Fourth-Order Gravitational Potential Based on Quantum Field Theory.

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There have been many attempts (1) to understand the gravitational interaction in terms of quantum field theory in flat Minkowskian space-time in analogy to the electromagnetic interaction. Since in the case of the electromagnetic interaction there is excellent agreement between the quantized theory and experiment (2), we also believe that the gravitational interaction can be and should be understood by means of quantum field theory. This is the starting point of our discussions.

The motion of the perihelion of a planet is described in Einstein's theory of gravitation by the Hamiltonian

$$H=\frac{p^2}{2m}-\frac{p^4}{8c^2m^3}-\frac{kmM}{r}-\frac{3kMp^2}{2c^2mr}+\frac{k^2M^2}{2c^2r^2}\,, \tag{1}$$

if we choose harmonic co-ordinates (3) and expand the Hamiltonian up to the order c^{-2}. The extension of eq. (1) to the two-body problem is given by (4)

$$H=\frac{1}{2}\left(\frac{p_1^2}{m_1}+\frac{p_2^2}{m_2}\right)-\frac{1}{8c^2}\left(\frac{p_1^4}{m_1^3}+\frac{p_2^4}{m_2^3}\right)-\frac{km_1m_2}{r}-\frac{km_1m_2}{2c^2r}\left[3\left(\left(\frac{p_1}{m_1}\right)^2+\left(\frac{p_2}{m_2}\right)^2\right)-\right.$$
$$\left.-7\frac{\boldsymbol{p}_1\cdot\boldsymbol{p}_2}{m_1m_2}-\frac{(\boldsymbol{p}_1\boldsymbol{r})(\boldsymbol{p}_2\boldsymbol{r})}{m_1m_2r^2}\right]+\frac{k^2m_1m_2(m_1+m_2)}{2c^2r^2}\,, \tag{2}$$

from which we can obtain the Einstein-Infeld-Hoffmann equation. In this letter we want to discuss the question whether or not Hamiltonians (1) and (2) are the same in quantum theory.

(1) See, *e.g.*, S. N. GUPTA: *Proc. Phys. Soc.*, A **65**, 608 (1952); W. E. THIRRING: *Ann. of Phys.*, **16**, 96 (1961); R. P. FEYNMAN: *Acta Phys. Polon.*, **24**, 697 (1963); B. S. DEWITT: *Phys. Rev.*, **162**, 1195, 1239 (1967).
(2) See, *e.g.*, S. J. BRODSKY and S. D. DRELL: *Ann. Rev. Nucl. Sci.* (to be published).
(3) V. FOCK: *The Theory of Space-Time and Gravitation*, 2nd Revised Edition (New York, 1964).
(4) See, *e.g.*, L. D. LANDAU and E. M. LIFSHITZ: *The Classical Theory of Fields*, Revised Second Edition (New York, 1962).

Here we want to point out that there seems to exist ([5]) an erroneous belief that only three diagrams contribute to the classical process. Although in eqs. (1) and (2) the terms linear in k correspond to a tree diagram (the Born term), the quadratic term in k corresponds to fourth-order diagrams each of which contains a closed loop; the latter is a « radiative correction » term. Since the quantum theory of gravitation is unrenormalizable by standard criteria, almost nothing is known about how to extract finite and physically meaningful radiative corrections from the results in higher orders ([6]). We will extract a meaningful term as a fourth-order potential.

On the other hand there seems to exist an argument that the quantum theory should coincide with the c-number theory in the classical limit because both are invariant with respect to the general co-ordinate transformation. As has been pointed out by FOCK ([3]), however, the invariance with respect to general co-ordinate transformations is not a strong constraint. If we choose harmonic co-ordinates, there remains only Lorentz invariance. We can indeed prove that the equation of motion derived from eq. (2) is Lorentz covariant up to the approximation considered above *irrespective of the numerical factor of the last term of eq.* (2). It will be shown further that in the case of the massless Yang-Mills field which possesses a non-Abelian invariant group like the gravitational field the quantum theory does not coincide with the c-number theory in the classical limit.

For a long time it was believed that the experimental value and the theoretical value resulting from Einstein's theory for the motion of the perihelion of Mercury were in excellent agreement. However, DICKE and GOLDENBERG ([7]) have claimed that there is a discrepancy between the values due to the solar oblateness. Therefore we want to discuss also the question whether or not we can explain the Dicke-Goldenberg experiment in the quantum theory.

Let us calculate explicitly the fourth-order potential from quantum theory in order to answer the questions mentioned above in the following steps: i) we construct the Lagrangian; ii) we calculate the fourth-order S-matrix by the covariant formalism ([1]) in the momentum representation; iii) we integrate over the energy variable of a closed loop by means of the contour method; iv) we expand the result in terms of the inverse of the two masses m_1 and m_2; v) by Fourier transformation we obtain the r-representation; vi) we finally extract the potentials of the form k^2m^3/c^2r^2. From the resulting potential we must subtract the second Born term using an expansion up to the order c^{-2}.

