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SCIENTIFIC AND TECHNICAL SECTION

EFFICIENCY OF THE METHOD OF SPECTRAL VIBRODIAGNOSTICS FOR FATIGUE DAMAGE OF STRUCTURAL ELEMENTS. PART 1. LONGITUDINAL VIBRATIONS, ANALYTICAL SOLUTION

V. V. Matveev

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On the basis of an analysis of the asymptotics of higher approximations to solutions of a differential equation of longitudinal vibrations of a macroelastic body whose nonlinearity is explained by the hysteresis and "breathing" of the damaged material in the process of its cyclic deformation under conditions of tension–compression, we estimate the applicability of the spectral characteristics of strain cycles of the body under conditions of its natural or resonance vibrations to the diagnostics of fatigue damage of structural elements. It is shown that if nonlinearity is explained solely by the hysteresis of the material, then even the most representative first higher odd harmonics revealed in the vibration spectrum have very small amplitudes and the analysis of their variations is, in fact, similar to the analysis of the variations of the logarithmic decrement of vibrations (regarded as an integral characteristic of the dissipative properties of the material of the investigated element) but has lower resolution. At the same time, the discontinuities of the material appearing in the process of strain cycling are characterized by formation of numerous microcracks or a macrocrack and result in more clear detection of the representative constant component and even harmonics (mainly the second and fourth ones) in the spectrum of strain cycles of a vibration-insulated structural element in the process of its natural or resonance vibrations. These characteristics may serve as sensitive and efficient parameters for detection of fatigue damage to the material.

Introduction. We consider a method of spectral vibrodiagnostics based on analysis of harmonic components (amplitudes and phases) of strain cycles of the material or displacements of a deformable body. The spectral analysis of strain cycles of the material used for assessment of the degree of damage to its structure is known as the method of higher harmonics (MHH). As indicated in [1], application of this method shows that, in the process of strain cycling, in particular, of ferromagnetic materials, the amplitudes of the third, fifth, and seventh harmonics behave as nonmonotonic functions of time in agreement with the fundamental periods of the generalized diagram of fatigue fracture. The suggestion has been made to use the factor of higher harmonics determined according to the spectral characteristics of strain cycles of a harmonically loaded material as a parameter of inelasticity of the material and evaluate the degree of fatigue damage to this material by analyzing the behavior of this parameter [2]. In the literature, one can also find works devoted to investigation of the possibility of detection of "breathing" transverse cracks in rotating multiple-seated rotors by analyzing the presence and variations of the amplitudes and phases of the third and fourth harmonics [3] or the second, third, and fourth harmonics [4] in the spectrum of vibratory displacements of the rotor.

The presence of higher harmonics revealed by spectral analysis of strain cycles of the material or displacements of a deformable structural element under harmonic loading is explained by distortion of strain cycles or displacements of the investigated macroelastic body caused by nonlinearities in its elastic and dissipative properties, which must also manifest themselves in the process of natural and resonance vibrations of the body. In this case, to detect fatigue damage to a body subjected to strain cycling, one must evaluate the effect of nonlinearity

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caused solely by the imperfect elasticity of the material and exclude effects induced by the fasteners of the body and its interaction with other bodies by satisfying the requirements traditionally imposed on the procedures used for analysis of dissipative properties of materials [5,6].

It seems reasonable to compare the informativity of results of harmonic analysis of actual strain cycles of a macroelastic body in the process of its natural or resonance vibrations with results obtained by using integral characteristics of the elastic and inelastic properties of the body, namely, the resonance frequency and the logarithmic decrement of vibrations, both in the stage of dispersed fatigue damage and in the stage of development of fatigue cracks. In addition, it is necessary to determine the type of nonlinearity responsible for the appearance of individual harmonics and predict their possible amplitudes.

Computational Procedure. As is known [7,8], imperfect elasticity of a material leads to fairly small deviations from Hooke's law. In this case, single-frequency longitudinal vibrations of a macroelastic rodlike element with distributed parameters can be described, for a given mode of its natural vibrations, by the differential equation

$$M\frac{d^2u}{dt^2} + K\left[u + \varepsilon f\left(u, \frac{du}{dt}\right)\right] = \varepsilon Q_0 \sin \omega t, \tag{1}$$

where M is the generalized mass, K is the generalized stiffness, Q_0 is the amplitude of the driving force, $\varepsilon f(u, du/dt)$ is a nonlinear function that takes into account the imperfect elasticity of the material, and ε is a small parameter reflecting the fact that the deviations from linearity are small.

