow$' was not a main connective of the original formula-scheme,
   (f) replace '$(\forall x_k)$' by '$\bigcap \{\ldots^b/\{j_1, \ldots, j_m): b \in D\}$', where the element of $\mathscr{F}$ corresponding to the scope formula-scheme of '$(\forall x_k)$' is inserted where the dots appear, and where $j_1, \ldots, j_m$ are the argument places of the scope formula-scheme which are bound by '$(\forall x_k)$', and also replace '$(i_1, \ldots, i_n)$' by '$(i_{k_1}, \ldots, i_{k_{n-m}})$', where $i_1, \ldots, i_n$ is the sequence of positive integers, one for each argument-place of the scope formula-scheme and $i_{k_1}, \ldots, i_{k_{n-m}}$ is the same sequence but with $i_{j_1}, \ldots, i_{j_m}$ removed,

(g) replace '$(\exists x_k)$' by '$\bigcup\{\ldots{}^b/\{j_1, \ldots, j_m\}: b \in D\}$' and '$(i_1, \ldots, i_n)$' by '$(i_{k_1}, \ldots, i_{k_{n-m}})$', in the same manner as for '$(\forall x_k)$' in case (f).

(iii) For formula-schemes (in rules or axioms),
(a) replace '$\rightarrow$' as main connective by '$\leqslant$'
or (b) if '$\rightarrow$' is not the main connective, add '$\in T$' at the end of the formula-scheme.

(iv) For rules, replace '$\ldots, \ldots, \ldots \ldots \Rightarrow \ldots$' by 'If ... and ... and ...... then ...', and, for construction using (ii), for any positive integer k in an antecedent which does not occur in the consequent, replace 's' by 's^b/k' and put 'for all $b \in D$' at the end of such an antecedent.

(v) Replace expressions of the form, 'x_k does not occur free in A', by '$\mathscr{R}(A)$ is k-constant', where $\mathscr{R}(A)$ is the expression replacing the formula-scheme A.

THEOREM 3. (SOUNDNESS) *For any extension LQ of BBQ, if a formula A is a theorem of LQ then A is valid in the content semantics for LQ, obtained by adding the corresponding semantic postulates, constructed as above, for each additional axiom and rule used to obtain LQ from BBQ.*

PROOF. It suffices to show that each additional axiom-scheme of LQ is valid in the BBQ content semantics with its corresponding postulate added, and that each additional rule of LQ preserves validity in the BBQ content semantics with its corresponding semantic postulate added.

We deal firstly with the case where for the axiom-scheme or rule, construction step (i) is appropriate.

Let $\mathscr{C}(A_1, \ldots, A_n)$ be such additional axiom-scheme of LQ. Let $\mathscr{C}''(c_1, \ldots, c_n)$ be the corresponding semantic postulate, constructed as above. Let $\mathscr{C}'(c_1, \ldots, c_n)$ be constructed from $\mathscr{C}(A_1, \ldots, A_n)$ by applying (i) (a), (b), (c), (d) and (e), including the case where '$\rightarrow$' is the main connective of $\mathscr{C}(A_1, \ldots, A_n)$. We first show that

$$(*) \quad I(\mathscr{C}(A_1, \ldots, A_n)) = \mathscr{C}'(I(A_1), \ldots, I(A_n)),$$

for any interpretation I of the appropriate content semantics. This is the case because of the definition of interpretations and of the construction of $\mathscr{C}'$.

Let '$\rightarrow$' not be the main connective of $\mathscr{C}(A_1, \ldots, A_n)$. Then by the semantic postulate $\mathscr{C}''(c_1, \ldots, c_n)$, we have $\mathscr{C}''(I(A_1), \ldots, I(A_n))$, which is $\mathscr{C}'(I(A_1), \ldots, I(A_n)) \in T$. Hence by (*), $I(\mathscr{C}(A_1, \ldots, A_n)) \in T$, as required.

We then let '$\rightarrow$' be the main connective of $\mathscr{C}(A_1, \ldots, A_n)$. Here, let $\mathscr{C}(A_1, \ldots, A_n) = \mathscr{C}_1(A_1, \ldots, A_k) \rightarrow \mathscr{C}_2(A_{k+1}, \ldots, A_n)$, and correspondingly, $\mathscr{C}''(c_1, \ldots, c_n) = \mathscr{C}'_1(c_1, \ldots, c_k) \leqslant \mathscr{C}'_2(c_{k+1}, \ldots, c_n)$. Hence, $\mathscr{C}'_1(I(A_1), \ldots, I(A_k)) \leqslant \leqslant \mathscr{C}'_2(I(A_{k+1}), \ldots, I(A_n))$. By p7a, $\mathscr{C}'_1(I(A_1), \ldots, I(A_k)) \Rightarrow \mathscr{C}'_2(I(A_{k+1}), \ldots, I(A_n)) \in T$ and hence $\mathscr{C}'(I(A_1), \ldots, I(A_n)) \in T$ and, by (*), $I(\mathscr{C}(A_1, \ldots, A_n)) \in T$, as required.

We now consider an additional rule of LQ, $\mathscr{C}_1(A_{i_{1,1}}, \ldots, A_{i_{1,k_1}}), \ldots, \mathscr{C}_n(A_{i_{n,1}}, \ldots, A_{i_{n,k_n}}) \Rightarrow \mathscr{D}(A_{j_m})$. Let the corresponding semantic postulate be: If

$\mathscr{C}''_1(c_{i_{1,1}}, \ldots, c_{i_{1,k_1}}), \ldots, \mathscr{C}''_n(c_{i_{n,1}}, \ldots, c_{i_{n,k_n}})$ then $\mathscr{D}''(c_{j_1}, \ldots, c_{j_m})$. Let $\mathscr{C}'_1, \ldots, \mathscr{C}'_n, \mathscr{D}'$ be determined as above, in which case (*) holds again. Let $I(\mathscr{C}_1(A_{i_{1,1}}, \ldots, A_{i_{1,k_1}})) \in T, \ldots, I(\mathscr{C}_n(A_{i_{n,1}}, \ldots, A_{i_{n,k_n}})) \in T$. Hence, $\mathscr{C}'_1(I(A_{i_{1,1}}), \ldots, I(A_{i_{1,k_1}})) \in T, \ldots, \mathscr{C}'_n(I(A_{i_{n,1}}), \ldots, I(A_{i_{n,k_n}})) \in T$. As previously shown, $\mathscr{C}''_1(I(A_{i_{1,1}}), \ldots, I(A_{i_{1,k_1}})), \ldots, \mathscr{C}''_n(I(A_{i_{n,1}}), \ldots, I(A_{i_{n,k_n}}))$. By the semantic postulate, $\mathscr{D}''(I(A_{j_1}), \ldots, I(A_{j_m}))$ and, as above, $\mathscr{D}'(I(A_{j_1}), \ldots, I(A_{j_m})) \in T$ and $I(\mathscr{D}(A_{j_1}, \ldots, A_{j_m})) \in T$, as required.