First of all we must choose the type of field by which we represent the matter. We calculate the potential in the cases of the scalar field and the Dirac field.

We begin with constructing the Lagrangian. Since the explicit form of the Lagrangian in the case of the scalar particle is well known ([1,8]) and is too lengthy to be written down here, we do not write it explicitly, but only comment that it can be obtained by two different approaches. One is to start from the general co-ordinate–transformation–invariant Lagrangian density

$$\mathscr{L} = -\frac{1}{16\pi k}\sqrt{g}\,R - \frac{1}{2}\sqrt{g}\,(\partial_\mu\varphi\partial_\nu\varphi g^{\mu\nu} + m^2\varphi^2)\,, \tag{3}$$

([5]) See, *e.g.*, E. CORINALDESI: *Proc. Phys. Soc.*, A **69**, 189 (1956). His calculations are incorrect. We obtain using his linear theory $\ddot{\boldsymbol{x}} = k^2m_2\boldsymbol{x}/c^2r^4\,(3m_2+4m_1)$ instead of his eq. (28), $\ddot{\boldsymbol{x}} = k^2m_2\boldsymbol{x}/c^2r^4\cdot(4m_2+5m_1)$. There are many other papers which state the belief mentioned in the text.

([6]) See, *e.g.*, B. S. DEWITT: *Phys. Rev.*, **162**, 1239 (1967), especially p. 1246.

([7]) R. H. DICKE and H. M. GOLDENBERG: *Phys. Rev. Lett.*, **18**, 313 (1967).

([8]) Even if we use « a new improved energy-momentum tensor », the final results do not charge. See C. G. CALLAN jr., S. COLEMAN and R. JACKIW: MIT preprint CTP No. 113.

and expand it in powers of the gravitational constant by substituting

$$g_{\mu\nu} = \delta_{\mu\nu} + \varkappa h_{\mu\nu} \qquad \text{(expansion } F\text{)}, \tag{4}$$

or

$$\sqrt{g}\, g^{\mu\nu} = \delta_{\mu\nu} + \varkappa h_{\mu\nu} \qquad \text{(expansion } G\text{)}, \tag{5}$$

where $h_{\mu\nu}$ is the field of the graviton and $32\pi\varkappa^2 = k$ (9).

The other way (10) is to start from Minkowskian space and to construct the Lagrangian by means of gauge invariance. This invariance is necessary for the theory to be compatible with Lorentz invariance (11). If we take into account self-interactions the invariant group is *uniquely* enlarged from the Abelian gauge group into a non-Abelian group.

Although the two approaches are equivalent, the latter seems to be more close to the method in the case of electromagnetic interaction. The explicit form of the interaction Lagrangian density in the case of the Dirac field is given by the latter approach as follows:

$$\mathscr{L} = \frac{\varkappa}{2} h_{\mu\nu}\{\overline{\Psi}(\!(\gamma_\mu \partial_\nu)\!)\psi - \lambda\delta_{\mu\nu}\overline{\Psi}[(\!(\gamma_\lambda\partial_\lambda)\!) + m]\psi\} - \tag{6}$$

$$- \frac{\varkappa^2}{32}\left[3\overline{\Psi}(\!(\gamma_\varrho\,\partial_\lambda)\!)\psi h_{\varrho\sigma}h_{\lambda\sigma} - \frac{1}{2}\overline{\Psi}(\gamma_\mu\gamma_\nu\gamma_\lambda - \gamma_\lambda\gamma_\nu\gamma_\mu)\psi h_{\mu\varrho}h_{\nu\varrho,\lambda} - 2\lambda\overline{\Psi}(\!(\gamma_\varrho\partial_\lambda)\!)h_{\varrho\lambda}h_{\sigma\sigma}\right],$$

where $(\!(\gamma_\varrho\partial_\lambda)\!) = \frac{1}{4}(\gamma_\varrho\overrightarrow{\partial}_\lambda - \overleftarrow{\partial}_\lambda\gamma_\varrho + \gamma_\lambda\overrightarrow{\partial}_\varrho - \overleftarrow{\partial}_\varrho\gamma_\lambda)$ and λ is an arbitrary constant. This Lagrangian corresponds to the expansion F in the former approach and is given *uniquely* except for the λ dependence.