According to asymptotic methods of nonlinear mechanics [7,8], the third approximation to the general solution of Eq. (1) for both natural ($Q_0 = 0$) and resonance vibrations with frequency $\omega \approx \omega_0$, where $\omega_0 = \sqrt{K/M}$, can be represented in the form

$$u = a \cos \varphi + \varepsilon u_1(a, \varphi) + \varepsilon^2 u_2(a, \varphi), \tag{2}$$

where the amplitude a and the phase $\varphi = \omega t + \psi$ can be found, with sufficiently high accuracy, from the equations of the first approximation

$$\frac{da}{dt} = \varepsilon A_1(a) - \frac{\varepsilon Q_0 \cos \psi}{M(\omega_0 + \omega)} \quad \text{and} \quad \frac{d\varphi}{dt} = \omega_0 + \varepsilon B_1(a) + \frac{\varepsilon Q_0 \sin \psi}{M(\omega_0 + \omega)a} \tag{3}$$

and the 2π-periodic functions of the angle φ in the second ($u_1(a, \varphi)$) and third ($u_2(a, \varphi)$) approximations are given by the expressions

$$u_1(a, \varphi) = -g_0(a) + \sum_{i=3,5,\ldots} \frac{g_i(a) \cos i\varphi + h_i(a) \sin i\varphi}{i^2 - 1}, \tag{4}$$

$$u_2(a, \varphi) = -s_0(a) + \sum_{j=2,4,\ldots} \frac{s_j(a) \cos j\varphi + p_j(a) \sin j\varphi}{j^2 - 1}. \tag{5}$$

Here,

$$\begin{aligned} A_1(a) &= \frac{\omega_0}{2\pi} \int_0^{2\pi} f_0(a, \varphi) \sin \varphi \, d\varphi, \\ B_1(a) &= \frac{\omega_0}{2\pi a} \int_0^{2\pi} f_0(a, \varphi) \cos \varphi \, d\varphi, \end{aligned} \tag{6}$$

$$\begin{cases} g_0(a) = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} f_0(a, \varphi) d\varphi, \\ g_i(a) = \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} f_0(a, \varphi) \cos i\varphi \, d\varphi, \\ h_i(a) = \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} f_0(a, \varphi) \sin i\varphi \, d\varphi, \end{cases} \tag{7}$$

$$\begin{cases} s_0(a) = \frac{1}{2\pi} \int_0^{2\pi} f_1(a, \varphi) d\varphi, \\ s_j(a) = \frac{1}{\pi} \int_0^{2\pi} f_1(a, \varphi) \cos j\varphi \, d\varphi, \\ p_j(a) = \frac{1}{\pi} \int_0^{2\pi} f_1(a, \varphi) \sin j\varphi \, d\varphi, \end{cases} \tag{8}$$

where

$$f_0(a, \varphi) = f(a \cos \varphi, -a\omega \sin \varphi), \tag{9}$$

$$f_1(a, \varphi) = u_1 f_u'(a, \varphi) + \left[A_1 \cos \varphi - aB_1 \sin \varphi + \omega \frac{\partial u_1}{\partial \varphi}\right] f_{u'}'(a, \varphi)$$

$$+ \left(aB_1^2 - A_1 \frac{dA_1}{da}\right) \cos \varphi + \left(2A_1B_1 + A_1 \frac{dB_1}{da} a\right) \sin \varphi - 2\omega A_1 \frac{\partial^2 u_1}{\partial a \partial \varphi} - 2\omega B_1 \frac{\partial^2 u_1}{\partial \varphi^2}. \tag{10}$$

Furthermore, it is necessary to determine the nonlinear function (9), which characterizes the imperfect elasticity of the investigated element. As the final result, we arrive at the following expression for the general solution:

$$u = a \cos \varphi + u_{st} + \sum_{i=3,5,\ldots} a_i \cos i\varphi + \sum_{i=3,5,\ldots} b_i \sin i\varphi + \sum_{j=2,4,\ldots} a_j \cos j\varphi + \sum_{j=2,4,\ldots} b_j \sin j\varphi. \tag{11}$$

Choice of the Nonlinear Function.

1. First, we consider the possibility of taking into account imperfect elasticity of a material that is responsible for its dissipative properties. Note that hysteresis dissipation of energy, independent of the frequency of strain cycling of the material, is typical of the majority of structural materials, especially metallic. For this type of imperfect elasticity, we can represent the nonlinear function $\varepsilon f(u, du/dt)$ in a fairly general form as [7]

$$\varepsilon \overset{\leftrightarrows}{f}\left(u, \frac{du}{dt}\right) = \mp \sum_{n=2,3,\ldots} \frac{\nu_n}{n} \left[(a \pm u)^n - 2^{n-1} a^n\right] \tag{12}$$

and, hence,

$$\varepsilon f_0(a,\varphi) = \varepsilon \overset{\leftrightarrows}{f_0}(a \cos \varphi) = \mp \sum_{n=2,3,\ldots} \frac{\nu_n a^n}{n} \left[(1 \pm \cos \varphi)^n - 2^{n-1}\right], \tag{13}$$

where n and ν_n are parameters of the hysteresis loop, which depend on the dissipative properties of the material.