We now deal with the case where, for the axiom-scheme or rule, construction step (ii) is appropriate. Let $\mathscr{C}(A_1^{k_1}, \ldots, A_n^{k_n})(x_{i_{1,1}}, \ldots, x_{i_{1,l_1}}, \ldots, x_{i_{n,1}}, \ldots, x_{i_{n,l_n}})$ be an additional axiom-scheme of LQ, where $A_j^{k_j}$ is a k_j-place formula, for each $j = 1, \ldots, n$, and $x_{i_{j,1}}, \ldots, x_{i_{j,l_j}}$ are the unsubstituted variables of $A_j^{k_j}$ or variables obtained from $A_j^{k_j}$ by a variable substitution, all of which are free in $\mathscr{C}(A_1^{k_1}, \ldots, A_n^{k_n})$, for $j = 1, \ldots, n$. Let $\mathscr{C}''(F_1^{k}, \ldots, F_n^{k_n})\ s\ (i_{1,1}, \ldots, i_{1,l_1}, \ldots, i_{n,1}, \ldots, i_{n,l_n})$ be the corresponding semantic postulate, constructed as above, where $F_j^{k_j} \in \mathscr{F}^{k_j}$, for all j, $s \in S$ and $i_{1,1}, \ldots, i_{n,l_n} \in N^+$. Let $\mathscr{C}'(F_1^{k_1}, \ldots, F_n^{k_n})$ be constructed from $\mathscr{C}(A_1^{k_1}, \ldots, A_n^{k_n})$ by applying (ii) (a), (b), (c), (d), (e), (f) and (g), including the case where '$\rightarrow$' is the main connective of $\mathscr{C}(A_1^{k_1}, \ldots, A_n^{k_n})$. We show:

(**) $I(\mathscr{C}(A_1^{k_1}, \ldots, A_n^{k_n})(x_{i_{1,1}}, \ldots, x_{i_{n,l_n}})) = \mathscr{C}'(F^{k_n}_{A_{1,I}}, \ldots, F^{k_n}_{A_{k,I}})\, s_I(i_{1,1}, \ldots, i_{n,l_n})$, for any interpretation I.

Much as for (*) above, this follows by the construction of $\mathscr{C}'$, Lemmas 3 and 7 and the definition of the functions $F_{A,I}$ for compound formulae A. Substitution of x_l for x_k in $A_j^{k_j}$ (say), registered in the list of variables $(x_{i_{1,1}}, \ldots, x_{i_{n,l_n}})$, takes effect in the list $(i_{1,1}, \ldots, i_{n,l_n})$ through the application of (ii) (a).

Let '$\rightarrow$' not be the main connective of $\mathscr{C}(A_1^{k_1}, \ldots, A_n^{k_n})(x_{i_{1,1}}, \ldots, x_{i_{n,l_n}})$. Then, by the above semantic postulate, $\mathscr{C}''(F_{A_1,I}, \ldots, F_{A_n,I})\ s_I(i_{1,1}, \ldots, i_{n,l_n})$, which is $\mathscr{C}'(F_{A_1,I}, \ldots, F_{A_n,I})\, s_I(i_{1,1}, \ldots, i_{n,l_n}) \in T$. Hence, by (**), $I(\mathscr{C}(A_1^{k_1}, \ldots, A_n^{k_n})(x_{i_{1,1}}, \ldots, x_{i_{n,l_n}})) \in T$, as required. Similarly, for the remaining case where '$\rightarrow$' is the main connective of $\mathscr{C}(A_1^{k_1}, \ldots, A_n^{k_n})(x_{i_{1,1}}, \ldots, x_{i_{n,l_n}})$, its interpretation I is a member of T, by similar steps to that used in the non-quantificational case.

In the case where a variable x_k (say) is stated as not occurring free in a formula A (say), in order to apply the corresponding semantic postulate we need to show that $\mathscr{R}(A)$ is k-constant, where $\mathscr{R}(A)$ is what replaces A in the construction. This is automatic by the way the variables x_i are replaced by the index i in the construction and by the definition of k-constancy.

Additional rules of LQ, for which construction step (ii) is appropriate, preserve membership of T, for all interpretations I, in a similar manner to those rules, for which construction step (i) is appropriate, given (**) and the reasoning used above for the axiom-schemes. This is provided that there are no variables which can occur free in any of the premises of the rule that are not free in the conclusion of the rule. In such a case, for a rule $A_1, \ldots, A_n \Rightarrow B$, where x_k can occur free in each of $A_{j_1}, \ldots, A_{j_m}$ for $\{j_1, \ldots, j_m\} \subseteq \{1, \ldots, n\}$ and x_k does not occur free in B, we show that if $I^b/x_k(A_{j_1}) \in T$, for all $b \in D, \ldots, I^b/x_k(A_{j_m}) \in T$, for all $b \in D$, and $I(A_i) \in T$ (for all

$i \in \{1, \ldots, n\} - \{j_1, \ldots, j_m\}$) then $I(B) \in T$, for any interpretation I. For this, we first show, with the same symbolism as for (**):

(**)′ $I^b/x_k(\mathscr{C}(A_1^{k_1}, \ldots, A_n^{k_n})(x_{i_{1,1}}, \ldots, x_{i_{n,l_n}})) = \mathscr{C}'(F^{k_1}_{A_{1,I}}, \ldots, F^{k_n}_{A_{n,I}})$
$s_I^b/k(i_{1,1}, \ldots, i_{n,l_n})$, for any interpretation I, and any $b \in D$.

This can be shown in the same manner as for (**), with use of Lemma 2. Then the familiar procedure for showing validity-preservation of such a rule is employed, assuming the appropriate semantic postulate, constructed using (iv).

THEOREM 4. (COMPLETENESS) *For any extension LQ of BBQ, if a formula A is valid in the content semantics for LQ (as described for Theorem 3), then A is a theorem of LQ.*

PROOF. We show that each additional semantic postulate, i.e. added to the semantics for BBQ to obtain the semantics for LQ, holds for the canonical model structure constructed for the logic BBQ, with the addition of the axiom-scheme or rule out of which the respective semantic postulate was constructed. As in the soundness proof, we deal firstly with axiom-schemes and rules for which construction step (i) is appropriate.

Let $\mathscr{C}(A_1, \ldots, A_n)$ be such an additional axiom-scheme, where we insert $A_1, \ldots, A_n$, which are sentences of LQ'. (LQ' is LQ with the individual constants of D_c added in the manner of BBQ'.) Let $\mathscr{C}''(c_1, \ldots, c_n)$ be the constructed semantic postulate, based on the axiom-scheme, $\mathscr{C}(A'_1, \ldots, A'_n)$, where $A'_1, \ldots, A'_n$ are formula-schemes of LQ. Let $\mathscr{C}'(c_1, \ldots, c_n)$ be constructed, as in the soundness proof. By definition of $|A|$, for sentences A of LQ', and the construction of $\mathscr{C}'$,

(+) $|\mathscr{C}(A_1, \ldots, A_n)| = \mathscr{C}'(|A_1|, \ldots, |A_n|)$.

