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EMBEDDING L_1 IN A BANACH LATTICE

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ABSTRACT
We show that if X is a Banach lattice containing no copy of c_0 and if Z is a subspace of X isomorphic to $L_1[0,1]$ then (a) Z contains a subspace Z_0 isomorphic to L_1 and complemented in X and (b) X contains a complemented sublattice isomorphic and lattice-isomorphic to L_1.

1. Introduction

In [5] (see also [6]) Lotz shows that if X is a Banach lattice such that X^* does not have the Radon–Nikodym property, then either c_0 embeds in X^* or L_1 embeds as a complemented sublattice. In consequence if X^* contains no copy of c_0 and L_1 can be embedded in X^* then L_1 can be embedded as a complemented sublattice.

In this paper we show that if X does not contain c_0 and X contains a subspace Z isomorphic to L_1 then X contains a complemented sublattice $\cong L_1$. We also show that Z has a subspace $Z_0 \cong L_1$ and complented in X. This result extends a result of Enflo and Starbird [2] who establish the same result for $X = L_1$ (see also [4]).

In place of $L_1[0,1]$ we shall work with $L_1(\Delta, \mathcal{B}, \lambda)$ where Δ is the Cantor group $\Pi_{i=1}^{\infty}\{-1, +1\}$ with λ Haar measure defined on the Borel sets $\mathcal{B}$ of Δ. Of course $L_1[0,1]$ and $L_1(\Delta)$ are isometrically lattice-isomorphic.

On Δ, we denote by ε_i the characters

$$\varepsilon_i(t) = t_i \qquad \text{where } t = (t_j)_{j=1}^{\infty} \in \Delta.$$

Let Δ_k^n, $1 \leqq k \leqq 2^n$ be the set of $t \in \Delta$ such that

$$\sum_{j=1}^{n} t_j 2^{j-1} = k - 1.$$

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Then each Δ_k^n is clopen and (Δ_k^N: $1 \leqq k \leqq 2^n$) forms the standard nth partition of Δ. We denote the characteristic function of Δ_k^n by χ_k^n ($1 \leqq k \leqq 2^n$, $1 \leqq n < \infty$).

Suppose X is a Banach lattice which does not contain c_0. Then if $x_\alpha \in X$ is an increasing net with $\sup \| x_\alpha \| < \infty$, we conclude that x_α converges and

$$\lim_\alpha x_\alpha = \sup_\alpha x_\alpha .$$

In particular X is order-complete.

Now if T: $L_1(\Delta) \to X$ is a bounded linear operator we may define an operator $|T|$: $L_1(\Delta) \to X$ such that

$$|T|(\chi_k^n)) = \sup_{m \geqq n} \sum_{\Delta_j^m \subset \Delta_k^n} |T\chi_j^m|$$

and $\| |T| \| = \| T \|$. It is clear that if U: $L_1(\Delta) \to X$ satisfies $U \geqq T$ and $U \geqq -T$ then $U \geqq |T|$ (so that $\mathcal{L}(L_1(\Delta), X)$ is a Banach lattice).

2. Preliminary results

Our starting point is the following theorem proved in [4].

THEOREM 2.1. *Suppose (Ω, Σ, μ) is a measure space with $\mu(\Omega) = 1$. Suppose T: $L_1(\Delta) \to L_1(\Omega)$ is a bounded linear operator. Then there is an essentially unique (i.e. up to sets of μ-measure zero) map $\omega \to \nu_\omega$ from Ω into the space $\mathcal{M}(\Delta)$ of regular Borel measures on Δ such that*

(2.1.1) *$\omega \to \nu_\omega$ is weak*-measurable with respect to Σ,*

(2.1.2) $\int_\Omega |\nu_\omega|(B) d\mu(\omega) \leqq M\lambda(B)$, $B \in \mathcal{B}$,

(2.1.3) $Tf(\omega) = \int_\Delta f(t) d\nu_\omega(t)$, μ-a.e., $f \in L_1(\Delta)$,

where $M = \| T \|$.

Conversely if $\omega \to \nu_\omega$ satisfies (2.1.1) and (2.1.2) then (2.1.3) defines a bounded linear operator from $L_1(\Delta)$ into $L_1(\Omega)$ with $\| T \| \leqq M$.