For subsequent analysis, we represent function (13), which can describe longitudinal, bending, and torsional vibrations, in terms of the corresponding components

$$\delta_n = \delta_n(a) = \frac{2^{n+1}(n-1)\nu_n}{n(n+1)} a^{n-1}$$

of the experimentally established amplitude dependence of the logarithmic decrement of vibrations of the structural element under consideration $\delta(a) = \sum_{n=2,3,\ldots} \delta_n(a)$:

$$\varepsilon \overset{\leftrightarrows}{f_0}(a \cos \varphi) = \mp \sum_{n=2,3,\ldots} \frac{(n+1)\delta_n a}{2^{n+1}(n-1)} \left[(1 \pm \cos \varphi)^n - 2^{n-1}\right]. \tag{14}$$

In the general case, the (possible) nonlinearity of viscous resistance can be represented in the form

$$\varepsilon f\left(u, \frac{du}{dt}\right) = \sum_{m=0,2,3,\dots} \alpha_m \omega_0^{-2} \frac{du}{dt} \left|\frac{du}{dt}\right|^{m-1} \tag{15}$$

and, therefore,

$$\varepsilon f_0(a, \varphi) = \sum_{m=0,2,3,\dots} \alpha_m a^m \omega_0^{m-2} \left|\sin\varphi\right|^{m-1} \sin\varphi, \tag{16}$$

where α_m is a coefficient that depends on the ductile properties of the material and m is the degree of nonlinearity. Note that $m = 0$ corresponds to the case of dry friction with relatively large displacements of contacting surfaces and $m = 1$ corresponds to linear viscous resistance, which is not responsible for the appearance of higher harmonics.

Function (16) can be expressed in terms of the components

$$\delta_m = \delta_m(\omega_0, a) = \alpha_m \omega_0^{m-2} a^{m-1} \frac{m!!}{(m+1)!!} \left\{2\left[1 + (-1)^m\right] + \pi\left[1 - (-1)^m\right]\right\}$$

of the logarithmic decrement of vibrations of the investigated element $\delta(a, \omega) = \sum_{m=0,2,3,\dots} \delta_m(a, \omega)$:

$$\varepsilon f_0(a, \varphi) = \sum_{m=0,2,3,\dots} \frac{(m+1)!!\delta_m a |\sin\varphi|^{m-1} \sin\varphi}{m!!\{2[1 + (-1)^m] + \pi[1 - (-1)^m]\}}. \tag{17}$$

2. We now consider possible deviations from perfect elasticity of the material that may result in nonlinearity of its elastic characteristics. It should be noted that the hysteresis function (13) presented above also describes certain deviations from linear elasticity that, according to Eq. (3), may be responsible for the experimentally corroborated dependence of the resonance frequency on the amplitude of vibrations realized via the function $B_1(a)$ and the parameters ν_n and n of the hysteresis loop [6]. In this case, the nonlinear component of the elastic characteristic of the vibrating system is given by the expression

$$\Delta f_0(a, \varphi) = \frac{1}{2}\left[\overleftarrow{f_0}(a, \varphi) + \overrightarrow{f_0}(a, \varphi)\right] = \sum_{n=2,3,\dots} \frac{\nu_n a^n}{2n}\left[(1 - \cos\varphi)^n - (1 + \cos\varphi)^n\right]. \tag{18}$$

Function (18) is symmetric $[\Delta f_0(a, \varphi) = -\Delta f_0(a, -\varphi)]$ and, hence, ignores any possible difference between the tensile and compressive strain half cycles.

However, it is known [9] that tensile and compressive half cycles induce different fatigue changes in the material. One must also mention the phenomenon of "breathing" of a transverse macrocrack, i.e., its successive opening and closing in the process of rotation of a massive rotor [10]. Here, we assume that the entire cross section of a crack can be regarded as a single whole only in the case where the crack lies in the region of compression. Similarly, according to the method of detection of fatigue cracks proposed in [11] and based on analysis of the difference between the compliances of specimens under tensile-compressive loading in the half cycles of different signs, it is supposed that changes in the compliance of the system are promoted by propagation of the crack in tensile half cycles. At the same time, in compressive half cycles, the fatigue crack is completely closed and does not propagate, and the compliance of the system remains constant. In [12], it was indicated that photograms of fatigue crack opening displacements may serve as experimental corroboration of the accepted scheme of changes in the compliance of the material.

This enables us to assume that the appearance of all types of discontinuities of the material (submicro-, micro-, and, especially, macrocracks) caused by accumulation of microplastic deformations in the process of strain cycling must lead not only to changes in the resonance frequency of the deformed element (i.e., in its elastic characteristic averaged over a cycle) but also to asymmetry of this characteristic, i.e., to a difference between the values of stiffness in tensile and compressive half cycles (more precisely, to lower values of stiffness observed under conditions of tension).

By analogy with simulation of a "breathing" macrocrack in a rotor, this phenomenon can be described by a function of "breathing" of discontinuities formed in the material or simply by a "breathing function" of the material, i.e., in terms of the difference between the values of stiffness in tensile and compressive half cycles, namely,

$$K = K(u) = K_0\left[1 - 0.5\alpha\left(1 + \frac{u}{|u|}\right)\right] = K_0[1 - 0.5\alpha(1 + \operatorname{sign} u)], \tag{19}$$

where α is the (small) relative difference between the values of stiffness of the deformed element under conditions of compression, or prior to the appearance of discontinuities (K_0), and under conditions of tension (K_0'), i.e.,

$$\alpha = \frac{K_0 - K_0'}{K_0}. \tag{20}$$

The breathing function is associated with the nonlinear function (9) represented in the form

$$\varepsilon f_0(a, \varphi) = -0.5\alpha a\left(1 + \frac{\cos\varphi}{|\cos\varphi|}\right)\cos\varphi = -0.5\alpha a\{1 + \operatorname{sign}(\cos\varphi)\}\cos\varphi. \tag{21}$$