Let '$\rightarrow$' not be the main connective of $\mathscr{C}(A_1, \ldots, A_n)$. Then, since $\vdash_{LQ'}\mathscr{C}(A_1, \ldots, A_n)$, $|\mathscr{C}(A_1, \ldots, A_n)| \in T_c$ and by (+), $\mathscr{C}'(|A_1|, \ldots, |A_n|) \in T_c$. By construction of $\mathscr{C}''$, $\mathscr{C}''(|A_1|, \ldots, |A_n|)$ holds, where $|A_j|$, for $j = 1, \ldots, n$, are arbitrary elements of C_c.

Let '$\rightarrow$' be the main connective of $\mathscr{C}(A_1, \ldots, A_n)$. We let $\mathscr{C}(A_1, \ldots, A_n) = \mathscr{C}_1(A_1, \ldots, A_k) \rightarrow \mathscr{C}_2(A_{k+1}, \ldots, A_n)$. As above, $|\mathscr{C}_1(A_1, \ldots, A_k)| \Rightarrow |\mathscr{C}_2(A_{k+1}, \ldots, A_n)| \in T_c$, and, by (+), $\mathscr{C}'_1(|A_1|, \ldots, |A_k|) \Rightarrow \mathscr{C}'_2(|A_{k+1}|, \ldots, |A_n|) \in T_c$. By p7(a), proved in Theorem 2, $\mathscr{C}'_1(|A_1|, \ldots, |A_k|) \leqslant \mathscr{C}'_2(|A_{k+1}|, \ldots, |A_n|)$, i.e. $\mathscr{C}''(|A_1|, \ldots, |A_n|)$, as required.

Let $\mathscr{C}_1(A_{i_{1,1}}, \ldots, A_{i_{1,k_1}}), \ldots, \mathscr{C}_n(A_{i_{n,1}}, \ldots, A_{i_{n,k_n}}) \Rightarrow \mathscr{D}(A_{j_1}, \ldots, A_{j_m})$ be an additional rule of LQ, with $A_{i_{1,1}}, \ldots, A_{i_{n,k_n}}, A_{j_1}, \ldots, A_{j_m}$, which are arbitrary sentences of LQ', inserted. The semantic postulate is as in the soundness proof. $\mathscr{C}'_1, \ldots, \mathscr{C}'_n, \mathscr{D}'$ are also constructed as in the soundness proof. Let $\mathscr{C}''_1(|A_{i_{1,1}}|, \ldots, |A_{i_{1,k_1}}|), \ldots, \mathscr{C}''_n(|A_{i_{n,1}}|, \ldots, |A_{n,k_n}|)$ all hold. By p7(a), $\mathscr{C}'_1(|A_{i_{1,1}}|, \ldots, |A_{i_{1,k_1}}|) \in T_c, \ldots, \mathscr{C}'_n(|A_{i_{n,1}}|, \ldots, |A_{i_{n,k_n}}|) \in T_c$, and hence, by (+), $|\mathscr{C}_1(A_{i_{1,1}}, \ldots, A_{i_{1,k_1}})| \in T_c, \ldots, |\mathscr{C}_n(A_{i_{n,1}}, \ldots, A_{i_{n,k_n}})| \in T_c$. By definition of T_c, $\vdash_{LQ'}\mathscr{C}_1(A_{i_{1,1}}, \ldots, A_{i_{1,k_1}}), \ldots, \vdash_{LQ'}\mathscr{C}_n(A_{i_{n,1}}, \ldots, A_{i_{n,k_n}})$, and hence, by the rule, $\vdash_{LQ'}\mathscr{D}(A_{j_1}, \ldots, A_{j_m})$. Then $|\mathscr{D}(A_{j_1}, \ldots, A_{j_m})| \in T_c$, $\mathscr{D}'(|A_{j_1}, \ldots, A_{j_m}|) \in T_c$ and $\mathscr{D}''(|A_{j_1}, \ldots, A_{j_m}|)$, as required.

We now proceed with axiom-schemes and rules for which construction (ii) is appropriate. We use the same terminology as in the soundness proof, where appropriate. So, let $\mathscr{C}(A_1^{k_1}, \ldots, A_n^{k_n})(s(i_{1,1}), \ldots, s(i_{1,l_1}), \ldots, s(i_{n,1}), \ldots, s(i_{n,l_n}))$ be an additional axiom-scheme of LQ, where $A_j^{k_j}$ is a k_j-place formula of LQ, for each $j = 1, \ldots, n$, and the elements $s(i_{1,1}), \ldots, s(i_{n,l_n})$ of D_c are inserted into the free argument places of $\mathscr{C}(A_1^{k_1}, \ldots, A_n^{k_n})$, some of which may be obtained by variable substitution into the $A_j^{k_j}$'s. We show:

$$(++) \quad |\mathscr{C}(A_1^{k_1}, \ldots, A_n^{k_n})(s(i_{1,1}), \ldots, s(i_{n,l_n}))| = \mathscr{C}'([A_1]^{k_1}, \ldots, [A_n]^{k_n})s(i_{1,1}, \ldots, i_{n,l_n}).$$

This follows by definition of $[A]^m$, for m-place formulae A of LQ', definition of $|A|$, for sentences A of LQ', the operations defined on C_c, and the construction of $\mathscr{C}'$. Individual variable substitution is taken into account in the manner indicated in the soundness proof. Then, by the axiom-scheme, definition of T_c and $(++)$, $\mathscr{C}'([A_1]^{k_1}, \ldots, [A_n]^{k_n})s(i_{1,1}, \ldots, i_{n,l_n}) \in T_c$. Whether '$\rightarrow$' is the main connective of the axiom-scheme or not, $\mathscr{C}''([A_1]^{k_1}, \ldots, [A_n]^{k_n})s(i_{1,1}, \ldots, i_{n,l_n})$, as required.

In the case where a variable x_k (say) is stated as not occurring free in a formula A (say), in order to use the axiom-scheme we need to show that this is the case using the corresponding k-constancy condition on $\mathscr{R}(A)$, where $\mathscr{R}(A)$ is what replaces A in the construction. This follows conversely to that similar step in the soundness proof.