We shall say that $T \in \mathcal{L}(L_1(\Delta), L_1(\Omega))$ is *atomic* if $\mu\{\omega: \nu_m \in \mathcal{M}_c(\Delta)\} < 1$ where $\mathcal{M}_c(\Delta)$ is the subset of $\mathcal{M}(\Delta)$ of all continuous measures. Since $\mathcal{M}_c(\Delta)$ is weak*-Borel (see [4]), the set $\{\omega: \nu_\omega \in \mathcal{M}_c(\Delta)\} \in \Sigma$.

PROPOSITION 2.2. *If $T \in \mathcal{L}(L_1(\Delta), L_1(\Omega))$ is atomic then:*

(2.2.1) *there is a Borel subset B of Δ with $\lambda(B) > 0$ such that $T \mid L_1(B)$ is an isomorphism and $T(L_1(B))$ is complemented in $L_1(\Omega)$,*

(2.2.2) *there is an operator S of the form*

$$Sf(\omega) = f(\sigma\omega), \qquad \omega \in \Omega_0,$$

$$Sf(\omega) = 0, \qquad \omega \notin \Omega_0,$$

where $\mu(\Omega_0) > 0$ *and* $\sigma: \Omega \to \Delta$ *is* Σ*-measurable, such that* $S \leqq m \mid T \mid$, *for some* $m \in \mathbf{N}$.

REMARK 1. Of course we assume in (2.2.2) that S is bounded, which implies some conditions on σ.

REMARK 2. It is clear that (2.2.2) implies T is atomic; in fact (2.2.1) also implies T is atomic. This last remark follows from Theorem 3.1 below.

PROOF. (2.2.1) is proved in [4] theorem 5.5, for the case when Ω is a compact metric space and Σ is the Borel σ-algebra on Ω. For the general case we observe that since $L_1(\Delta)$ is separable, we may suppose $T(L_1(\Delta))$ is contained in $L_1(\Omega, \Sigma_0, \mu)$ where Σ_0 is a sub-σ-algebra of Σ such that $L_1(\Omega, \Sigma_0, \mu)$ is separable. Now let Σ_1 be a countable μ-dense sub-algebra of Σ_0 and let K be its Stone space; denote by $j: \Omega \to K$ the natural map.

Then j is measurable for the Borel σ-algebra on K and we may define a Borel measure $\hat{\mu}$ on K by

$$\hat{\mu}(B) = \mu(j^{-1}(B))$$

for B a Borel subset of K. It is easily checked that the map $J: f \to f \circ j$ defines an isometric isomorphism between $L_1(K, \hat{\mu})$ and $L_1(\Omega, \Sigma_0, \mu)$.

Consider $J^{-1}T$: $L_1(\Delta) \to L(K, \hat{\mu})$; this has the form

$$J^{-1}Tf(s) = \int_\Delta f(t)d\nu_s(t), \qquad \hat{\mu}\text{-a.e.},$$

where $s \to \nu_s$ is a Borel-measurable map from K into $\mathcal{M}(\Delta)$, satisfying the conditions of Theorem 2.1.

Then

$$JJ^{-1}Tf(\omega) = Tf(\omega) = \int_\Delta f(t)d\nu_{j\omega}(t), \qquad \mu\text{-a.e.}$$

Hence $\omega \to \nu_{j\omega}$ "represents" T; by the essential uniqueness of this representation we conclude

$$\mu\{\omega: \nu_{j\omega} \in M_c(\Delta)\} < 1.$$

Hence

$$\hat{\mu}\{s: \nu_s \in M_c(\Delta)\} < 1$$

and so $J^{-1}T$ is atomic and we may appeal to [3] theorem 5.5 for the result (note that $L_1(\Omega, \Sigma_0, \mu)$ is complemented in $L_1(\Omega)$).