By substituting relation (21) in expression (6) we obtain $B_1(a) = 0.25\alpha\omega_0$. According to (3), this enables us to determine the resonance ($\psi = 0$) frequency ω_0' of the element for the indicated nonlinear function $\omega_0' = (1 - 0.25\alpha)\omega_0$. Hence, we arrive at a simple relation for evaluation of the parameter α via the relative difference β between the frequencies of natural or resonance vibrations of the element prior to ω_0 and after ω_0' the appearance of discontinuities in the material:

$$\alpha = 4\beta, \tag{22}$$

where

$$\beta = \frac{\omega_0 - \omega_0'}{\omega_0}. \tag{23}$$

Note that, by using the expression for the natural frequency of a nonlinear system with asymmetric elastic characteristic from [13], we obtain the following relation for α:

$$\alpha = \frac{4\beta}{(1 + \beta)^2}. \tag{24}$$

It is clear that for small β, the values of α given by relations (22) and (24) are close.

It should be taken into account that the indicated breathing function of the material (19) is true for longitudinal vibrations under conditions of symmetric strain cycling of elastic elements. In the presence of the static component of strains (displacements) u_{st}, the breathing function (21) takes the form

$$\varepsilon f_0(a, \varphi) = -0.5\alpha a\left(1 + \frac{\Delta + \cos\varphi}{|\Delta + \cos\varphi|}\right)\cos\varphi,$$

where

$$\Delta = \frac{u_{st}}{a}.$$

For $u_{st} \geq a$ $(\Delta \geq 1)$, we have $\varepsilon f_0(a, \varphi) = -\alpha a \cos\varphi$, whereas for $u_{st} \leq -a$ $(\Delta \leq -1)$, we can write $\varepsilon f_0(a, \varphi) = 0$. It is clear that, in these cases, we do not detect any asymmetry of the elastic characteristic in the process of vibrations.

A somewhat different situation is observed for discontinuities of the material appearing in the process of bending and torsional vibrations of elastic elements. Thus, if a defect of the material subjected to bending is formed on one side of the neutral layer, then the breathing function of the material has the same form. However, this situation is possible only in the special case of the formation of one or several main cracks. For the period of diffused fatigue damage to the material, the elastic characteristic of the deformed element seems to be the same in the half cycles of its positive and negative displacements, although its values differ slightly from the values recorded prior to damage.

In this case, when the element passes through the equilibrium position in the process of vibrations, we observe jumpwise displacements of the neutral layer either toward compressed fibers or in the direction of the curvature of the direct axis of the element. However, this problem should be a subject of special investigation.

Numerical Results.

1. *Nonlinearity Described by the Hysteresis Function (14).* By substituting (14) in (7), in view of (4), we obtain

$$\varepsilon u_1(a, \varphi) = \sum_{i=3,5,\ldots} \alpha_i a \cos i\varphi + \sum_{i=3,5,\ldots} \beta_i a \sin i\varphi. \tag{25}$$

Here,

$$\alpha_i = \frac{1}{i^2-1} \sum_{n=3,4,\ldots} \frac{(n+1)\delta_n}{2^{n+i}(n-1)} \sum_{k=0}^{\frac{n-i-e}{2}} \frac{\binom{n}{k}\binom{n-k}{i+k}}{2^{2k}} \quad (n \geq i), \tag{26}$$

$$\beta_i = \frac{1}{i^2-1} \sum_{n=2,3,\ldots} \frac{(n+1)\delta_n}{2^{n-1}(n-1)\pi} \left\{ \frac{2^{n-1}-1}{i} + (-1)^{(i+1)/2} \sum_{q=2,4,\ldots}^{n-\chi} \frac{n(n-1)\ldots(n-q+1)}{q!} \right.$$

$$\left. \times \left[\frac{1}{q+1} + \sum_{k=1}^{\frac{i-1}{2}} (-1)^k \frac{(i^2-1)(i^2-3^2)\ldots[i^2-(2k-1)^2]}{(2k)!(2k+q+1)} \right] \right\}, \tag{27}$$

where

$$\chi = \frac{1-(-1)^n}{2}, \quad e = \frac{1+(-1)^n}{2}, \quad \binom{p}{\eta} = \frac{p(p-1)\ldots(p-\eta+1)}{1\cdot 2\cdot 3\ldots\cdot \eta}, \quad \binom{p}{0} = 1.$$

For the first three higher harmonics and $n = 2, 3, \ldots, 6$, the general solution of the differential equation (1) in the second approximation takes the form

$$u(a, \varphi) = a[\cos\varphi - (0.003906\delta_3 + 0.00651\delta_4 + 0.00824\delta_5 + 0.007263\delta_6)\cos 3\varphi$$

$$- (0.0000061\delta_5 + 0.0001171\delta_6)\cos 5\varphi + (0.007958\delta_2 + 0{,}007958\delta_3 + 0.006947\delta_4 + 0.005684\delta_5$$

$$+ 0.004421\delta_6)\sin 3\varphi + (0.0003789\delta_2 + 0.0003789\delta_3 + 0.0004912\delta_4 + 0.00063155\delta_5 + 0.00075555\delta_6)\sin 5\varphi$$

$$+ (0.0000631\delta_2 + 0.0000631\delta_3 + 0.0000733\delta_4 + 0.0000861\delta_5 + 0.000102\delta_6)\sin 7\varphi]. \tag{28}$$