The corresponding semantic postulates can also be shown for additional rules of LQ, much in the way that they are shown for those rules for which construction step (i) is used. The essential differences are the use of $(++)$ instead of $(+)$ and the use of reasoning steps that occur in the above proof of semantic postulates for axiom-schemes. As mentioned in the soundness proof, we have to also consider the special case where a variable x_k (say) can occur free in any of the premises of the rule but is not free in the conclusion. For this, we need:

$$(++)' \quad |\mathscr{C}(A_1^{k_1}, \ldots, A_n^{k_n})(s(i_{1,1}), \ldots, s(i_{n,l_n}))^b/x_k| =$$
$$\mathscr{C}'([A_1]^{k_1}, \ldots, [A_n]^{k_n})s^b/k(i_{1,1}, \ldots, i_{n,l_n}),$$

with symbolism as for $(++)$. This can be proved in the manner of $(++)$. Then the usual procedure can be employed for proving the corresponding semantic postulate using the rule, except that a substitution needs to be made in the free x_k-places in each of the premises of the rule, once established using the antecedents of the semantic postulate. As in the completeness argument for pIIa of Theorem 2, each of these free x_k-places are substituted by an element b of D_c which is new to all the premises. This can be done since each of these premises are provable in LQ' for all such substitutions from D_c. Then we replace all these substituted occurrences of b by a variable z (say) which is new to the proofs of these premises and apply the rule to obtain the conclusion, which is then used to derive the consequent of the semantic postulate in the usual way. This completes the proof of Theorem 4.

We still need to compare the quantificational semantic postulates, constructed as above, against those given for the specific extensions of BBQ in the introduction to (ii) of this paper. Q1, with x_k for x, has a constructed postulate, $\bigcap\{(F^n \Rightarrow G^m)^b/\{l_1, \ldots, l_q\}: b \in D\}\, s(i_{r_1}, \ldots, j_{r_{n+m-q}}) \leqslant (F^n \Rightarrow \bigcap\{G^{mb}/\{k_1, \ldots, k_l\}: b \in D\})\, s(i_1, \ldots, i_n, j_{p_1}, \ldots, j_{p_{m-1}})$, where $F_n s(i_1, \ldots, i_n)$ is k-constant. $l_1, \ldots, l_q$ are the argument places of $A \rightarrow B$ at which x_k is bound by the quantifier $(\forall x_k)$ and $k_1, \ldots, k_l$ are the argument places of B at which x_k is so bound. $(i_{r_1}, \ldots, j_{r_{n+m-q}})$ represent the free variable places in $(\forall x_k)(A \rightarrow B)$ and $(i_1, \ldots, i_n, j_{p_1}, \ldots, j_{p_{m-l}})$ represent those in $A \rightarrow (\forall x_k)B$. It can be seen by definition of the respective functions in $\mathscr{F}$, their combination with s and the definition of s^b/k, that $\bigcap\{(F^n \Rightarrow G^m)^b/\{l_1, \ldots, l_q\}: b \in D\}\, s(i_{r_1}, \ldots, j_{r_{n+m-q}}) = \bigcap\{(F^n \Rightarrow G^m)\, s^b/k(i_1, \ldots, i_n, j_1, \ldots, j_m): b \in D\}$ and $(F^n \Rightarrow \bigcap\{G^{mb}/\{k_1, \ldots, k_l\}: b \in D\})\, s(i_1, \ldots, i_n, j_{p_1}, \ldots, j_{p_{m-l}}) = F^n s(i_1, \ldots, i_n) \Rightarrow \bigcap\{G^m s^b/k(j_1, \ldots, j_m): b \in D\}$. This then will yield our semantic postulate q1. There is a similar transformation for q2. The advantages in expression can be seen for q1 and q2, but the constructed postulates are more systematic in that the forms $F^n s(i_1, \ldots, i_n)$ for some $F^n \in \mathscr{F}^n$, $s \in S$, $i_1, \ldots, i_n \in N^+$ are maintained wherever possible.

III. Reduced content semantics for BB^dQ and its extensions

We first present the motivation for introducing reduced modelling in the Routley-Meyer sentential relational semantics. There, the reduced modelling simplifies the semantics in that the set of regular set-ups 0 in which all theorems are true becomes a singleton set $\{T\}$ (cf. [22], p. 299). As a result, some of the semantic postulates simplify and the Entailment Lemma (cf. [22], p. 302) simplifies. Due to this latter simplification, the Detachment Rule, A, $A \rightarrow B \Rightarrow B$, and the Substitution of Equivalents Rule in either the form, $A \leftrightarrow B \Rightarrow \mathscr{C}(A) \leftrightarrow \mathscr{C}(B)$, or the form , $A \leftrightarrow B$, $\mathscr{C}(A) \Rightarrow \mathscr{C}(B)$, preserve truth in the base set-up T. This seems quite intuitive and, in fact, it seems unintuitive for any of these rules not to preserve truth in the base set-up, which represents the actual world as far as the definition of validity is concerned.

With respect to the content semantics of I and II, these rules do preserve membership of the truth-filter T, as can be seen by p6(a), p7(a), (b) and (c), especially. However, there are other properties that are satisfied in the base set-up of the Routley-Meyer semantics (with the constant-domain quantificational extension) that are not satisfied in the truth-filter of the above content semantics. These are the primeness and saturation properties, i.e. if $A \vee B$ is true at the base set-up T then either A is true or B is true at T, and if $(\exists x)\, A$ is true at T then A^d/x is true at T for some domain object d. These two properties seem quite intuitive for a truth-filter T to satisfy if such a set T does represent the body of truths. So, we form the reduced content semantics by adding semantic postulates corresponding to primeness and saturation. Both the reduced content semantics and the reduced Routley-Meyer semantics now have their key properties in common.

On the syntactic side, the meta-rules that are added to *BBQ* to yield BB^dQ (see below) produce a desirable duality between conjunction and disjunction and between universal and existential quantification in the context of rules (cf. [6], Chapter 1, for discussion). The completeness theorem, proved below, will be the first completeness test these meta-rules will have, since the Routley-Meyer constant-domain quantificational semantics is incomplete, at least for some strong relevant logics, and Fine's stratified domain semantics has not as yet been given a reduced modelling.

The system $BB^dQ = BBQ + \text{MR1} + \text{MR2}$, where:

MR1. If $A \Rightarrow B$ then $C \vee A \Rightarrow C \vee B$.
MR2. If $A \Rightarrow B$ then $(\exists x_k)A \Rightarrow (\exists x_k)B$.

MR1 and MR2 both carry the proviso that R6 is not used in the derivation $A \Rightarrow B$ to generalize on any free variables in A, except to prove a theorem.

The BB^dQ model structures (BB^dQ m.s.) for the reduced content semantics are set out exactly as for *BBQ* model structures, but with the two following additional semantic postulates:

p6(c) If $c \cup d \in T$ then $c \in T$ or $d \in T$.
p11(b) If $\bigcup \{F^n s^b/k(i_1, \ldots, i_n): b \in D\} \in T$ then $F^n s^b/k(i_1, \ldots, i_n) \in T$, for some $b \in D$.

