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FINITE REPRESENTABILITY OF $l_p(X)$ IN ORLICZ FUNCTION SPACES

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ABSTRACT

We show that if $l_p(X)$, $p \neq 2$, is finitely crudely representable in an Orlicz space L_φ (which does not contain c_0) then the Banach space X is isomorphic to a subspace of L_p. The same remains true for $p = 2$ when L_φ is 2-concave or 2-convex, or if X has local unconditional structure. We extend a theorem of Guerre and Levy to Orlicz function spaces.

Introduction

Let X be a Banach space and $1 \leqq p < 2$. It was proved by N. Kalton ([K]) that if $l_p(X)$ isomorphically embeds into L_0 then X embeds into L_p. The same remains true for $p = 2$ as was shown by B. Maurey ([M]). Here we want to give an analogous statement for an Orlicz space L_φ instead of L_0. We consider only normed Orlicz spaces (i.e. associated to a convex Orlicz function) although the results easily extend to the quasinormed case.

In the frame of Orlicz spaces, it is more natural to take as hypothesis that $l_p(X)$ is finitely crudely representable in L_φ (cf. [JMST], p. 170 for a definition), which is equivalent to say that it is C-isomorphically embeddable in some ultrapower of L_φ (for some $C < \infty$).

Now we suppose $1 \leqq p < \infty$, $p \neq 2$ and obtain that X embeds in L_p. For $p = 2$, we have to suppose moreover the Orlicz space L_φ to be 2-concave (hence embeddable in L_1) or 2-convex. This restriction can be avoided when X has local unconditional structure. As an application we give an extension to Orlicz spaces of a result of S. Guerre and M. Lévy (see [GL]), concerning l_p spaces in

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subspaces of L_1: an infinite dimensional subspace E of L_φ contains $l_{p(E)}$ (resp. $l_{q(E)}$) when $p(E)$ (resp. $q(E)$), the g.l.b. (resp. l.u.b.) of type (resp. cotype) exponents of E, is different from 2. See also [R] for a less-refined version of this last result.

For $p > 2$ these results were announced in [R2]. A preliminary and shortened version of this work was given also in [R3].

Let us now recall some definitions concerning Musielak–Orlicz spaces (cf. [Mu]). Let $(\Omega, \mathscr{A}, \mu)$ be a measure space. A Musielak–Orlicz function is a measurable function $\psi : \Omega \times \mathbf{R}_+ \to \mathbf{R}_+$ with partial functions $\psi_\omega = \psi(\omega, .)$ being Orlicz. For $f \in L_0(\Omega)$ define the "modular":

$$\Psi(f) = \int_\Omega \psi(\omega, |f(\omega)|) d\mu(\omega).$$

Then $\| f \|_\psi = \inf\{a : \Psi(f/a) \leqq 1\}$ and the Musielak–Orlicz space is $L_\psi = \{ f \in L_0 / \| f \|_\psi < \infty\}$. Now if ψ is uniformly moderate, i.e.

$$\operatorname*{Ess\,sup}_{\omega} \sup_{t} \frac{\psi(\omega, 2t)}{\psi(\omega, t)} < \infty,$$

then $\| f \|_\psi$ is defined by $\Psi(f / \| f \|_\psi) = 1$.

If φ is a moderate Orlicz function $(\sup_t(\varphi(2t)/\varphi(t)) < \infty)$ then ultrapowers of $L_\varphi(\Omega, \mathscr{A}, \mu)$ are Musielak–Orlicz spaces $L_\psi(\tilde{\Omega}, \tilde{\mathscr{A}}, \tilde{\mu})$ (associated to an uniformly moderate M.–O. function and a "bigger" measure space): see e.g. [W] or [HLR]. So the finite representability of $l_p(X)$ in L_φ is equivalent to its embeddability in $L_\psi(\tilde{\Omega}, \tilde{\mathscr{A}}, \tilde{\mu})$. If X is assumed to be separable, then $(\tilde{\Omega}, \tilde{\mathscr{A}}, \tilde{\mu})$ may be supposed σ-finite, in fact (using a change of density) a probability space.