We now simplify solution (28) by neglecting the difference between the coefficients of δ_n and taking into account the fact that $\sum \delta_n$ determines the logarithmic decrement of vibrations of the investigated element $(\delta = \sum \delta_n)$, whose maximum value specified by the dissipative properties of the material is approximately equal to 1% for the majority of structural materials if the maximum amplitude of stresses excludes the possibility of damage to the material in the process of vibrodiagnostics and, hence, is lower than its fatigue strength. As a result, we obtain

$$u(a, \varphi) = a(\cos\varphi - 0.0000065\theta_{3,\dots,6}\cos 3\varphi - 0.0000010\theta_{5,6}\cos 5\varphi + 0.000066\sin 3\varphi$$

$$+ 0.0000053\sin 5\varphi + 0.00000078\sin 7\varphi), \tag{29}$$

where

$$\theta_{3,\dots,6} \approx \frac{\sum_{m=3}^{6}\delta_m}{\delta} < 1, \qquad \theta_{5,6} \approx \frac{\sum_{n=5}^{6}\delta_m}{\delta} < 1. \tag{30}$$

It should also be noted that the constant component and even harmonics may appear in the spectrum for the indicated type of nonlinearity. To reveal these components, one must deduce explicit expressions for $f_1(a, \varphi)$, $s_0(a)$, $s_j(a)$, and $p_j(a)$ by using relations (10) and (8) and find the solution of the equation in the third approximation, i.e., function (5) responsible for the terms of order of smallness ε^2.

For the nonlinear function (14), we obtain

$$\varepsilon^2 u_2(a, \varphi) = \alpha_0 a + \sum_{j=2,4,\dots} \alpha_j a\cos j\varphi + \sum_{j=2,4\dots} \beta_j a\sin j\varphi. \tag{31}$$

Here,

$$\alpha_0 = \sum_{i=3,5,\dots}\left\{-\alpha_i^{(n)}\sum_{i=2,3,\dots}\frac{n(n+1)\delta_n}{2^n(n-1)}C^{(n)}_{v=i} + \frac{2}{\pi}\beta_i^{(n)}\sum_{n=2,3,\dots}\frac{n(n+1)\delta_n}{2^n(n-1)}\left[\frac{1}{i} - D^{(n)}_{v=i}\right]\right\},$$

$$\alpha_j = \frac{1}{j^2-1}\sum_{i=3,5,\dots}\left\{-\alpha_i^{(n)}\sum_{n=2,3,\dots}\frac{n(n+1)\delta_n}{2^{n+1}(n-1)}\left(C^{(n)}_{v=|(i-j)|} + C^{(n)}_{v=i+j}\right)\right.$$

$$\left.+\frac{\beta_i^{(n)}}{\pi}\sum_{n=2,3,\dots}\frac{n(n+1)\delta_n}{2^n(n-1)}\left[\frac{2i}{i^2-j^2} - \frac{i-j}{|(i-j)|}D^{(n)}_{v=|(i-j)|} - D^{(n)}_{v=i+j}\right]\right\},$$

$$\beta_j = \frac{1}{j^2-1}\sum_{i=3,5,\dots}\left\{\frac{\alpha_i^{(n)}}{\pi}\sum_{n=2,3,\dots}\frac{n(n+1)\delta_n}{2^n(n-1)}\left[\frac{2j}{j^2-i^2} - \frac{j-i}{|(j-i)|}D^{(n)}_{v=|(j-i)|} - D^{(n)}_{v=j+i}\right]\right.$$

$$\left.-\beta_i^{(n)}\sum_{n=2,3,\dots}\frac{n(n+1)\delta_n}{2^{n+1}(n-1)}\left(C^{(n)}_{v=|(i-j)|} + C^{(n)}_{v=i+j}\right)\right\}, \tag{32}$$

where

$$C_v^{(n)} = \frac{1}{2^v}\sum_{k=0}^{\frac{n-v-\bar{\chi}}{2}}\binom{n-1}{k}\binom{n-k-1}{v+k}\frac{1}{2^{2k}}, \quad \bar{\chi} = \frac{3-(-1)^n}{2},$$

$$D_v^{(n)} = (-1)^{(v+1)/2}\sum_{q=2,4,\dots}^{n-\bar{e}}\frac{(n-1)(n-2)\dots(n-q)}{q!}\left[\frac{1}{q+1} + \sum_{k=1}^{\frac{v-1}{2}}(-1)^k\frac{(v^2-1)(v^2-3^2)\dots[v^2-(2k-1)^2]}{(2k)!(2k+q+1)}\right],$$

$$\bar{e} = \frac{3+(-1)^n}{2}. \tag{33}$$

For approximate assessment of the values of the constant component $a_0 = \alpha_0 a$ and the amplitudes of the even harmonics $a_j = \alpha_j a$ and $b_j = \beta_j a$, we can omit the amplitudes $a_i = \alpha_i a$ and $b_i = \beta_i a$ of harmonics whose order is higher than three because, according to relation (28), they are weaker than a_3 at least by an order of magnitude ($a_5 \approx 0{,}1 a_3$ and $a_7 \approx 0{,}01 a_3$). Moreover, since the coefficients of the components of the amplitudes a_i for different n are close to each other, we can, for the sake of simplicity, restrict ourselves to analysis of a single component of the amplitudes a_j and a_3 ($a_j^{(n)}$ and $a_3^{(n)}$) for any fixed n, i.e., omit the summation over n in expressions (26) and (32).