It remains to show soundness and completeness of BB^dQ with respect to the BB^dQ content semantics. For soundness, we first show the following lemma:

LEMMA 8. *If $A \Rightarrow B$, provided* R6 *is not used to generalize on any free variable of A, except to prove a theorem, then, if $I(A) \in T$ then $I(B) \in T$, for any interpretation I of a BB^dQ m.s. M.*

PROOF. Let $A \Rightarrow B$ with the proviso and let $I(A) \in T$. Let $x_{k_1}, \ldots, x_{k_m}$ be the variables occurring free in the derivation, $A \Rightarrow B$, but are not free in A. Then $I^{b_1} x_{k_1} \ldots {}^{b_m}/x_{k_m}(A) \in T$, for all $b_1, \ldots, b_m \in D$. If C is an axiom, then $I^{b_1}/x_{k_1} \ldots {}^{b_m}/x_{k_m}(C) \in T$, for all $b_1, \ldots, b_m \in D$. All the rules except R6, MR1 and MR2 clearly preserve membership of T, for any interpretation I in M.

R6. If R6 is used to prove a theorem, $(\forall x_k)C$(say), then $I^b/x_k(C) = T$, for all $b \in D$, for all interpretations I in M, since C is a theorem. Then, by the soundness argument for R6, $I((\forall x_k)C) = T$, for all such I. If R6 is not used to prove a theorem then it is not used to generalize on any free variable of A and must generalize one of the variables $x_{k_1}, \ldots, x_{k_m}$, say x_{k_j}. Since we assume that $I^{b_1}/x_{k_1} \ldots {}^{b_m}/x_{k_m}(E) \in T$, for all $b_1, \ldots, b_m \in D$, where $E \Rightarrow (\forall x_{k_j})E$ is the application of R6, $I^{b_1}/x_{k_1} \ldots {}^{b_m}/x_{k_m}$ (without ${}^{b_j}/x_{k_j}$)$((\forall x_j)E) \in T$, by the soundness of R6. Hence, by redundancy, $I^{b_1}/x_{k_1} \ldots {}^{b_m}/x_{k_m}((\forall x_j)E) \in T$, for all $b_1, \ldots, b_m \in D$.

MR1. Let $C \vee F$ be derived from $C \vee E$ using MR1, where $E \Rightarrow F$, with the proviso. We inductively apply Lemma 8 to $E \Rightarrow F$, and so, if $I(E) \in T$ then $I(F) \in \mathrm{T}$, for any I in M. We let $I(C \vee E) \in T$. By Lemma 7, $I(C) \cup I(E) \in T$. By p6(c), $I(C) \in T$ or $I(E) \in T$. Hence, $I(C) \in T$ or $I(F) \in T$, and, by p3(a), 3(b) and 6(a), $I(C \vee F) \in T$.

MR2. Let $(\exists x_k)E$ be derived from $(\exists x_k)C$, using MR2, where $C \Rightarrow E$ as above. Again, by Lemma 8, if $I(C) \in T$ then $I(E) \in T$, for any I in M. Let $I((\exists x_k)C) \in T$. By Lemma 7, $\bigcup\{I^b/x_k(C): b \in D\} \in T$ and, by Lemmas 2 and 3, $\bigcup\{F^n_{C,I}s^b_I/k(i_1, \ldots, i_n): b \in D\} \in T$. By p11(b), $F^n_{C,I}s^b_I/k(i_1, \ldots, i_n) \in T$, for some $b \in D$. By Lemmas 2 and 3, $I^b/x_k(C) \in T$, for some $b \in D$. By Lemma 8, $I^b/x_k(E) \in T$, for this $b \in D$. Hence, $F^n_{E,I}s^b_I/k(j_1, \ldots, j_m) \in T$, for this b, using Lemmas 2 and 3. Then, by p9a and 6a, $\bigcup\{F^m_{E,I}s^b_I/k(j_1, \ldots, j_m): b \in D\} \in T$ and $\bigcup\{I^b/x_k(E): b \in D\} \in T$ and hence $I((\exists x_k)E) \in T$.

Thus, $I^{b_1}/x_{k_1} \ldots {}^{b_m}/x_{k_m}(B) \in T$, for all $b_1, \ldots, b_m \in D$, and, in particular, $I(B) \in T$, as required.

THEOREM 5. (SOUNDNESS) *For all formulae A, if A is a theorem of BB^dQ then A is valid in the BB^dQ content semantics.*

PROOF. Using the proof of Theorem 1, all the axioms of BB^dQ are valid and all rules of BB^dQ preserve validity. We need check only the two meta-rules MR1 and MR2. Moreover, by the proof of Lemma 8, both MR1 and MR2 can be seen to preserve validity.

We follow the completeness preliminaries and the completeness proof given for Theorem 2. We indicate changes where appropriate. As before, we introduce the added individual constants, $a_1, a_2, \ldots, a_n, \ldots$, which form the domain D_c, and extend BB^dQ to BB^dQ' by modifying A6 to A6′ and adding these constants to the syntax. The essential difference in this proof is that the use of the set of theorems of BBQ' in defining $|A|$, T_c and $|A| \leqslant |B|$ is replaced by the set T' of BB^dQ' formulae, where T' satisfies the following properties: For formulae A, B, C, D of BB^dQ',

(i) if $\vdash_{BB^dQ'} A \to B$ and $A \in T'$ then $B \in T'$.
(ii) if $A \in T'$ and $B \in T'$ then $A \& B \in T'$,
(iii) if $A \in T'$ and $A \to B \in T'$ then $B \in T'$,
(iv) if $A \to B \in T'$ and $C \to D \in T'$ then $B \to C \to. A \to D \in T'$.
(v) if $A \to B \in T'$ and $A \to C \in T'$ then $A \to B \& C \in T'$,
(vi) if $A \to {\sim}B \in T'$ then $B \to {\sim}A \in T'$,
(vii) if $(\forall x_k)(A \to B) \in T'$ then $A \to (\forall x_k)B \in T'$, where x_k is not free in A,
(viii) if $A \vee B \in T'$ then $A \in T'$ or $B \in T'$.
(ix) if $A^a/x_k \in T'$, for all $a \in D_c$, then $(\forall x_k)A \in T'$,
(x) if $(\exists x_k)A \in T'$ then $A^a/x_k \in T'$, for some $a \in D_c$,
(xi) if $\vdash_{BB^dQ'} A$ then $A \in T'$, and
(xii) $A'' \notin T'$, where $A'' = A'^{a_{i_1}}/x_{i_1} \ldots {}^{a_{i_n}}/x_{i_n}$, $x_{i_1}, \ldots, x_{i_n}$ being the free individual variables of A', and A' being our arbitrary non-theorem of BB^dQ. So A'' is A' with its free individual variables replaced by their canonical valuations under I_c.

