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ALMOST PERIODIC SETS AND MEASURES ON THE TORUS

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ABSTRACT

We establish some "number theoretical" results about a continuous function h from the circle $\mathbf{T}$ into itself, which generalize Kronecker's theorem in several ways. These results are used to characterize the almost periodic sets of the flow on the torus $\mathbf{T}^2$ generated by $(\theta, \phi) \to (\theta + \alpha, \phi + h(\theta))$, where α is irrational. The almost periodic measures are characterized in the case $h(\theta) = \theta$.

§1. Introduction

If X is a compact Hausdorff space and T a homeomorphism of X onto itself then the flow on X generated by the powers of T will be denoted (T, X). With this flow we associate the flows induced by T on the space 2^X of closed subsets of X, and on the space $\mathcal{P}(X)$ of probability measures on X, denoting them $(T, 2^X)$ and $(T, \mathcal{P}(X))$ respectively. One of the questions that one would like to answer is what is the effect of distality of the flow (T, X) on the minimal subflows of $(T, 2^X)$ and $(T, \mathcal{P}(X))$. It is easy to see that the full flows are not distal in general. It was shown in [6] that when (T, X) is distal and minimal every minimal subflow of $(T, 2^X)$ is distal and moreover every such subflow is a factor of the enveloping semigroup of (T, X). It is not known yet whether this is true for minimal subflows of $(T, \mathcal{P}(X))$.

In order to study the minimal subflows of $(T, 2^X)$ and $(T, \mathcal{P}(X))$ we must determine which points are almost periodic points of these flows. We say that a point x of the flow (T, X) is *almost periodic* (a.p.) if given a neighbourhood U of x, a finite subset K of the integers $\mathbf{Z}$ exists such that $K + L(U) = \mathbf{Z}$, where $L(U) = \{n \in \mathbf{Z}: T^n x \in U\}$. It is well known that x is a.p. iff its orbit closure in X, denoted $\bar{O}(X)$, is a minimal set. We say that a closed subset of X (a probability

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measure on X) is a.p. if it is an almost periodic point of the flow $(T, 2^X)$ $((T, \mathcal{P}(X))$.

We shall study the problem of determining the a.p. sets and measures of (T, X) in the case of the simplest non-equicontinuous distal flow—a flow on the torus which extends the irrational rotation of the circle. In this paper we study mainly the flow $(T, 2^X)$. We achieve a complete characterization of a.p. sets only under a further restriction on (T, X). As a by-product we shall also obtain some generalizations of Kronecker's theorem. In the special case in which we can characterize the a.p. measures it will be easy to verify that the corresponding minimal flow is again distal.

Let $h: \mathbf{T} \to \mathbf{T}$ be a continuous map of the 1-torus $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ and let α be an irrational number. We let $T: \mathbf{T}^2 \to \mathbf{T}^2$ be the map $(\theta, \phi) \to (\theta + \alpha, \phi + h(\theta))$. Clearly $(T, \mathbf{T}^2)$ is distal and *we always assume that the flow* $(T, \mathbf{T}^2)$ *is not equicontinuous*. In particular this implies that $(T, \mathbf{T}^2)$ is minimal. If h is an essential map the assumption is automatically satisfied. Let $\pi: \mathbf{T}^2 \to \mathbf{T}$ and $\sigma: \mathbf{T} \to \mathbf{T}$ be defined by $\pi(\theta, \phi) = \theta$ and $\sigma(\theta) = \theta + \alpha$. For every $\zeta \in \mathbf{T}$ let $g_\zeta: \mathbf{T} \to \mathbf{T}$ be given by

$$g_\zeta(\theta) = h(\zeta + \theta) - h(\theta).$$

For an integer n put

$$h_n(\theta) = \begin{cases} h(\theta + (n-1)\alpha) + \cdots + h(\theta + \alpha) + h(\theta) & n > 0, \\ 0 & n = 0, \\ -h(\theta + n\alpha) - \cdots - h(\theta - \alpha) & n < 0. \end{cases}$$

Notice that $T^n(\theta, \phi) = (\theta + n\alpha, \phi + h_n(\theta))$. We let

$$H_n(\theta) = h_n(\theta) - h_n(0) \qquad \text{and} \qquad A(\theta) = \{H_n(\theta): n \in \mathbf{Z}\}.$$

We shall use script letters for pointed flows. For each $\zeta \in \mathbf{T}$ let $\mathcal{X}_\zeta = (T, \mathbf{T}^2, (\zeta, 0))$,

$$\mathcal{X}_0 \vee \mathcal{X}_\zeta = (T \times T, \bar{O}((0,0)(\zeta,0)), (0,0)(\zeta,0)).$$

If we define a flow on $\mathbf{T}^3$ by

$$R_\zeta(\theta, \phi, \psi) = (\theta + \alpha, \phi + h(\theta), \psi + h(\zeta + \theta))$$

then it is easy to see that the orbit closure of $(0,0,0)$ in $\mathcal{Y}_\zeta = (R_\zeta, \mathbf{T}^3, (0,0,0))$ is isomorphic to $\mathcal{X}_0 \vee \mathcal{X}_\zeta$. We say that $\mathcal{X}_0$ and $\mathcal{X}_\zeta$ are *disjoint* over $\mathcal{W} = (\sigma, \mathbf{T}, 0)$ if $\mathcal{Y}_\zeta$ is minimal.