Thus, if $n = 3$ or $n = 4$, then, for the lowest harmonic ($j = 2$), we find

$$\alpha_2^{(3)} = (-0.125\alpha_3^{(3)} + 0.13945\beta_3^{(3)})\delta_3, \quad \beta_2^{(3)} = -(0.07276\alpha_3^{(3)} + 0.125\beta_3^{(3)})\delta_3,$$
$$\alpha_2^{(4)} = (-0.52083\alpha_3^{(4)} + 0.12631\beta_3^{(4)})\delta_4, \quad \beta_2^{(4)} = (0.37894\alpha_3^{(4)} - 0.13021\beta_3^{(4)})\delta_4, \tag{34}$$

and, for the constant component,

$$\alpha_0^{(3)} = 0.382\beta_3^{(3)}\delta_3, \quad \alpha_0^{(4)} = (-0.05208\alpha_3^{(4)} + 0.21221\beta_3^{(4)})\delta_4. \tag{35}$$

Since $\delta_n < 1\%$, it follows from the example presented above that the amplitudes of the second harmonic are smaller than the amplitudes of the third harmonic by more than two orders of magnitude, i.e., they are comparable with the amplitudes of the seventh harmonic and, hence, can be neglected.

2. *Dissipative Nonlinearity Described by Expression (17).* By substituting relations (17) and (7) in (4) and integrating the expression obtained, we obtain

$$\varepsilon u_1(a, \varphi) = \sum_{i=3,5,\ldots} \lambda_i \, a \cos i\varphi, \tag{36}$$

where

$$\lambda_i = -\sum_{m=0,2,3,\ldots} \frac{(1-m)(3-m)\ldots[(i-2)-m]\delta_m}{\pi(i^2-1)(i+m)(i+m-2)\ldots[i+m-(i-3)]}. \tag{37}$$

For the first three harmonics ($i = 3, 5, 7$) and $m = 0, 2, 3, \ldots, 6$, we arrive at the following general solution of the differential equation (1) in the second approximation:

$$u = a[\cos\varphi + (-0.013263\delta_0 + 0.0079577\delta_2 + 0.013263\delta_3 + 0.01705\delta_4 + 0.019894\delta_5 + 0.022105\delta_6)\sin 3\varphi$$
$$+ (-0.0026525\delta_0 + 0.0003789\delta_2 - 0.0006315\delta_4 - 0.00132629\delta_5 - 0.0020095\delta_6)\sin 5\varphi$$
$$+ (-0.0009473\delta_0 + 0.0000631\delta_2 - 0.0000287\delta_4 + 0.000077\delta_6)\sin 7\varphi]. \tag{38}$$

It is easy to see that the amplitudes of the higher harmonics are approximately equal to those obtained in Subsection 1.

3. *Nonlinear Elasticity Described by Relation (18).* The indicated type of nonlinearity is responsible for the appearance of a symmetric elastic characteristic and the odd higher harmonics $a_j a \cos j\varphi$ automatically obtained in the second approximation [see relation (29)] if we use the hysteresis function (12). In view of relations (3) and (6), in this case, the resonance frequency is a function of the amplitude of vibrations a or the components of the decrement of vibrations δ_n:

$$\omega = \omega_0 + \varepsilon B_1(a) = \omega_0\left(1 - \sum_{n=2} b_n a^{n-1}\right) = \omega_0\left(1 - \sum_{n=2} b_n \delta_n\right), \tag{39}$$

where

$$b_n = \frac{\nu_n}{2n} \sum_{k=0}^{\frac{n-1-e}{2}} 2^{-2k} \binom{n}{k} \binom{n-k}{1+k}, \quad d_n = \frac{n+1}{2^{n+2}(n-1)} \sum_{k=0}^{\frac{n-1-e}{2}} 2^{-2k} \binom{n}{k} \binom{n-k}{k+1}. \tag{40}$$

4. *"Breathing" Discontinuities in the Material of a Macroelastic Element Described by Function (21).* By substituting (21) in (7) and then in (4) and integrating the relation obtained, we find

$$\varepsilon u_1(a, \varphi) = \nu_0 a + \sum_{j=2,4,\ldots} \nu_j a \cos j\varphi, \tag{41}$$

where

$$\nu_0 = \frac{\alpha}{\pi}, \quad \nu_j = (-1)^{j/2} \frac{2\alpha}{\pi(j^2-1)^2}. \tag{42}$$

For the first four higher harmonics ($j = 2, 4, 6, 8$), we arrive at the following expression for the general solution (2) in the second approximation:

$$u = a(0.31831\alpha + \cos\varphi - 0.07074\,\alpha\cos 2\varphi + 0.00283\,\alpha\cos 4\varphi - 0.00052\,\alpha\cos 6\varphi + 0.00016\,\alpha\cos 8\varphi). \tag{43}$$

It is clear that, for evaluation of the constant component and the cosine harmonics, one must know the quantity α. Approximate values of α can be obtained by using relation (22) [or (24)] and existing experimental data on the variation of the resonance frequency of vibrations ω_0 and the dynamic (shear) modulus of elasticity ($E_{sh} = \text{const}\,\omega_0^2$) in the course of fatigue testing of metal specimens.