For (2.2.2) observe that by the remarks after theorem 3.2 of [4]

$$\nu_\omega = \sum_{n=1}^{\infty} a_n(\omega)\delta(\sigma_n\omega) + \rho_\omega$$

where

(1) $a_n : \Omega \to \mathbf{R}$ is Σ-measurable, $n \in \mathbf{N}$,

(2) $\sigma_n : \Omega \to \Delta$ is Σ-measurable, $n \in \mathbf{N}$,

(3) $|a_n(\omega)| \geqq |a_{n+1}(\omega)|$, $n \in \mathbf{N}$, $\omega \in \Omega$,

(4) $\sigma_n(\omega) \neq \sigma_m(\omega)$, $m \neq n$,

(5) $\rho_\omega \in \mathcal{M}_c(\Delta)$, $\omega \in \Omega$.

Since $\nu_\omega \neq \rho_\omega$ on a set of positive measure there is a set Ω_0 with $\mu(\Omega_0) > 0$ such that

$$|a_1(\omega)| \geqq \varepsilon > 0, \qquad \delta \in \Omega_0.$$

Define

$$Sf(\omega) = f(\sigma_1\omega), \qquad \omega \in \Omega_0,$$
$$= 0, \qquad \omega \notin \Omega_0.$$

Then for $f \geqq 0$

$$Sf \leqq \varepsilon^{-1} \int_\Delta f(t)\, d\,|\nu_\omega|(t)$$
$$= \varepsilon^{-1} |T| f$$

(since it is easy to show that $\omega \to |\nu_\omega|$ represents $|T|$, cf. [4]).

From this it easily follows that S is bounded; and for (2.2.2) choose $m > 1/\varepsilon$. Of course, $S \neq 0$ since $S\chi_\Delta \neq 0$.

Proposition 2.3. *Suppose $T \in \mathcal{L}(L_1(\Delta), L_1(\Omega))$ and that $T\chi_k^n = b_k^n$. Define*

$$h_n(\omega) = \max_{1 \leqq k \leqq 2^n} |b_k^n(\omega)|.$$

If $\limsup_{n\to\infty} h_n(\omega) > 0$ on a set of positive μ-measure, then T is atomic.

Proof. Consider $|T|$; it is enough to show that $|T|$ is atomic. Let

$$|T|(\chi_k^n) = c_k^n$$

and

$$g_n(\omega) = \max_{1 \leqq k \leqq 2^n} c_k^n(\omega).$$

Then $g_n \geqq h_n$ and hence for some Ω_0 with $\mu(\Omega_0) > 0$

$$\limsup g_n(\omega) > 0, \qquad \omega \in \Omega_0.$$

For $\omega \in \Omega_0$ choose $0 < \xi_\omega < \limsup g_n(\omega)$ and choose $n(j) \to \infty$ such that $g_{n(j)}(\omega) > \xi_\omega$. Then there exists $k(j)$, $1 \leqq k(j) \leqq 2^{n(j)}$ such that

$$c_{k(j)}^{n(j)} > \xi_\omega.$$

If we choose $\tau_k^n \in \Delta_k^n$, as in the proof of theorem 3.1 of [4] if

$$\alpha_\omega^n = \sum_{k=1}^{2^n} c_k^n(\omega) \delta(\tau_k^n)$$

then $\alpha_\omega^n \to |\nu_\omega|$ μ-a.e. in the weak*-topology. Now

$$\alpha_\omega^{n(j)} \geqq \xi_\omega \delta(\tau_{k(j)}^{n(j)})$$

and hence if τ_ω is a limit point of $\{\tau_{k(j)}^{n(j)}: j \in \mathbf{N}\}$,

$$|\nu_\omega| \geqq \xi_\omega \delta(\tau_\omega) \qquad (\mu\text{-a.e. } \omega \in \Omega_0)$$

(since the positive cone of $\mathcal{M}(\Delta)$ is weak* closed). Thus $|\nu_\omega| \not\in \mathcal{M}_c(\Delta)$, μ-a.e., $\omega \in \Omega_0$.