Let $\mathscr{X}_\zeta = (S_\zeta, \mathbf{T}^2, (0,0))$ where

$$S_\zeta(\theta, \phi) = (\theta + \alpha, \phi + g_\zeta(\theta)).$$

The map $F: \mathbf{T}^3 \to \mathbf{T}^2: (\theta, \phi, \psi) \to (\theta, \psi - \phi)$ is a homomorphism of $\mathscr{Y}_\zeta$ onto $\mathscr{X}_\zeta$. Notice that

$$R_\zeta^n(\theta, \phi, \psi) = (\theta + n\alpha, \phi + h_n(\theta), \psi + h_n(\zeta + \theta)).$$

$$S_\zeta(0, \phi) = (n\alpha, \phi + H_n(\zeta)).$$

We let π be also the map $(\theta, \phi, \psi) \to \theta$; then $\pi: \mathscr{Y}_\zeta \to \mathscr{W}$ as well as $\pi: \mathscr{X}_\zeta \to \mathscr{W}$ are homomorphisms of flows.

In section 2 we show that for a residual subset Γ of $\mathbf{T}$, $\zeta \in \Gamma$ implies $\mathscr{X}_0$ is disjoint from $\mathscr{X}_\zeta$ over $\mathscr{W}$. From this we conclude that for every $\zeta \in \Gamma$, $A(\zeta)$ is dense in $\mathbf{T}$. In section 3 we prove a proposition which states that a subset $\Theta \subseteq \mathbf{T}$ which satisfies a certain condition with respect to h, has the property that for every $\varepsilon > 0$ there exists an integer n for which $H_n(\Theta)$ is ε-dense. We give Kozma's proof of the fact that when h is essential every subset Θ accumulating at zero satisfies the condition. (Originally we could prove this only for monotone h.) In section 4 we use the results of sections 2 and 3 to show that an a.p. $A \subseteq \mathbf{T}^2$ has the property $A \supseteq \{\theta\} \times \mathbf{T}$ whenever $\pi(A)$ is second category at θ (i.e. for every open interval V containing θ, $V \cap \pi(A)$ is second category). We also show that when h is essential $A \subseteq \mathbf{T}^2$ is a.p. iff $A \supseteq \{\theta\} \times \mathbf{T}$ whenever θ is an accumulation point of $\pi(A)$. An example of a non-essential function h for which this latter statement fails is given.

In the final section we consider a probability measure μ on the flow $(T, \mathbf{T}^2)$, where T is given by the function $h(\theta) = \theta$. We show that μ is a.p. iff it has the form $\mu = \mu_d + \nu_c \times m$ where $\pi(\mu) = \nu$, $\nu = \nu_d + \nu_c$ is the decomposition of ν into purely discontinuous and continuous parts, $\pi(\mu_d) = \nu_d$, and m is Lebesgue measure on $\mathbf{T}$. This theorem is not true for the general h.

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§2. Disjointness over $\mathscr{W}$

2.1. Theorem. *For a residual subset Γ of $\mathbf{T}$, $\zeta \in \Gamma$ implies $\mathscr{X}_0$ is disjoint from $\mathscr{X}_\zeta$ over $\mathscr{W}$. In other words $\mathscr{Y}_\zeta$ is minimal.*

We first prove a number of lemmas.

Let E be the enveloping semigroup of $(T, \mathbf{T}^2)$; E is thus a compact group of 1–1 transformations of $\mathbf{T}^2$ onto itself which is the closure of $\{T^n : n \in \mathbf{Z}\}$, considered as a subset of $(\mathbf{T}^2)^{\mathbf{T}^2}$, in the topology of pointwise convergence. E is also the enveloping semigroup of each of the flows $(\mathcal{R}_\zeta, \mathbf{T}^3)$, and it acts on $(S_\zeta, \mathbf{T}^2)$ and $(\sigma, \mathbf{T})$ as well. Notice however that if $p \in E$ and x is a point of $\mathbf{T}^3$, $\mathbf{T}^2$ or $\mathbf{T}$ then the point px depends upon the particular flow which is being considered. There is a weaker topology on E, called the τ-topology, in which E is compact T_1 and not Hausdorff. For details on the τ-topology the reader is referred to [2] or [5].

Put

$$H_\zeta = \{p \in E : p(\zeta, 0) = (\zeta, 0) \quad \text{in} \quad (T, \mathbf{T}^2)\},$$

$$G = \{p \in E : p(0) = 0 \quad \text{in} \quad (\sigma, \mathbf{T})\}$$

$$= \{p \in E : \pi(p(0,0)) = 0 \quad \text{in} \quad (T, \mathbf{T}^2)\}.$$

The following statements are easily verified:

(i) G is a τ-closed normal subgroup of E. The τ-topology on G coincides with the pointwise convergence topology, and in this topology G is a compact Hausdorff topological group. E/G is topologically isomorphic to $\mathbf{T}$, in both topologies.

(ii) For each $\zeta \in \mathbf{T}$, H_ζ is a closed normal subgroup of G and G/H_ζ is topologically isomorphic to $\mathbf{T}$.

(iii) $\mathcal{X}_0$ and $\mathcal{X}_\zeta$ are disjoint over $\mathcal{W}$ iff $H_0 H_\zeta = H_\zeta H_0 = G$.