In [14], it was indicated that, for the alloys studied by Lazan and Demer, the maximum relative decrease in the value of the modulus of elasticity attained in the process of cyclic loading under the action of stresses higher than the fatigue limit was never more than 6–8% on a base of 10^5, which corresponds to changes in the frequency of vibrations by 3–4%. At the same time, an analysis of the dependences of the period of resonance vibrations on the length of fatigue cracks recorded near stress concentrators in the course of fatigue testing of tubular specimens of ÉP-609Sh steel demonstrated that the reliably detected minimum relative increase in the period of vibrations is as high as 0.07–0.2% if the length of the crack is ≥ 0.13 mm. Thus, the experimentally established range of the parameter β defined by (23) is 0.001–0.04, which specifies, in view of relation (22) [or (24)], the maximum range of the parameter α, namely, 0.004–0.15, and, hence, the following values of the harmonics:

$$u = a(0.00127\ldots0.0178 + \cos\varphi - 0.000283\ldots0.0106\cos 2\varphi + 0.0000113\ldots0.000425\cos 4\varphi$$

$$- 0.000002\ldots0.000078\cos 6\varphi + 0.00000064\ldots0.000024\cos 8\varphi). \tag{44}$$

Let us now estimate the values of α on the basis of concepts of linear fracture mechanics. Thus, as indicated in [15], if two identical plates of thickness t are subjected to the action of the same tensile load P and one of these plates has no cracks and the other is weakened by a symmetric transverse through crack of length $2l$, then an additional amount of energy $\Delta\Pi = \pi\sigma^2 l^2 t/E$ is spent in the process of loading of the cracked plate; here, σ is the nominal stress related to unit area of the cross section F of the plate without cracks and E is the modulus of elasticity of the material of the plates. The expression $\pi\sigma^2 l$ can be rewritten in terms of the stress intensity factor K_{I}. This gives

$$\Delta\Pi = \frac{K_{\mathrm{I}}^2 F_c}{2E}, \tag{45}$$

where F_c is the area of the crack (in this case, it is equal to $2tl$). Note that, under conditions of plane deformation, we must replace the quantity E by $E/(1-\mu^2)$. Somewhat arbitrarily, we assume that relation (45) is true for various transverse cracks and boundary conditions provided that we know the value of K_{I}. Thus, for a flat specimen of length L, width b, and thickness t weakened by a transverse crack of area F_c, the expression for its potential strain energy Π under the action of a tensile load $P = \sigma F$ can be represented in the form

$$\Pi = \frac{P^2}{2K_0} + \frac{K_I^2 F_c}{2E} = \frac{P^2}{2K_0'}, \tag{46}$$

where $K_0 = EF/L$ is the stiffness of the intact specimen and K_0' is the stiffness of the cracked specimen in tension. We neglect the possibility of the appearance of bending strains caused by asymmetric location of the crack with respect to the direct axis of the specimen.

By using relation (20), we represent the stiffness K_0' in the form $K_0' = K_0(1 - \alpha)$. We substitute this expression in (46) and arrive at the following relation for the parameter α:

$$\alpha = \frac{\left(\frac{K_I}{\sigma}\right)^2 \frac{F_c}{F}}{L + \left(\frac{K_I}{\sigma}\right)^2 \frac{F_c}{F}}. \tag{47}$$

Let us now analyze some special cases of tension–compression of elastic elements described by Eq. (1). We consider a flat specimen with dimensions: $t = 2$ mm, $b = 10$ mm, and $L = 25$ mm. The specimen is weakened by a transverse main crack. The relevant values of K_I are determined from reference data [16].

For edge cracks of length $l = 1$–3 mm, the parameter α varies within the range 0.0173–0.2365. For central through cracks of length $2l = 2$–4 mm, it varies within the range 0.0257–0.1106. For a central semielliptic surface crack whose length-to-depth ratio $2l/a$ guarantees attainment of almost equal values of K_I along the crack front, we have $\alpha = 0.00355$–0.0219 for $l = 1$–2 mm. For a through crack of length l appearing on the contour of a circular hole of radius R, this parameter varies from 0.028 to 0.0093 if $l = 0.2$–0.5 and $R = 1$ mm.

It is worth noting that if we analyze vibrations of an elastic element regarded as a system with distributed parameters, then the values of the parameter α also depend on the location of the crack and the modes of vibration.

Let us also estimate the value of α in the case of diffused fatigue damage by assuming that the methods of linear fracture mechanics are applicable to observed microcracks. According to [17], in the course of tensile–compressive fatigue testing of thin-walled tubular specimens of 45 steel, one observes 200–900 cracks of average length $2l = 0.02$ mm on a surface of area 1 mm^2 if the number of loading cycles is equal to half the number of cycles to failure. If we assume that the system of surface cracks of the same length forms a doubly periodic orthogonal lattice, then relation (47) gives $\alpha \approx 0.001$–0.003 for a 2-mm-thick flat specimen independently of its length.