T' would then have sufficient properties to substitute for the set of theorems of BB^dQ' and have additional properties, viz. (viii) and (x), which would enable p6(c) and p11(b) to be shown. Before constructing T', we show its use in defining $|A|$, T_c and $|A| \leqslant |B|$.

$|A| =_{df} \{C : C$ is a sentence of BB^dQ' and $A \leftrightarrow C \in T'\}$.

$T_c =_{df} \{|A| : A$ is a sentence of BB^dQ' and $A \in T'\}$.

$|A| \leqslant |B| =_{df} A \to B \in T'$.

Further, the definitions of $|A|^*$, $|A| \cap |B|$, $|A| \cup |B|$, $|A| \Rightarrow |B|$ are independent of the particular formula A and B chosen to represent $|A|$ and $|B|$, due to Substitution of Equivalents in T'. The definitions of $\bigcap \{[A]^{n\,b}/\{j_1, \ldots, j_m\} (a_{i_1}, \ldots, a_{i_n}) : b \in D_c\}$ and $\bigcup \{[A]^{n\,b}/\{j_1, \ldots, j_m\} (a_{i_1}, \ldots, a_{i_n}) : b \in D_c\}$ are also independent of the particular formula A chosen to represent $[A]^n$, due to property (ix) and to the following:

If $(\forall x_k)(A \leftrightarrow B) \in T'$ then $(\forall x_k)A \leftrightarrow (\forall x_k)B \in T'$, and
If $(\forall x_k)(A \leftrightarrow B) \in T'$ then $(\exists x_k)A \leftrightarrow (\exists x_k)B \in T'$.

We proceed to construct T'. The procedure follows the pattern for the construction of prime extensions to theories, as in [22], Chapter 4, modified to take in the quantificational properties given for T' by (ix) and (x) above. This modification will be based on the one used by Gabbay in [11]. In order to ensure that T' is also closed under the rules of BB^dQ (except R6), we need to employ a special notion of deducibility in the construction process, illustrated on pp. 337-9 of [22], this notion having the respective rules built into it.

We introduce *D-*BB^dQ'*-theories* and the notions of primeness, richness, saturation and regularity for them. A *D-*BB^dQ'*-* theory S is a set of formulae of BB^dQ' satisfying the following conditions:

(i) if $\vdash_{BB^dQ'} A \to B$ and $A \in S$ then $B \in S$,
(ii) if $A \in S$ and $B \in S$ then $A \& B \in S$,
(iii) if $A \in S$ and $A \to B \in S$ then $B \in S$,
(iv) if $A \to B \in S$ and $C \to D \in S$ then $B \to C \to . A \to D \in S$,
(v) if $A \to B \in S$ and $A \to C \in S$ then $A \to B \& C \in S$,
(vi) if $A \to {\sim} B \in S$ then $B \to {\sim} A \in S$,
(vii) if $(\forall x_k)(A \to B) \in S$ then $A \to (\forall x_k)B \in S$, where x_k is not free in A.

A *D-*BB^dQ'-theory S is *prime* iff, whenever $A \vee B \in S$, $A \in S$ or $B \in S$.

A *D-*BB^dQ'-theory S is *rich* iff, whenever $A^a/x_k \in S$ for all $a \in D_c$, $(\forall x_k)A \in S$.

A *D-*BB^dQ'-theory S is *saturated* iff, whenever $(\exists x_k)A \in S$, $A^a/x_k \in S$ for some $a \in D_c$.

A D-BB^dQ'-theory S is *regular* iff, whenever A is a theorem of BB^dQ', $A \in S$.

We need to establish a prime, rich, saturated and regular *D-*BB^dQ'-theory T' such that $A'' \notin T'$. We introduce a notion of derivability based on the conditions (i)-(vii) satisfied by *D-*BB^dQ'-theories. A formula B of BB^dQ' is *D-*BB^dQ'*-derivable* from a formula A of BB^dQ', written $A \Rightarrow_{BB^dQ'} B$, iff B is derivable from A by successive applications of the following rules.

(I) $A \Rightarrow B$, where $\vdash_{BB^dQ'} A \to B$,
(II) $A, B \Rightarrow A \& B$,

(III) $A,\ A \to B \Rightarrow B$,
(IV) $A \to B,\ C \to D \Rightarrow B \to C \to .\ A \to D$,
(V) $A \to B,\ A \to C \Rightarrow A \to B \& C$,
(VI) $A \to \sim B \Rightarrow B \to \sim A$,
(VII) $(\forall x_k)(A \to B) \Rightarrow A \to (\forall x_k)B$, where x_k is not free in A,
(VIII) $C \vee A \Rightarrow C \vee B$, where $A \Rightarrow B$,
(IX) $(\forall x_k)A \Rightarrow (\forall x_k)$B, where $A \Rightarrow B$,
(X) $(\exists x_k)A \Rightarrow (\exists x_k)B$, where $A \Rightarrow B$.

A set T of formulae of BB^dQ' is *D-BB^dQ'-derivable* from a set S of formulae of BB^dQ', written $S \Rightarrow_{BB^dQ'} T$, iff, for some $A_1, \ldots, A_m \in S$ and some $B_1, \ldots, B_n \in T$, $A_1 \& \ldots \& A_m \Rightarrow_{BB^dQ'} B_1 \vee \ldots \vee B_n$. A pair of sets of formulae $\langle S, T \rangle$ of BB^dQ' is *D-BB^dQ'-maximal* iff

(1) T is not D-BB^dQ'-derivable from S,
(2) $S \cup T =$ set of all formulae of BB^dQ' (and $S \cap T = \emptyset$, since this follows from (1)),
(3) if $(\forall x_k)A \in T$ then $A^a/x_k \in T$, for some $a \in D_c$, and
(4) if $(\exists x_k)A \in S$ then $A^a/x_k \in S$, for some $a \in D_c$.

LEMMA 9. *If $\langle S, T \rangle$ is D-BB^dQ-maximal then S is a prime, rich and saturated D-BB^dQ'-theory.*

PROOF. The proof follows the method used on p.307 of [22].