2.2. Lemma. *If $\mathcal{V} = (\tau, V, v_0)$ is a pointed equicontinuous flow which is an extension of $\mathcal{W}$ and a factor of $\mathcal{X}_0$, then the extension $\mathcal{V} \to \mathcal{W}$ is trivial* (*i.e.* 1–1).

Proof. Let $H = \{p \in E : pv_0 = v_0\}$; H is a τ-closed subgroup of G and $H_0 \subseteq H \subseteq G$. Since $G/H_0 \cong \mathbf{T}$, H/H_0 is a subgroup of $\mathbf{T}$. If this is a proper subgroup it is finite and $\mathcal{X}_0$ is a finite group extension of $\mathcal{V}$. Since $\mathcal{V}$ is equicontinuous so is $\mathcal{X}_0$, [7], a contradiction. Thus $H = G$ and $\mathcal{V}$ is isomorphic to $\mathcal{W}$.

For each $\zeta \in \mathbf{T}$ put $G_\zeta = H_0 H_\zeta$. Since H_0 is normal in G this is a τ-closed subgroup of G. Since $G/H_0 \cong \mathbf{T}$, either $G_\zeta = G$ or G_ζ / H_0 is a finite subgroup $\{0, 1/l, 2/l, \cdots, (l-1)/l\}$ of $\mathbf{T}$. In the first case we write $l(\zeta) = \infty$ and in the second $l(\zeta) = l$.

2.3. Lemma. *The function $l(\zeta)$ on $\mathbf{T}$ is unbounded.*

Proof. Assume $\{l(\zeta) : \zeta \in \mathbf{T}\}$ is bounded say by l_0. Then the τ-closed subgroup H of G generated by $\cup \{G_\zeta : \zeta \in \mathbf{T}\}$ satisfies

$$H/H_0 \subseteq \left\{0, \frac{1}{l_0!}, \frac{2}{l_0!}, \cdots, \frac{l_0! - 1}{l_0!}\right\}.$$

Thus $H_0 \subseteq H \subsetneqq G$. We claim that H is a normal subgroup of E. In fact let $p \in E$ and $q \in H_\zeta$ then if in $(T, \mathbf{T}^2)$, $p^{-1}(\zeta, 0) = (\xi, \phi)$, we have (in $(\mathbf{T}^2, T)$)

$$p^{-1}qp(\xi, \phi) = p^{-1}q\,(\zeta, 0) = p^{-1}(\zeta, 0) = (\xi, \phi).$$

Since subtraction of ϕ from the second coordinate is an automorphism of $(T, \mathbf{T}^2)$ it follows that $p^{-1}qp \in H_\xi \subseteq G_\xi$, and H is normal in E. This implies that the unique pointed flow $\mathscr{V} = (\tau, V, v_0)$ which is a factor of $\mathscr{X}_0$, an extension of $\mathscr{W}$ and for which $\{p \in E : pv_0 = v_0\} = H$, is a regular flow. Now $\mathscr{V}$ is regular, distal, and by Lemma 2.2 is not equicontinuous. By [1, p. 609] such a flow cannot be metric; a contradiction.

We notice that since the pointed flows $\mathscr{X}_\zeta$ and $\mathscr{X}_{\zeta+\alpha}$ are isomorphic, $H_\zeta = H_{\zeta+\alpha}$ and $l(\zeta) = l(\zeta + \alpha)$. Thus we have the following

2.4. Corollary. *Either $l(\zeta) = \infty$ for every $\zeta \in \mathbf{T}$ or for every l_0 there exists $l_1 > l_0$, $l_1 \leqq \infty$, for which $\{\zeta \in \mathbf{T} : l(\zeta) = l_1\}$ is dense in $\mathbf{T}$.*

2.5. Lemma. *Given $\zeta_0 \in \mathbf{T}$, an open set $U \subseteq \mathbf{T}^3$ which contains a point of the form $(0, \phi, 0)$, and an $\varepsilon > 0$, there exists $\zeta \in \mathbf{T}$ and $n \in \mathbf{Z}$ such that*

$$R_\zeta^n(0,0,0) \in U \quad \textit{and} \quad |\zeta - \zeta_0| < \varepsilon.$$

Proof. Let l_0 be a positive integer such that $|\chi - \phi| < 1/l_0$ implies $(0, \chi, 0) \in U$. Let $\zeta \in \mathbf{T}$ be chosen so that $l = l(\zeta) > l_0$ and $|\zeta - \zeta_0| < \varepsilon$. Such a ζ exists by Corollary 2.4. If $l = \infty$, $\mathscr{Y}_\zeta$ is minimal and the lemma follows. Thus we can assume that $l < \infty$ so that $H_\zeta H_0/H_0 = \{0, 1/l, \cdots, (l-1)/l\}$. Given $p \in H_\zeta$, such that pH_0 generates G_ζ/H_0 we have for some $\chi \in \mathbf{T}$ (in $\mathscr{Y}_\zeta$)

$$p(0,0,0) = (0, \chi, 0), \quad p^2(0,0,0) = (0, 2\chi, 0), \quad \cdots, \quad p^l(0,0,0) = (0,0,0),$$

so that $\chi = k_0/l$ $(1 \leqq k_0 \leqq l-1)$ and for some k, $|k\chi - \phi| < 1/l < 1/l_0$. If $p^k = \lim R_\zeta^{n_i}$ then eventually $R_\zeta^{n_i}(0,0,0) \in U$.