3. The main results

THEOREM 3.1. *Suppose X is a Banach lattice which does not contain c_0. Suppose T: $L_1(\Delta) \to X$ is an isomorphism of $L_1(\Delta)$ onto a subspace Z of X. Then*:

(3.1.1) *there exists a subset B of Δ with $\lambda(B) > 0$ such that $T(L_1(B))$ is complemented in X,*

(3.1.2) *there exists an isomorphism S of $L_1(\Delta)$ onto a complemented subspace Y of X which is also a lattice isomorphism, i.e.*

$$S(f \wedge g) = Sf \wedge Sg, \qquad f, g \in L_1(\Delta).$$

PROOF. Let Y be the smallest closed sublattice of X containing Z. The Y is separable and order-complete. Consider $|T|$: $L_1(\Delta) \to X$: then $|T|(L_1) \subset Y$. Let $u = |T|(\chi_\Delta)$, and let Y_u be defined by

$$Y_u = \bigcup_{n \in \mathbf{N}} n([-u, u] \cap Y),$$

where $[-u, u]$ is the order-interval $-u \leqq x \leqq u$; taking $[-u, u] \cap Y$ as the unit ball Y_u becomes an order-complete Banach lattice which is an AM-space. Hence

Y_u is isometrically lattice-isomorphic to a space $C(\Omega)$ where Ω is a Stonian space. We denote by $\theta: C(\Omega) \to Y_u \subset Y$ this isomorphism.

Since Y is separable, there exists a positive linear functional $\phi \in X^*$ such that $\phi(u) = 1$ and ϕ is strictly positive on Y, i.e. $\phi(y) = 0$, $y \geqq 0$ imply $y = 0$ ($y \in Y$). Clearly $\phi \circ \theta \in C(\Omega)^*$ and $\|\phi \circ \theta\| = 1$; hence there is a probability measure μ on Ω such that

$$\phi(\theta f) = \int_\Omega f(\omega) d\mu(\omega), \qquad f \in C(\Omega).$$

In fact μ is normal, since if f_α is an increasing net bounded above in $C(\Omega)$ then $\theta f_\alpha \to \theta(\sup_\alpha f_\alpha)$ in Y as X contains no copy of c_0; thus if $G \subset \Omega$ is meagre, $\mu(G) = 0$. Also the support of μ is dense in Ω since ϕ is strictly positive on Y_u. This means that $L_\infty(\Omega, \mathscr{B}(\Omega), \mu)$ may be identified with $C(\Omega)$ both as a Banach space and as a lattice; for if f is a bounded Borel function on Ω then there is a unique $g \in C(\Omega)$ such that $f = g$ μ-a.e. Note that in $L_\infty(\Omega, \mu)$ the lattice supremum of a countable subset is the point-wise supremum. Thus we shall regard θ as a lattice isomorphism of $L_\infty(\Omega, \mu)$ onto Y_u.

If $f \in C(\Delta)$ and $\|f\| \leqq 1$, then $Tf \in [-u, u] \cap Y$ and hence we may induce a map T_0: $C(\Delta) \to L_\infty(\Omega, \mu)$ so that $\|T_0\| \leqq 1$ and $\theta T_0 = T$.

Denote by j the natural inclusion map $L_\infty(\Omega, \mu) \hookrightarrow L_1(\Omega, \mu)$. Suppose $B \subset \Omega$ is Borel, and denote by P_B the band projection on X induced by $\theta\chi_B$. Thus for $v \in X$, $v \geqq 0$

$$P_B v = \sup_n (v \wedge n\theta\chi_B).$$

Then $\|P_B\| \leqq 1$ and it is easy to show that since X does not contain c_0, each $x \in X$, the map $B \to P_B x$ is a countably additive vector measure on $\mathscr{B}(\Omega)$. In particular $B \to \phi(P_B x)$ is a μ-continuous measure, and hence by taking its Radon–Nikodym derivative we may induce an operator V: $X \to L_1(\Omega, \mu)$ such that

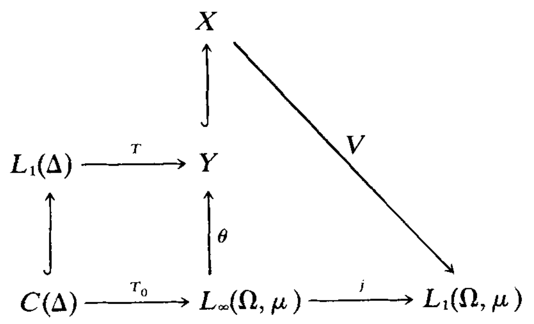
$$\phi(P_B x) = \int_B Vx(\omega) d\mu(\omega).$$

If A and B are disjoint

$$\begin{aligned} |\phi(P_A x - P_B x)| &\leqq \phi(|P_A x| + |P_B x|) \\ &\leqq \phi(|x|) \\ &\leqq \|\phi\| \|x\| \end{aligned}$$

so that $\|V\| \leqq \|\phi\|$.