It is easy to see that the values of α are in reasonable agreement with its ranges indicated earlier.

Discussion. As follows from solution (28) and relation (38), for the type of nonlinearity caused by the dissipative properties of a material with an expected value of their integral characteristic (i.e., the decrement of vibrations) of about 1%, the amplitudes of the first higher odd harmonics in the vibration spectrum are quite small, and the amplitudes of even harmonics and the constant component are negligible. Thus, the maximum amplitudes of the third harmonics are smaller than the amplitude of the first harmonic by about five orders of magnitude ($a_3 \approx b_3 \approx 6.5 \cdot 10^{-5} a$) and the amplitude of the second harmonic is smaller than the amplitude of the third harmonic by more than two orders of magnitude ($a_2 \approx 1 \cdot 10^{-8} a$). Moreover, in this case, the presence of higher harmonics does not actually serve as an indication of the presence of fatigue damage to the material because the dissipation of energy is also typical of its initial state (prior to cycling). In this case, the value of the decrement may both increase and decrease in the process of cycling, and the fatigue damage to the material can be more efficiently detected by analyzing the behavior of the decrement of vibrations and the amplitude of higher harmonics as functions of the number of strain cycles rather than their absolute values [18]. It is also worth noting that if the first harmonic obeys the cosine law of variation, then the odd sinusoidal harmonics reflect the imperfect elasticity of the material responsible for the amplitude dependence of the relative dissipation of energy. At the same time, the odd cosinusoidal harmonics describe pure elastic and symmetric nonlinearity.

Results of numerical calculations also demonstrate that it is practically impossible to evaluate the degree of damage to the material from the changes in the mechanism of dissipation of energy or in the character of the amplitude dependence of the decrement of vibrations because the amplitudes of harmonics, including the amplitude of the most representative, third harmonic, are determined by the sum of the components of the decrement of vibrations δ_n.

In fact, the analysis of the behavior of the amplitudes of higher harmonics is, in this case, similar to the analysis of the behavior of the integral characteristic of dissipative properties of the material of the element, i.e., of its logarithmic decrement of vibrations $\delta = \sum \delta_n$, with the only difference that the former is characterized by a much lower resolution because it is based on an analysis of the behavior of very small quantities.

More promising results are obtained in analyzing "breathing" discontinuities of the material.

First, the spectrum of solution (44) obtained in this case is characterized by the presence of a pronounced constant component whose values [even for insignificant variations of the resonance frequency (of about 0.1%)] are higher than the values of this component determined by possible nonlinearities of the dissipative properties of the material by about four orders of magnitude. Hence, the constant component may serve as a quite sensitive parameter for both detection of damage and evaluation of the degree of fatigue damage. This confirms the potential efficiency of the experimental procedure suggested in [11] for detection of fatigue cracks by instrument-assisted separation of the constant component of forces acting on the specimen under strain-controlled harmonic loading.

Second, in this solution, we readily detect even harmonics whose amplitudes (for the indicated changes in the resonance frequency) are higher than the amplitudes of the same harmonics induced by possible nonlinearities of the dissipative properties of the material by about three orders of magnitude. Moreover, the maximum of these amplitudes, i.e., the amplitude of the second harmonic, is greater than the maximum (third) amplitude of odd harmonics caused by nonlinearities of dissipative properties by an order of magnitude.

Third, unlike the nonlinearity of dissipative properties, the process of breathing of the material is observed only after the appearance of breathing discontinuities in the process of cycling, i.e., both higher harmonics and the constant component are small or almost absent in the intact material. Furthermore, as the degree of damage increases and, hence, the resonance frequency of the element decreases in the process of strain cycling, we observe a well-pronounced increase in the amplitudes of even harmonics. According to relation (43), this effect is proportional to the relative change in the degree of damage of the material described by the parameter α. Therefore, the increment of the amplitudes of even harmonics recorded in the process of spectral analysis of strain cycles of the investigated element both after cycling and in the intact state may serve as a representative informative parameter of discontinuities of the material formed in the course of strain cycling. It should also be noted that the analysis of the behavior of this parameter is clearly characterized by higher resolution than the analysis of the behavior of the integral characteristic of elasticity of the investigated element, i.e., its resonance frequency. Thus, for a relative variation of the resonance frequency of about 0.1%, the amplitudes of even harmonics change by a factor of ten or even more and, for a relative variation of the resonance frequency of about 4%, the amplitudes vary by a factor 300.

Conclusion. The presence of a constant component of a certain magnitude and even harmonics whose amplitudes exceed certain prescribed levels in the spectrum of strain cycles of the material or vibratory displacements of the investigated element may serve as an indicator of breathing discontinuities in the material of the vibration-insulated structural element in the process of its natural or resonance single-frequency longitudinal vibration. The values of the constant component and the amplitudes of even harmonics (primarily, the second and fourth) exceeding the corresponding values recorded for the intact material of the investigated element may serve as a sensitive and efficient parameter for detection of fatigue damage to the material and evaluation of its level.

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