2.6. Lemma. *Let $\zeta \in \mathbf{T}$ be such that the orbit closure of $(0,0,0)$ in $\mathscr{Y}_\zeta$ contains the subset $\{0\} \times \mathbf{T} \times \{0\}$ of $\mathbf{T}^3$. Then $\mathscr{Y}_\zeta$ is minimal.*

Proof. Let $\phi, \psi \in \mathbf{T}$. Let $q \in G$ be such that in $(T, \mathbf{T}^2)$, $q(\zeta, 0) = (\zeta, \psi)$ and let $q(0,0) = (0, \chi)$. By our assumption there exists $p \in E$ such that in $\mathscr{Y}_\zeta$, $p(0,0,0) = (0, \phi - \chi, 0)$. Now in $\mathscr{Y}_\zeta$

$$qp(0,0,0) = q(0, \phi - \chi, 0) = (0, \phi, \psi).$$

Thus $\{0\} \times \mathbf{T} \times \mathbf{T}$ is contained in the orbit closure of $(0,0,0)$ in $\mathcal{Y}_\zeta$. It follows easily that the orbit closure of $(0,0,0)$ in $\mathcal{Y}_\zeta$ is all of $\mathbf{T}^3$. Since $\mathcal{Y}_\zeta$ is distal, hence semisimple, our lemma follows.

We can now prove Theorem 2.1.

PROOF. Let U be an open subset of $\mathbf{T}^3$ which intersects $\{0\} \times \mathbf{T} \times \{0\}$ and put

$$\Gamma(U) = \{\zeta \in \mathbf{T} : \exists n, R_\zeta^n(0,0,0) \in U\}.$$

Clearly $\Gamma(U)$ is an open subset of $\mathbf{T}$; by Lemma 2.5 it is also dense in $\mathbf{T}$. Now let $\{U_i\}$ be a countable collection of open sets in $\mathbf{T}^3$ such that $\{U_i \cap (\{0\} \times \mathbf{T} \times \{0\})\}$ is a basis for open sets in $\{0\} \times \mathbf{T} \times \{0\}$. Then

$$\Gamma = \cap \Gamma(U_i)$$

is a residual subset of $\mathbf{T}$ and $\zeta \in \Gamma$ implies

$$\text{cls}\{R_\zeta^n(0,0,0) : n \in \mathbf{Z}\} \supseteq \{0\} \times \mathbf{T} \times \{0\}.$$

By Lemma 2.6 $\mathcal{Y}_\zeta$ is minimal.

2.7. THEOREM. *Let $h : \mathbf{T} \to \mathbf{T}$ be continuous and such that the corresponding flow $(T, \mathbf{T}^2)$ is not equicontinuous. Then for a residual sub-set Γ of $\mathbf{T}$, $\zeta \in \Gamma$ implies $A(\zeta)$ is dense in $\mathbf{T}$.*

PROOF. We recall that the map $F : \mathbf{T}^3 \to \mathbf{T}^2$, $F(\theta, \phi, \psi) = (\theta, \psi - \phi)$ is a homomorphism of $\mathcal{Y}_\zeta$ onto $\mathcal{X}_\zeta$. If $\Gamma \subseteq \mathbf{T}$ is the residual subset of Theorem 2.1, then it follows that $\mathcal{X}_\zeta$ is minimal whenever $\zeta \in \Gamma$. Since

$$S_\zeta^n(0,0) = (n\alpha, H_n(\zeta))$$

the theorem follows.

REMARK. The fact that for $\zeta \in \Gamma$, $\mathcal{X}_\zeta$ is minimal can be stated as follows. For each positive integer l the equation $lg_\zeta(\theta) = f(\theta + \alpha) - f(\theta)$ has no continuous solution, [3].

PROBLEM. When $h(\theta) = \theta$, $H_n(\theta) = n\theta$ and the set Γ is the set of irrationals in $[0,1)$, is $\mathbf{T} \backslash \Gamma$ always countable?

§3. The asymptotic behavior of $H_n(\Theta)$

Let Θ be a subset of $\mathbf{T}$ having 0 as an accumulation point. We say that Θ satisfies condition (*) *if for every $\eta > 0$ and a non-empty open subset V of $\mathbf{T}$, there exist $\theta \in \Theta$ and an integer n such that $|\theta| < \eta$ and $H_n(\theta) \in V$.*

3.1. Proposition. *Let* $\Theta \subseteq \mathbf{T}$ *satisfy condition* $(*)$, *then for every* $\varepsilon > 0$ *there exists an integer* n *such that* $H_n(\Theta) = \{H_n(\theta): \theta \in \Theta\}$ *is* ε*-dense in* $\mathbf{T}$.

Proof. We first show that the following statement is true. Given an open non-empty subset V of $\mathbf{T}$ and a cofinite subset N of $\mathbf{Z}$, there exists a subset $N' \subseteq N$ and $\theta \in \Theta$ such that N' is cofinite in $\mathbf{Z}$ and $H_k(\theta) \in V$ for every $k \in N'$.