It is easy to see that for $f \in C(\Delta)$, $jT_0 f = VTf$; thus the following diagram commutes:



At this point we recall that T is an isomorphism so that for some $c > 0$

$$\| Tf \| \geqq c \| f \|, \qquad f \in L_1(\Delta).$$

We now claim the existence of $\delta > 0$ such that if $B \subset \Omega$ is a Borel set with $\mu(B) < \delta$ then $\| \theta \chi_B \| < c/2$. If this were false then we could find B_n with $\mu(B_n) \leqq 2^{-n}$ and $\| \theta \chi_{B_n} \| \geqq c/2$. If $C_n = \bigcup_{k \geqq n} B_k$, then $\mu(C_n) \to 0$ and hence $\inf_n \chi_{C_n} = 0$ in $L_\infty(X, \mu)$. Thus $\| \theta(\chi_{C_n}) \| \to 0$ and so $\| \theta \chi_{B_n} \| \to 0$ which is a contradiction.

We shall now show that VT is atomic as an operator from $L_1(\Delta)$ into $L_1(\Omega)$. Let

$$T_0 \chi_k^n = b_k^n, \qquad 1 \leqq k \leqq 2^n, \quad 1 \leqq n < \infty$$

and

$$h_n(\omega) = \max_{1 \leqq k \leqq 2^n} | b_k^n(\omega) |, \qquad \omega \in \Omega.$$

It suffices to show $\limsup_{n \to \infty} h_n > 0$ on a set of positive μ-measure. Our argument here is based on one of Dor [1].

If $s \in \Delta$, $1 \leqq n < \infty$,

$$\left\| \sum_{k=1}^{2^n} \varepsilon_k(s) T \chi_k^n \right\| \geqq c.$$

Define $A_n \subset \Delta \times \Omega$ by

$$A_n = \left\{ (s, \omega) \colon \delta \left| \sum_{k=1}^{2^n} \varepsilon_k(s) b_k^n(\omega) \right|^2 > \sum_{k=1}^{2^n} | b_k^n(\omega) |^2 \right\}.$$

Since each $b_k^n \in C(\Omega)$, A_n is open and hence $\lambda \times \mu$-measurable. For each $\omega \in \Omega$ let $A_n(\omega)$ be the set of $s \in \Delta$ for which $(s, \omega) \in A_n$. Then

$$\lambda(A_n(\omega)) \sum_{k=1}^{2^n} |b_k^n(\omega)|^2 \leqq \delta \int_\Delta \left| \sum_{k=1}^{2^n} \varepsilon_k(s) b_k^n(\omega) \right|^2 d\lambda(s)$$

$$= \delta \sum_{k=1}^{2^n} |b_k^n(\omega)|^2$$

and hence $\lambda(A_n(\omega)) \leqq \delta$. Thus $(\lambda \times \mu)(A_n) \leqq \delta$ and by Fubini's theorem there exists $s = s(n) \in \Delta$ such that if $C = C_n = \{\omega : (s, \omega) \in A_n\}$ then $\mu(C) \leqq \delta$.

Thus

$$\left\| \theta \left[\chi_C \sum_{k=1}^{2^n} \varepsilon_k(s) b_k^n \right] \right\| \leqq \| \theta \chi_C \| \leqq c/2.$$

Hence

$$\left\| \theta \left[\chi_{\Omega - C} \sum_{k=1}^{2^n} \varepsilon_k(s) b_k^n \right] \right\| \geqq c/2.$$

Let

$$g_n(\omega) = \sqrt{\left(\sum |b_k^n(\omega)|^2 \right)}.$$

For $\omega \in \Omega \setminus C$,

$$g_n(\omega) \geqq \sqrt{\delta} \left| \sum_{k=1}^{2^n} \varepsilon_k(s) b_k^n(\omega) \right|.$$