LEMMA 10. (EXTENSION) *If the set U, which is the singleton, $\{A''\}$, is not D-BB^dQ'-derivable from the set T of all theorems of BB^dQ then there is a D-BB^dQ'-maximal pair $\langle T', U' \rangle$ such that $T \subseteq T'$ and $U \subseteq U'$.*

PROOF. We enumerate all the formulae of BB^dQ': $A_1, A_2, \ldots, A_n, \ldots$ We then define sets T_i and U_i of formulae of BB^dQ' as follows:

(i) $T_o = T$ and $U_o = U(=\{A''\})$.
(ii) $T_{i+1} = T_i$ and $U_{i+1} = U_i \cup \{A_{i+1}\}$, if $T_i \cup \{A_{i+1}\} \Rightarrow_{BB^dQ'} U_i$ and A_{i+1} is not of the form $(\forall x_k)A$, for any k.
(iii) $T_{i+1} = T_i$ and $U_{i+1} = U_i \cup \{(\forall x_k)A, A^a/x_k\}$, if $T_i \cup \{A_{i+1}\} \Rightarrow_{BB^dQ'} U_i$ and $A_{i+1} = (\forall x_k)A$, for some k, a being the first new individual constant not occurring in $\{T_i, U_i, A_{i+1}\}$.
(iv) $T_{i+1} = T_i \cup \{A_{i+1}\}$ and $U_{i+1} = U_i$ if $T_i \cup \{A_{i+1}\} \not\Rightarrow_{BB^dQ'} U_i$ and A_{i+1} is not of the form $(\exists x_k)A$, for any k.
(v) $T_{i+1} = T_i \cup \{(\exists x_k)A, A^a/x_k\}$ and $U_{i+1} = U_i$, if $T_i \cup \{A_{i+1}\} \not\Rightarrow_{BB^dQ'} U_i$ and $A_{i+1} = (\exists x_k)A$, for some k. a is again the first new individual constant not occurring in $\{T_i, U_i, A_{i+1}\}$.

There are always constants not occurring in $\{T_i, U_i, A_{i+1}\}$, since T has no constants, U has finitely many and each of T_i and U_i includes only finitely many more than T_{i-1} and U_{i-1}, respectively.

We let $T' = \bigcup_i T_i$ and $U' = \bigcup_i U_i$. Then clearly $T \subseteq T'$, $U \subseteq U'$ and conditions (2), (3) and (4) of D-BB^dQ'-maximality for $\langle T', U' \rangle$ all hold. We establish condition (1). We prove this by induction on i. U_o is not D-BB^dQ'-derivable from T_o, by assumption. We prove the induction step by contraposition. So, we let U_{i+1} be D-BB^dQ'-derivable from T_{i+1}.

Let $T_i \cup \{A_{i+1}\} \Rightarrow_{BB^dQ'} U_i$. So, by (ii) and (iii) above, $T_{i+1} = T_i$ and $U_{i+1} = U_i \cup \{A_{i+1}\}$ or $U_{i+1} = U_i \cup \{(\forall x_k) A, A^a/x_k\}$, according as A_{i+1} is not or is of the form $(\forall x_k) A$, for some k. So, $T_i \Rightarrow_{BB^dQ'} U_i \cup \{A_{i+1}\}$ or $T_i \Rightarrow_{BB^dQ'} U_i \cup \{(\forall x_k) A, A^a/x_k\}$, accordingly.

We show that if $T_i \Rightarrow_{BB^dQ'} U_i \cup \{(\forall x_k) A, A^a/x_k\}$ then $T_i \Rightarrow_{BB^dQ'} U_i \cup \{(\forall x_k) A\}$. Let $B_1, \ldots, B_m \in T_i$ and $C_1, \ldots, C_n \in U_i$ such that $B_1 \& \ldots \& B_m \Rightarrow_{BB^dQ'} C_1 \vee \ldots \vee C_n \vee (\forall x_k) A \vee A^a/x_k$. Since a is new to $B_1, \ldots, B_m$, $C_1, \ldots, C_n$, $(\forall x_k) A$, we can replace a in the derivation by an individual variable x_l which is new to the derivation and obtain $B_1 \& \ldots \& B_m \Rightarrow_{BB^dQ'} C_1 \vee \ldots \vee C_n \vee (\forall x_k) A \vee A^{x_l}/x_k$. By rule (IX), $(\forall x_l)(B_1 \& \ldots \& B_m) \Rightarrow$ $\Rightarrow_{BB^dQ'} (\forall x_l)(C_1 \vee \ldots \vee C_n \vee (\forall x_k) A \vee A^{x_l}/x_k)$, and hence, since x_l is not free in $B_1, \ldots, B_m$, $C_1, \ldots, C_n$ and $(\forall x_k) A$, $B_1 \& \ldots \& B_m \Rightarrow_{BB^dQ'} C_1 \vee \ldots \vee C_n \vee (\forall x_k) A \vee (\forall x_l)(A^{x_l}/x_k)$. Since $\vdash_{BB^dQ'} (\forall x_l)(A^{x_l}/x_k) \to (\forall x_k) A$, $B_1 \& \ldots \& B_m \Rightarrow_{BB^dQ'} C_1 \vee \ldots$ $\ldots \vee C_n \vee (\forall x_k) A$ and hence $T_i \Rightarrow_{BB^dQ'} U_i \cup \{A_{i+1}\}$, which now holds whether A_{i+1} has the form $(\forall x_k) A$, for some k, or not.

Combining $T_i \cup \{A_{i+1}\} \Rightarrow_{BB^dQ'} U_i$ and $T_i \Rightarrow_{BB^dQ'} U_i \cup \{A_{i+1}\}$, we can derive $T_i \Rightarrow_{BB^dQ'} U_i$, as is required, in the manner of [22], Chapter 4, using rule (VIII).

Let $T_i \cup \{A_{i+1}\} \Rightarrow_{BB^dQ'} U_i$. So, by (iv) and (v) above, $U_{i+1} = U_i$ and $T_{i+1} = T_i \cup \{A_{i+1}\}$ or $T_{i+1} = T_i \cup \{(\exists x_k) A, A^a/x_k\}$, according as A_{i+1} is not or is of the form $(\exists x_k) A$, for some k. Since $T_{i+1} \Rightarrow_{BB^dQ'} U_{i+1}$, $T_i \cup \{A_{i+1}\} \Rightarrow_{BB^dQ'} U_i$ or $T_i \cup \{(\exists x_k) A, A^a/x_k\} \Rightarrow_{BB^dQ'} U_i$. Since $T_i \cup \{A_{i+1}\} \not\Rightarrow_{BB^dQ'} U_i$, $T_i \cup \{(\exists x_k) A, A^a/x_k\} \Rightarrow_{BB^dQ'} U_i$. Let $B_1, \ldots, B_m \in T_i$ and $C_1, \ldots, C_n \in U_i$ such that $B_1 \& \ldots \& B_m \& (\exists x_k) A \& A^a/x_k \Rightarrow_{BB^dQ'} C_1 \vee \ldots \vee C_n$. Since a is new to $B_1, \ldots, B_m, C_1, \ldots, C_n$ and $(\exists x_k) A$, we can replace a in the derivation by an individual variable x_l which is new to the derivation and obtain $B_1 \& \ldots$ $\ldots \& B_m \& (\exists x_k) A \& A^{x_l}/x_k \Rightarrow_{BB^dQ'} C_1 \vee \ldots \vee C_n$. By rule (X), $(\exists x_1)(B_1 \& \ldots$ $\ldots \& B_m \& (\exists x_k) A \& A^{x_l}/x_k) \Rightarrow_{BB^dQ'} (\exists x_l)(C_1 \vee \ldots \vee C_n)$, and, since x_l is not free in $B_1, \ldots, B_m$, $C_1, \ldots, C_n$ and $(\exists x_k) A$, $B_1 \& \ldots \& B_m \& (\exists x_k) A \& (\exists x_l)(A^{x_l}/x_k) \Rightarrow$ $\Rightarrow_{BB^dQ'} C_1 \vee \ldots \vee C_n$. Since $\vdash_{BB^dQ'} (\exists x_k) A \to (\exists x_l)(A^{x_l}/x_k)$, $B_1 \& \ldots \& B_m \&$ $\& (\exists x_k) A \Rightarrow_{BB^dQ'} C_1 \vee \ldots \vee C_n$ and hence $T_i \cup \{A_{i+1}\} \Rightarrow_{BB^dQ'} U_i$, contradicting $T_i \cup \{A_{i+1}\} \not\Rightarrow_{BB^dQ'} U_i$.