Let $\mathbf{Z} = N + \{1, \cdots, d\}$, let $\theta_0 \in V$ and choose $\delta > 0$ such that $\{\theta: |\theta - \theta_0| < \delta\} \subseteq V$. Put $U = \{\theta: |\theta - \theta_0| < \frac{1}{2}\delta\}$. Since Θ satisfies condition $(*)$ there exists $\theta \in \Theta$ such that for every $\zeta \in \mathbf{T}$, $|h(\zeta + \theta) - h(\zeta)| < \delta/2d$ and for which $H_n(\theta) \in U$ for some integer n. We observe that in the flow $\mathscr{X}_\theta$, $S_\theta^k(0,0) = (k\alpha, H_k(\theta))$. Since $\mathscr{X}_\theta$ is a semi-simple flow, we have $H_k(\theta) \in U$ for k in a cofinite subset N_1 of $\mathbf{Z}$, say $\mathbf{Z} = N_1 + \{1, \cdots, t\}$.

For every $k \in N_1$ choose $n_k \in N$ such that $|k - n_k| < d$ and let $N' = \{n_k : k \in N_1\}$. Then $N' \subseteq N$, $\mathbf{Z} = N' + \{1, \cdots, d + t\}$ and if $n_k \in N'$ then (without loss of generality we assume $n_k \geqq k > 0$)

$$\begin{aligned}|H_{n_k}(\theta) - H_k(\theta)| &= |h((k-1)\alpha + \theta) + \cdots + h(n_k\alpha + \theta) \\ &\qquad - h((k-1)\alpha) \cdots - h(n_k\alpha)| \\ &\leqq d\frac{\delta}{2d} = \tfrac{1}{2}\delta.\end{aligned}$$

Thus $H_k(\theta) \in U$ implies $H_{n_k}(\theta) \in V$ and our claim is proved.

Now let $\{V_1, \cdots, V_l\}$ be a covering of $\mathbf{T}$ by open intervals of length $< \varepsilon$. Inductively we choose elements $\theta_1, \theta_2, \cdots, \theta_l$ in Θ and cofinite subsets of $\mathbf{Z}$, $N_1 \supseteq N_2 \supseteq \cdots \supseteq N_l$ such that $H_n(\theta_i) \in V_i$ whenever $n \in N_j$ and $j \geqq i$. If $n \in N_l$ then $H_n(\theta_i) \in V_i$ for $i = 1, \cdots, l$ and consequently $H_n(\Theta)$ is ε-sence in $\mathbf{T}$.

3.2. Corollary. *Let* Θ *be an infinite subset of* $\mathbf{T}$ *and* $\varepsilon > 0$, *then there exists a positive integer* n *such that* $n\Theta = \{n\theta: \theta \in \Theta\}$ *is* ε*-dense in* $\mathbf{T}$.

Proof. Take $h(\theta) = \theta$, then $H_n(\theta) = n\theta$. If Θ accumulates at zero it clearly satisfies condition $(*)$ and by Proposition 3.1 $n\Theta$ is ε-dense for some n. If Θ accumulates at ζ then $\{\theta - \zeta: \theta \in \Theta\}$ accumulates at 0 and for some n, $\{n\theta - n\zeta: \theta \in \Theta\}$ and hence also $n\Theta$ are ε-dense.

Let $h: \mathbf{T} \to \mathbf{T}$ be a continuous function of index $d \geqq 1$. h can be lifted to a function $\tilde{h}: \mathbf{R} \to \mathbf{R}$ and then for every $\zeta \in \mathbf{R}$, $\tilde{h}(\zeta + 1) - \tilde{h}(\zeta) = d$. Notice that for a fixed $\lambda \in \mathbf{R}$, $\tilde{h}(\zeta + \lambda) - \tilde{h}(\zeta)$ is a periodic function of period 1. Put

$$\tilde{H}_n(\theta) = \sum_{k=0}^{n-1} [\tilde{h}(\theta + k\alpha) - \tilde{h}(k\alpha)] \quad (\theta \in \mathbf{R}).$$

The following theorem is due to I. Kozma.

3.3. THEOREM. *Let h be essential. If Θ is a subset of* **T** *accumulating at zero then for every $\varepsilon > 0$ there exists an integer n such that $H_n(\Theta)$ is ε-dense in* **T**.

PROOF. By Proposition 3.1 all we have to show is that Θ satisfies condition (*). Without loss of generality we can assume that h is of index $d \geqq 1$. We can also assume that $\Theta \subseteq [0,1)$ and accumulates at zero as a subset of **R**. Let $\eta > 0$ and an interval V of length $\delta > 0$ in **T** be given. By a compactness argument we deduce the existence of a $\beta > 0$ such that $0 \leqq \gamma < \beta$ implies $\tilde{h}(\zeta + (1-\gamma)) - \tilde{h}(\zeta) > d - \frac{1}{2}\delta$, for every $\zeta \in \mathbf{R}$. Choose $\theta \in \Theta$ such that (i) $0 < \theta < \eta$, (ii) $\theta < \beta$, and (iii) for every $\zeta \in \mathbf{R}$, $|\tilde{h}(\zeta + \theta) - \tilde{h}(\zeta)| < \frac{1}{2}\delta$.