Hence

$$\| \theta g_n \| \geqq \tfrac{1}{2} c \sqrt{\delta}.$$

Now let $E = \{x : g_n(x) > c\sqrt{\delta}/4\|u\|\}$. Then

$$\theta(g_n \chi_{\Omega - E}) \leqq \frac{c\sqrt{\delta}}{4\|u\|} u$$

and hence

$$\| \theta(g_n \chi_{\Omega - E}) \| \leqq \tfrac{1}{4} c \sqrt{\delta}.$$

For $\omega \in E$

$$|g_n(\omega)|^2 \leqq h_n(\omega) \sum_{k=1}^{2^n} |b_k^n(\omega)|$$

and

$$\theta \left[\sum_{k=1}^{2^n} |b_k^n(\omega)| \right] = \sum_{k=1}^{2^n} |\theta b_k^n| = \sum_{k=1}^{2^n} |T \chi_k^n| \leqq |T|(\chi_\Delta) = u.$$

Hence

$$\sum_{k=1}^{2^n} |b_k^n(\omega)| \leqq 1.$$

Thus for $\omega \in E$

$$h_n(\omega) \geqq |g_n(\omega)|^2$$

$$\geqq \frac{c\sqrt{\delta}}{4\|u\|} g_n(\omega).$$

Hence

$$\|\theta h_n\| \geqq \frac{c^2\delta}{16\|u\|}.$$

Now

$$\left\|\theta\left(\limsup_{n\to\infty} h_n\right)\right\| = \lim_{n\to\infty}\left\|\theta\left(\sup_{k\geqq n} h_k\right)\right\|$$

$$\geqq \limsup_{n\to\infty} \|\theta h_n\|$$

$$\geqq \frac{c^2\delta}{16\|u\|}.$$

Hence $\limsup_{n\to\infty} h_n > 0$ on a set of positive measure, and so VT is atomic.

Thus we conclude by Proposition 2.2 that there is a Borel subset B of Δ of positive measure such that VT maps $L_1(B)$ isomorphically to a complemented subspace of $L_1(\Omega)$. Suppose P is a projection of $L_1(\Omega)$ onto $VT(L_1(B))$; then $V^{-1}PV$ projects X onto $T(L_1(B))$, since V maps $T(L_1(B))$ isomorphically onto its image.

For our second claim we appeal to (2.2.2) to produce a non-zero operator $S: L_1(\Delta) \to L_1(\Omega)$ of the form

$$Sf(\omega) = f(\sigma\omega), \qquad \omega \in \Omega_0,$$

$$Sf(\omega) = 0, \qquad \omega \notin \Omega_0,$$

with $S \leqq m|T|$ for some $m \in \mathbf{N}$. Again by results of [3] there is a subset C of Δ of positive measure such that $S \mid L_1(C)$ is an isomorphism onto a complemented subspace of $L_1(\Omega)$.

Let S_0 be the restriction of S, $S_0: C(\Delta) \to L_\infty(\Omega, \mu)$. If $f \in C(\Delta)$, $f \geqq 0$

$$mV(|T|f) \geqq m|VT|f$$

$$\geqq S_0 f$$

$$= V\theta S_0 f$$

$$= V\theta S_0 f$$

and hence $m|T|f \geqq \theta S_0 f$. Thus θS_0 extends to a map S_1: $L_1(\Delta) \to Y$; clearly $VS_1 = S$. As before $S_1(L_1(C))$ is a complemented subspace of X. To complete the proof we show that S_1 is a lattice homomorphism (since of course $L_1(C)$ is isomorphically lattice-isomorphic to $L_1(\Delta)$). If $f, g \in L_1(C)$ then

$$\begin{aligned} VS_1(f \wedge g) &= S(f \wedge g) \\ &= VS_1 f \wedge VS_1 g. \end{aligned}$$

Let $h = S_1(f \wedge g)$. Then $h \leqq S_1 f \wedge S_1 g$ and hence $Vh \leqq V(S_1 f \wedge S_1 g) \leqq VS_1 f \wedge VS_1 g = Vh$. Thus

$$\phi(S_1 f \wedge S_1 g - h) = \int (V(S_1 f \wedge S_1 g) - Vh) d\mu = 0.$$

Since ϕ is strictly positive on Y

$$S_1 f \wedge S_1 g = S_1(f \wedge g)$$

as required.