Using these Lagrangians and the well-known propagators we can calculate the S-matrix. Some of the fourth-order Feynman diagrams are given in Fig. 1. Here only *a*), *b*), *c*) and *d*) contribute to the potential considered (12). There are no ultraviolet or infra-red divergences in the quantities which we want to calculate. This is to be expected; the former should not exist if the theory works, since the fourth-order diagrams are the lowest-order ones which contribute to the potential of the type r^{-2}, and the latter also should not exist in the order c^{-2} since the form of radiation of gravitational waves contains a factor c^{-5}.

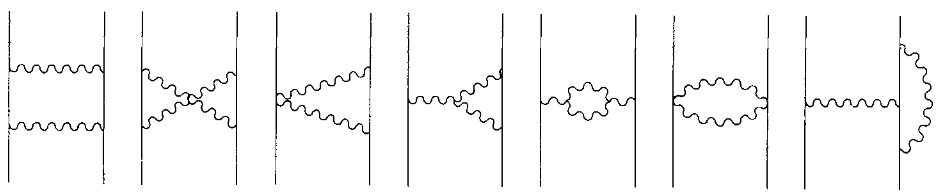

Fig. 1. – Some of the fourth-order diagrams. The wavy line represents the graviton and the solid line matter.

(9) The normalization constant of the propagator of the graviton is different from the case of $16\pi\varkappa^2 = k$.

(10) See, *e.g.*, W. Wyss: *Helv. Phys. Acta*, **38**, 469 (1965); S. N. Gupta: *Phys. Rev.*, **96**, 1683 (1954).

(11) S. Weinberg: *Phys. Rev.*, **138**, B 988 (1965).

(12) The diagram which contains a loop of fictitious quanta introduced by Feynman does not contribute to the potential considered here.

After lengthy calculations we obtain the spin-averaged results given in Table I. We summarize the final results for the fourth-order potential by

$$V = \frac{1}{2}\frac{k^2}{c^2 r^2} m_1 m_2 (m_1 + m_2)\,, \tag{7}$$

for both the scalar field and the Dirac field.

TABLE I. – *The contributions from graphs to the potential. Each value represents the numerical factor of the potential of the form* $m_1 m_2 (m_1 + m_2) k^2/c^2 r^2$. We enumerate in column a' the value to which we subtract the second-Born term from the contribution from graph a).

		a'	b	c	d	Total
scalar	F	$\frac{3}{2}$	2	-4	1	$\frac{1}{2}$
	G	$\frac{3}{2}$	2	-1	-2	$\frac{1}{2}$
Dirac	F	$\frac{1}{4}+\lambda$	$\frac{3}{4}+\lambda$	$-\frac{3}{2}-2\lambda$	1	$\frac{1}{2}$
	G	$\frac{1}{4}+\lambda$	$\frac{3}{4}+\lambda$	$+\frac{3}{2}-2\lambda$	-2	$\frac{1}{2}$

Before discussing the results, we make a few comments. The total results do not depend on the expansion F or G as should be the case and they do not depend on the constant λ in the case of the Dirac field. It is worth mentioning that the terms quadratic in k and trilinear in m are given exactly by eq. (7); terms such as $k^2 m^3/c^2 r^2 \times f(h/pr)$ do not appear, where f is an arbitrary function and not a constant.

We can see from eq. (7) that the resulting potential is the same as the last term of eq. (2). Thus *we can extract a finite and physically meaningful radiative correction in spite of the unrenormalizability of the theory*; the last term of eq. (2) can be understood as a radiative correction in our approach. This term coincides with the c-number theory, while it cannot explain the Dicke-Goldenberg experiment. Even if we take into account the effects of other interactions such as the strong interaction, the situation is still the same because of the gravitational Ward identity for the two-graviton vertex. On the other hand the scalar-tensor theory ([13]) cannot be consistently quantized at least in the perturbation approach ([14]). Therefore *if* the Dicke-Goldenberg experiment and its interpretation are correct, we cannot explain the experiment on our fundamental assumption that quantum theory is able to describe correctly classical processes in the classical limit. This is a very serious problem.

Similarly we can calculate the fourth-order potential in the cases of the electromagnetic field and the massless Yang-Mills field. The Hamiltonian obtained is the same as Darwin's formula ([4]) in the case of the electromagnetic field, while in the massless Yang-Mills field it differs from the corresponding « classical » one. In general, in the case of fields introduced artificially, the fourth-order potential obtained in the quantum theory differs from the « classical » one. Thus *the correspondence principle does not hold in general in the quantum field theory.* It is remarkable that it holds both in the case of the electromagnetic field and the gravitational field which really exist in nature.

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([13]) C. BRANS and R. H. DICKE: *Phys. Rev.*, **124**, 925 (1961).
([14]) Y. IWASAKI: *Phys. Rev. D*, **2**, 2255 (1970); *Progr. Theor. Phys.*, **44**, 1376 (1970).