Let N be a positive integer such that $1 - \beta < N\theta \leqq 1$. For each $\zeta \in \mathbf{R}$ consider the sum

$$\sum_{k=0}^{N-1} [\tilde{h}(\theta + \zeta + k\theta) - \tilde{h}(\zeta + k\theta)] = \tilde{h}(\zeta + N\theta) - \tilde{h}(\zeta) > d - \tfrac{1}{2}\delta.$$

There exists a positive integer m such that $|m\alpha - \theta - p|$, for some integer p, is so small that also

$$\sum_{k=0}^{N-1} [\tilde{h}(\theta + \zeta + km\alpha) - \tilde{h}(\zeta + km\alpha)] > d - \tfrac{1}{2}\delta \qquad (\zeta \in \mathbf{R}).$$

Now

$$\begin{aligned}\tilde{H}_{mN+1}(\theta) &= \sum_{k=0}^{mN} [\tilde{h}(\theta + k\alpha) - \tilde{h}(k\alpha)] \\ &= \sum_{l=0}^{m} \sum_{k=0}^{N-1} [\tilde{h}(\theta + l\alpha + km\alpha) - \tilde{h}(l\alpha + km\alpha)] \\ &\geqq m(d - \tfrac{1}{2}\delta) \geqq d - \tfrac{1}{2}\delta.\end{aligned}$$

Since, for every k, $|\tilde{H}_{k+1}(\theta) - \tilde{H}_k(\theta)| < \frac{1}{2}\delta$, we conclude that the set $\{H_k(\theta)\}_{k=0}^{mN+1}$, is δ-dense in **T**. Thus for some k, $H_k(\theta) \in V$ and the proof is completed.

§4. Almost periodic subsets

Let A be a closed subset of $\mathbf{T}^2$, put $\Theta = \pi(A)$ and let $\Theta = \Theta' \cup \Theta''$ where Θ'' is the set of isolated points of Θ. For every $\theta \in \Theta''$ let $A_\theta = A \cap \pi^{-1}(\theta)$ and let $A'' = \cup\{A_\theta : \theta \in \Theta''\}$, $A' = A \setminus A''$.

4.1. THEOREM. *If $A' = \Theta' \times \mathbf{T} = \pi^{-1}(\Theta')$ then A is an a.p. set.*

PROOF. It suffices to show that for every net $\{n_i\}$ in $\mathbf{Z}$ such that $T^{n_i}x \to x$ for every $x \in \mathbf{T}^2$ and for which $\lim T^{n_i}A = B$ exists, $B = A$. Consider an arbitrary subnet $\{n_j\}$ and a net $(\theta_j, \phi_j) \in A$. We have to show that $\lim T^{n_j}(\theta_j, \phi_j) = (\theta, \phi) \in A$. We can assume that $(\theta_j, \phi_j) \to (\theta_0, \phi_0) \in A$. Clearly $\theta = \theta_0$. Now if $\theta_0 \in \Theta''$ then ultimately $(\theta_j, \phi_j) = (\theta_0, \phi_j)$ and $T^{n_j}(\theta_0, \phi_j) \to (\theta_0, \phi) = (\theta, \phi) \in A$. If $\theta_0 \in \Theta'$ then by our assumption $(\theta_0, \phi) = (\theta, \phi) \in A$ and we conclude that $B \subseteq A$. Since clearly $A \subseteq B$ we have our desired equality.

4.2. THEOREM. *Let A be a closed subset of $\mathbf{T}^2$ which is almost periodic in the flow $(T, \mathbf{T}^2)$. If $\pi(A)$ is of second category at $\theta \in \mathbf{T}$ then $\{\theta\} \times \mathbf{T} \subseteq A$.*

PROOF. Assume first that $\pi(A)$ is of second category at zero. Let (θ_i, ϕ_i) be a sequence of points in A which converges to $(0, \phi)$, and such that $\theta_i \in \Gamma$ for every i. Such a sequence exists by our assumption on $\pi(A)$ and by the fact that Γ is residual (Theorem 2.7). Since for each i, $A(\theta_i)$ is dense in $\mathbf{T}$, each of the sets $\Theta_k = \{\theta_i\}_{i=k}^{\infty}$ clearly satisfy condition (*). By Proposition 3.1, given an $\varepsilon > 0$ we can find n for which $H_n(\Theta_k)$ is ε-dense.

Now

$$T^n(\theta_i, \phi_i) = (\theta_i + n\alpha, \phi_i + h_n(\theta_i)) = (\theta_i + n\alpha, \phi_i + H_n(\theta_i) + h_n(0))$$

and it follows that for some sequence n_j

$$B = \lim T^{n_j}A \supseteq \zeta \times \mathbf{T},$$

where $\zeta = \lim n_j\alpha$. Since A is a.p. this implies $A \supseteq \{0\} \times \mathbf{T}$. The general case follows since if $\pi(A)$ is second category at ζ then for some B in the orbit closure of A in $2^{\mathbf{T}^2}$, $\pi(B)$ is second category at zero.

4.3. THEOREM. *Let h be essential; a closed subset A of $\mathbf{T}^2$ is a.p. iff $A' = \Theta' \times \mathbf{T}$. Each orbit closure in $2^{\mathbf{T}^2}$ contains a unique minimal set.*

PROOF. By Theorem 4.1 the condition is sufficient. The necessity follows as in the proof of Theorem 4.2, only one uses Theorem 3.3 instead of Theorem 2.7. The second statement in the theorem follows immediately from the first.