Then $T_i \cup \{A_{i+1}\} \Rightarrow_{BB^dQ'} U_i$ and, as already proved, $T_i \Rightarrow_{BB^dQ'} U_i$. This establishes the induction step, since then $T_i \not\Rightarrow_{BB^dQ'} U_i$ implies $T_{i+1} \not\Rightarrow_{BB^dQ'} U_{i+1}$. Then $T' \not\Rightarrow_{BB^dQ'} U'$, since if $T' \Rightarrow_{BB^dQ'} U'$ then $D_1 \& \ldots \& D_m \Rightarrow_{BB^dQ'} E_1 \vee \ldots \vee E_n$, for some $\{D_1, \ldots, D_m\} \subseteq T'$ and $\{E_1, \ldots, E_n\} \subseteq U'$, and hence $T_i \Rightarrow_{BB^dQ'} U_i$, for some i such that $\{D_1, \ldots, D_m\} \subseteq T_i$ and $\{E_1, \ldots, E_n\} \subseteq U_i$. Thus, condition (1) of D-BB^dQ'-maximality is established for $\langle T', U' \rangle$ and the lemma is proved.

LEMMA 11. *$\{A''\}$ is not D-BB^dQ'-derivable from the set of theorems of BB^dQ.*

PROOF. Let $T \Rightarrow_{BB^dQ'} \{A''\}$, where T is the set of theorems of BB^dQ. Replace each individual constant occurring in this derivation by a distinct individual variable new to the proof. Thus, we obtain $T \Rightarrow_{BB^dQ'} \{A'^{x_{j_1}}/x_{i_1} \ldots {}^{x_{j_n}}/x_{i_n}\}$, where $x_{i_1} \ldots x_{i_n}$ are the original free variables of A' and $x_{j_1}, \ldots, x_{j_n}$ are the substituted variables new to the derivation. Given that all formulae in the derivation are of BB^dQ, each of the rules (I)–(X) preserve theoremhood in BB^dQ since they each constitute derived rules of BB^dQ or derived meta-rules of BB^dQ. Hence $A'^{x_{j_1}}/x_{i_1} \ldots {}^{x_{j_n}}/x_{i_n}$ is a theorem of BB^dQ, and by R6 and A6, A' is also a theorem, contradicting our assumption.

Lemma 12. *The set T' of formulae of BB^dQ', constructed by Lemma* 10, *satisfies the properties* (i)-(xii).

Proof. By Lemmas 11 and 10, $\langle T', U' \rangle$ is D-BB^dQ'-maximal, $T \subseteq T'$ and $U \subseteq U'$, where T and U are as given. By Lemma 9, T' is a prime, rich and saturated D-BB^dQ'-theory, and thus T' satisfies properties (i)-(x). Since T' and U are disjoint, $A'' \notin T'$, which yields property (xii). For the remaining property (xi), we need to show that T' contains all theorems of BB^dQ'. Let $A(a_{i_1}, \ldots, a_{i_n})$ be such a theorem. Replace all individual constants in the proof of $A(a_{i_1}, \ldots, a_{i_n})$ by variables new to the proof, obtaining $A(x_{j_1}, \ldots, x_{j_n})$, a theorem of BB^dQ. By R6, $(\forall x_{j_1}, \ldots, x_{j_n})A(x_{j_1}, \ldots, x_{j_n}) \in T$ and, since $T \subseteq T'$, $(\forall x_{j_1}, \ldots, x_{j_n})A(x_{j_1}, \ldots, x_{j_n}) \in T'$. By A6′ and property (i), $A(a_{i_1}, \ldots, a_{i_n}) \in T'$, as required.

Theorem 6 (Completeness). *For all formulae A, if A is valid in the BB^dQ content semantics then A is a theorem of BB^dQ.*

Proof. Using the above preliminaries and our arbitrary non-theorem A' of BB^dQ, we proceed to check out the canonical BB^dQ model structure M_c with canonical interpretation I_c. These are set up exactly as for Theorem 2 with T', constructed above, used in lieu of the set of theorems of BB^dQ'. The closure conditions follow, as shown for Theorem 2. The (old) semantic postulates p1(a)-p7(c) all hold as a result of properties (ii)-(vi) and (xi) of T'. This also applies to postulates, p8(a), 9(a) and 10. Property (ix) is also needed for p8(b), 9(b) and 11(a), in lieu of R6. Postulate p6(c) requires property (viii) and p11(b) requires (x). Thus M_c is a BB^dQ model structure. I_c again satisfies: $I_c(A'(x_{i_1}, \ldots, x_{i_n})) = |A'(a_{i_1}, \ldots, a_{i_n})|$. To show that $I_c(A'(x_{i_1}, \ldots, x_{i_n})) \notin T_c$, we need to show that $|A''| \notin T_c$, i.e. $A'' \notin T'$. But this is property (xii) of T'.

Hence, A' is not valid in M_c and A' is invalid in the BB^dQ content semantics, as required.

IV. Reduced content semantics for extensions of BB^dQ.

Generally, the results of II for unreduced content semantics for extensions of BBQ also apply here for reduced content semantics for extensions of BB^dQ. The general soundness proof follows as in II, as here we just have the two additional semantic postulates p6(c) and p11(b). The general completeness proof also follows as in II, except that we use appropriate properties of T', which is constructed in the same manner as in III, but is an extension of L^dQ instead of BB^dQ, with the additional rules of L^dQ being preserved in T'.

However, in the special case where there is a variable x_k (say) which can occur free in any of the premises of a rule but is not free in the conclusion of the rule, it is easier to restate the rule by generalizing with respect to x_k each premise in which it can occur free and then construct the semantic postulate for this restated rule. This avoids the problem of T' not preserving R6 and the restated rule is deductively equivalent in L^dQ anyway.

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