Now suppose (Ω, Σ, μ) is a measure space in the $\mu(\Omega) = 1$. A re-arrangement invariant Banach function space X on (Ω, Σ, μ) is an order-ideal of $L_0(\Omega, \Sigma, \mu)$ equipped with a Banach lattice norm such that if $f \in X$ and $g \in L_0$ are such that $|f|$ and $|g|$ have the same distribution, i.e.

$$\mu\{\omega: |f(\omega)| > a\} = \mu\{\omega: |g(\omega)| > a\} \quad (a \in \mathbf{R}),$$

then $g \in X$ and $\|g\| = \|f\|$.

Under these circumstances X is automatically contained in $L_1(\Omega, \mu)$. Examples are Orlicz spaces and Lorentz spaces.

THEOREM 3.2. *Suppose X is a re-arrangement invariant Banach function space on (Ω, Σ, μ) not containing c_0. Then if X contains a subspace isomorphic to $L_1(\Delta)$, $X = L_1(\Omega, \mu)$.*

PROOF. By Theorem 3.1 there is a lattice isomorphic embedding S: $L_1(\Delta) \to X$. Consider S as a map from $L_1(\Delta)$ into $L_1(\Omega)$. Then S is a lattice homomorphism.

Suppose

$$Sf(\omega) = \int_\Delta f(t) d\nu_\omega(t)$$

as in Theorem 3.1. If we pick $\tau_k^n \in \Delta_k^n$ then $\alpha_\omega^n \to \nu_\omega$ μ-a.e. where

$$\alpha_\omega^n = \sum_{k=1}^{2^n} S\chi_k^n(\omega)\delta(\tau_k^n).$$

Since S is a lattice homomorphism

$$|S\chi_k^n| \wedge |S\chi_j^n| = 0$$

and hence α_ω^n is a single non-negative point mass: μ-a.e. Thus ν_ω is a single non-negative point mass μ-a.e., and it follows that

$$Sf(\omega) = a(\omega)f(\sigma\omega), \qquad \mu\text{-a.e.},$$

where $a(\omega) \geqq 0$ and $\sigma: \Omega \to \Delta$ is Σ-measurable.

Since $S \neq 0$, $a \not\equiv 0$ and so there exists $\varepsilon > 0$ such that $\mu(A) > 0$ where $A = \{\omega: a(\omega) > \varepsilon\}$. Then

$$Tf(\omega) = \chi_A f(\sigma\omega), \qquad \mu\text{-a.e.}$$

defines a non-zero bounded linear operator of $L_1(\Delta)$ into X. Since $T: L_1(\Delta) \to L_1(\Omega)$ is atomic, there is a subset C of Δ of positive measure such that $T \mid L_1(C)$ is an isomorphism, i.e. for some $c > 0$

$$\int_\Omega |Tf(\omega)|\, d\mu(\omega) \geqq c\|f\|, \qquad f \in L_1(C).$$

Now suppose $D \in \Sigma$ and $\mu(D) < \mu(\sigma^{-1}C \cap A)$; observe that $\mu(\sigma^{-1}C \cap A) > 0$. Then there exists $B \subset C$ such that $\mu(\sigma^{-1}B \cap A) = \mu(D)$ as the measure $B \to \mu(\sigma^{-1}B \cap A)$ is clearly non-atomic. Thus

$$\begin{aligned}\|\chi_D\| &= \|T\chi_B\| \\ &\leqq \|T\|\lambda(B) \\ &\leqq c^{-1}\|T\|\mu(\sigma^{-1}B \cap A) \\ &= c^{-1}\|T\|\mu(D).\end{aligned}$$

It follows now easily for simple functions and hence for any $f \in X$ that

$$\|f\| \leqq c^{-1}\|T\| \int_\Omega |f(\omega)|\, d\mu(\omega).$$

This implies $X = L_1(\Omega)$.

NOTE. After the initial preparation of this paper, the author learned that Theorem 3.2 had also been obtained by Johnson, Maurey, Schechtman and Tzafriri [3] under the slightly stronger assumption that c_0 is not finitely representable in X.

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