Next we show an example of a function $h: \mathbf{T} \to \mathbf{T}$ for which the flow $(T, \mathbf{T}^2)$ is not equicontinuous, yet there exists a sequence $\Theta = \{\theta_i\} \subseteq \mathbf{T}$, which converges to zero and for which the sets $A(\theta_i) = \{H_n(\theta_i): n \in \mathbf{Z}\}$ converge to a point. If we let $A = \{(\theta_i, 0): \theta_i \in \Theta\} \cup \{(0,0)\}$ then A is an a.p. set which does not contain $\{0\} \times \mathbf{T}$.

Consider the function h constructed in [3, p. 585]. This generates a minimal flow which is not uniquely ergodic hence not equicontinuous. Using the notations of [3] we have for $\zeta = \zeta_{i_0} = n_{i_0}^{-1}$

$$H_n(\zeta) = \sum_{k=0}^{n-1} [h(\zeta + k\alpha) - h(k\alpha)]$$

$$= \sum_{k=0}^{n-1} \left[\sum_{l \neq 0} \frac{1}{|l|} (e^{2\pi i n_l \alpha} - 1) e^{2\pi i n_l (\zeta + k\alpha)} \right.$$

$$\left. - \sum_{l \neq 0} \frac{1}{|l|} (e^{2\pi i n_l \alpha} - 1) e^{2\pi i n_l k\alpha} \right]$$

$$= \sum_{k=0}^{n-1} \left[\sum_{l \neq 0} \frac{1}{|l|} (e^{2\pi i n_l \alpha} - 1) e^{2\pi i n_l k\alpha} (e^{2\pi i n_l \zeta} - 1) \right]$$

$$= \sum_{0<|l|<l_0} \frac{1}{|l|} (e^{2\pi i n_l \alpha} - 1) \left(\sum_{k=0}^{n-1} e^{2\pi i n_l k\alpha} \right) (e^{2\pi i n_l \zeta} - 1)$$

$$= \sum_{0<|l|<l_0} \frac{1}{|l|} (e^{2\pi i n_l \zeta} - 1)(e^{2\pi i n_l n\alpha} - 1).$$

Hence

$$\left| H_n(\zeta) + \sum_{0<|l|<l_0} \frac{1}{|l|} (e^{2\pi i n_l \zeta} - 1) \right| =$$

$$= \left| \sum_{0<|l|<l_0} \frac{1}{|l|} e^{2\pi i n_l n\alpha} (e^{2\pi i n_l \zeta} - 1) \right|$$

$$\leqq 2 \sum_{0<l<l_0} \frac{1}{|l|} |e^{2\pi i n_l \zeta} - 1| \leqq 4\pi (2^{-n_{l_0}-1}).$$

Thus the diameter of the sets $A(\zeta_l)$ tends to zero as l tends to infinity. Choosing a subsequence $\Theta = \{\zeta_{l_i}\}$ so that $A(\zeta_{l_i})$ converge to a point, we obtain the set we are looking for.

§5. Almost periodic measures

Let $\mu \in \mathcal{P}(\mathbf{T}^2)$, write $\nu = \pi(\mu) \in \mathcal{P}(\mathbf{T})$ and let $\nu = \nu_c + \nu_d$ be the decomposition of ν into continuous and purely discontinuous parts. If $\nu_d \neq 0$ let $\nu_d = \Sigma\, a_i \delta_{\theta_i}$ where δ_{θ_i} is the point mass at θ_i and $a_i > 0$. For each θ_i let μ_i be the restriction of μ to $\pi^{-1}(\theta_i)$, then put $\mu'' = \Sigma \mu_i$ and $\mu' = \mu - \mu''$. Clearly $\pi(\mu') = \nu_c$ and $\pi(\mu'') = \nu_d$. Let $a = \Sigma a_i$ then $a^{-1}\mu''$ and $(1-a)^{-1}\mu'$ are probability measures.

5.1. THEOREM. *If $\mu' = \nu_c \times m$ then μ is a.p.*

PROOF. By [4, lemma 3.6.] each of the measures $a_i^{-1}\mu_i$ is an almost periodic point of $\mathcal{P}(\mathbf{T}^2)$ and moreover its orbit closure in $\mathcal{P}(\mathbf{T}^2)$ is distal. It therefore follows that the point $(a_1^{-1}\mu_1, a_2^{-1}\mu_2, \cdots)$ of the flow $\mathcal{P}(\mathbf{T}^2)^{\mathbf{Z}^+}$ is an almost periodic point. Now it is easy to see that a sequence $\{T^{n_i}a^{-1}\mu''\}$ converges to a measure γ iff $\gamma = \Sigma\gamma_i$ where $\gamma_i = \lim T^{n_i}a^{-1}\mu_i$. Hence the orbit closure of $(a_1^{-1}\mu_1, a_2^{-1}\mu_2, \cdots)$ in $\mathcal{P}(\mathbf{T}^2)^{\mathbf{Z}^+}$ is isomorphic with the orbit closure of $a^{-1}\mu''$ in $\mathcal{P}(\mathbf{T}^2)$. In particular $a^{-1}\mu''$ is a.p. Finally it is clear that $(1-a)^{-1}(\nu_c \times m)$ is a.p. and its orbit closure is distal; thus $\mu = \nu_c \times m + \mu''$ is a.p.

Following the example of R. Jewett presented in [6] we now show that for the function $h(\theta) = \theta$ the converse is also true.

5.2. THEOREM. *Let $h(\theta) = \theta$; then a measure $\mu \in \mathcal{P}(\mathbf{T}^2)$ is a.p. iff $\mu' = \nu_c \times m$. Each orbit closure in $\mathcal{P}(\mathbf{T}^2)$ contains a unique minimal set.*

PROOF. Assume μ is a.p. and write $\mu = \mu' + \mu''$ as above. Since $a^{-1}\mu''$ is also a.p. (Theorem 5.1) it follows that so is $(1-a)^{-1}\mu'$. Thus it suffices to show that if $\mu \in \mathcal{P}(\mathbf{T}^2)$ is a.p. and $\pi(\mu) = \nu$ is continuous then $\mu = \nu \times m$.

Let $\mu = \int \mu_\theta d_\nu(\theta)$, where $\theta \to \mu_\theta$ is a ν-measurable map of $\mathbf{T}$ into $\mathcal{P}(\mathbf{T})$, be a disintegration of μ. We compute the Fourier transform of $T^n\mu$. Put $f(\theta,\phi) = \exp[2\pi i(k\theta + l\phi)]$ and notice that $T^n(\theta,\phi) = (\theta + n\alpha, \phi + n\theta + \frac{1}{2}n(n-1)\alpha)$. Now

$$(T^n\mu)^\wedge(k,l) = \int f(\theta,\phi)dT^n\mu = \int f(T^n(\theta,\phi))d\mu$$

$$= e^{2\pi i(k+(l/2)(n-1))n\alpha}\int\int e^{2\pi i(k+ln)\theta}e^{2\pi i l\phi}d\mu_\theta(\phi)\,d\nu(\theta)$$

$$= e^{2\pi i(k+(l/2)(n-1))n\alpha}\int e^{2\pi i(k+ln)\theta}\hat{\mu}_\theta(l)d\nu(\theta).$$

The function $\theta \to \hat{\mu}_\theta(l)$ is clearly in $L^1(\nu)$ and $\hat{\mu}_\theta(0) \equiv 1$. Let $d\gamma_l = \hat{\mu}_\theta(l)d\nu$; then γ_l is continuous and by Wiener's theorem

$$\lim \frac{1}{2N+1}\sum_{-N}^{N} |\hat{\gamma}_l(n)|^2 = 0.$$

This implies that there exists a subsequence n_j for which $\hat{\gamma}_l(k + ln_j) \to 0$ $\forall k \forall l \neq 0$ (for the complement of

$$J_j^{k,l} = \left\{n \in \mathbf{Z}: |\hat{\gamma}_l(k+ln)|^2, \cdots, |\hat{\gamma}_l(k + l(n+j))|^2 < \frac{1}{j}\right\}$$

$j = 1, 2, \cdots$ has density zero; thus choosing $n_j \in \bigcap_{|l|+|k|<j} J_j^{k,l}$ we obtain the desired subsequence). So that for $l \neq 0$,

$$\widehat{T^{n_j}\mu}(k,l) = e^{2\pi i(k+(l/2)(n_j-1))n_j\alpha}\hat{\gamma}_l(k+ln_j)\to 0,$$

while when $l=0$, $\widehat{T^{n_j}\mu}(k,0) = e^{2\pi i k n_j\alpha}\hat{\nu}(k)$.

Computing $\widehat{T^n(\nu\times m)}\,(k,l)$ we see that for $l\neq 0$ this is zero, while for $l=0$ the expression is $e^{2\pi i k n_j\alpha}\hat{\nu}(k)$. Thus $\lim T^{n_j}\mu = \lim T^{n_j}(\nu\times m)$. Since both μ and $\nu\times m$ are a.p. and both project onto ν, we conclude that $\mu=\nu\times m$. The last assertion in the theorem is now clear. The proof is completed.

Using again the example in [3, p. 585], we see that Theorem 5.2 is not true in the general case, for the flow $(T,\mathbf{T}^2)$ which corresponds to the function h produced in this example is not uniquely ergodic. In particular there exists a measure $\mu\in\mathcal{P}(\mathbf{T}^2)$ such that $T\mu=\mu$, and hence μ is a.p., but $\mu\neq m\times m$. Since $\sigma\pi(\mu)=\pi(T\mu)=\pi(\mu)$, it follows that $\pi(\mu)=m$. Thus μ is an almost periodic measure which projects onto m and does not equal $m\times m$.

PROBLEM. Is the conclusion of Theorem 5.2 true for $(T,\mathbf{T}^2)$ which is uniquely ergodic?

We conclude with the following observation. There exists a flow $(T,\mathbf{T}^2)$ and an a.p. measure $\mu\in\mathcal{P}(\mathbf{T}^2)$ such that $\operatorname{supp}(\mu)=\mathbf{T}^2$, but for some measure η in the orbit closure of μ the interior of $\operatorname{supp}(\eta)$ is empty. In fact let h be any essential function, let $\Theta=\{\theta_i\}_{i=1}^{\infty}$ be a dense subset of $[0,1)$. Let $z_i=(\theta_i,0)$ and put $\eta=\Sigma a_i\delta_{z_i}$ where $\Sigma a_i=1$ and $a_i>0$ for every i. Then $\operatorname{supp}(\eta)=\{(\theta,0):\theta\in\mathbf{T}\}$, and η is a.p. (Theorem 5.1). By Theorem 4.3 there exists μ in the orbit closure of η for which $\operatorname{supp}(\mu)=\mathbf{T}^2$.

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