I. l_p sequences in a Musielak–Orlicz space $L_\psi(\Omega, \mathscr{A}, \mathbf{P})$

As we will be concerned with the asymptotic properties of such sequences, we will make use of the extension S_ψ of the space of random measures introduced (for Orlicz spaces) by Garling ([Ga]). Let N_ψ be the set of random probability measures μ on $(\Omega, \mathscr{A}, \mathbf{P})$ such that:

$$\Psi(\mu) := \mathbf{E} \int_{\mathbf{R}} \psi(\omega, |t|) d\mu_\omega(t) < \infty.$$

Let $\mathcal{O}_K$ be the set of K-moderate Orlicz functions f $(\operatorname{Sup}_{t>0}(f(2t)/f(t)) \leqq K)$. N_ψ is equipped with the w.m. topology (see [Ga], [A]) and $\mathcal{O}_K$ with the topology of uniform convergence on compact sets. Then $S_\psi = N_\psi \times \mathcal{O}_K$ is equipped with the (metrizable) topology such that:

$$\sigma_n = (\mu_n, f_n) \xrightarrow[n\to\infty]{} \sigma = (\mu, f) \qquad \text{iff } \mu_n \xrightarrow{\text{w.m.}} \mu \quad \text{and}$$

$$f_n + \Psi_{\mu_n} \xrightarrow[n\to\infty]{} f + \Psi_\mu \qquad (\text{where } \Psi_\mu(\lambda) = \mathbf{E} \int \psi(\omega, \lambda\,|\,t\,|)d\mu_\omega(t)).$$

Recall that S_ψ is locally compact (i.e. the sets $\{\sigma = (\mu, f) \in S_\psi, \Psi(\mu) + f(1) \leqq C\}$ are compact) and there is a natural homeomorphic embedding i of L_ψ in S_ψ: $i(x) = (\delta_x, 0)$, where $(\delta_x)_\omega = \delta_{x(\omega)}$ is the evaluation measure at the point $x(\omega)$. In particular if $i(x_n) \xrightarrow[n\to\infty]{} \sigma = (\mu, f)$ then for each $x \in L_\psi$:

$$\Psi(\lambda(x + x_n)) \xrightarrow[n\to\infty]{} \mathbf{E} \int \psi(\omega, \lambda\,|\,x(\omega) + t\,|)d\mu_\omega(t) + f(\lambda)$$

which allows one to calculate $t(x) = \lim_{n\to\infty} \| x + x_n \|$, the "type" defined by $(x_n)_{n=1}^\infty$ (in the sense of [KM]): see [Ga], th. 36.

On S_ψ are defined the operations of scaling and convolution:

- If $a \in \mathbf{R}$ and $\sigma = (\mu, f) \in S_\psi$ then $a \,.\, \sigma = (s_a\mu, s_a f)$ where $s_a\mu$ is the image of the random measure μ by the scaling $t \to at$; and $s_a f(t) = f(\,|\,a\,|\,t)$.
- If $\sigma = (\mu, f)$ and $\tau = (\nu, g)$ then $\sigma * \tau = (\mu * \nu, f + g)$.

If $\sigma \in S_\psi$ let $K_\psi(\sigma)$ be the closure of $\{\sigma\}$ under the scaling and convolution operations in S_ψ. Let $\bar{K}_\psi(\sigma)$ be its (topological) closure in S_ψ.

By a p-stable element of S_ψ we mean a couple (μ, f) where $f(t) = a^p t^p$ ($a \in \mathbf{R}_+$) and μ is a random p-stable symmetric probability distribution ($\hat{\mu}_\omega(t) = e^{-A^p(\omega)t^p}$) when $p \leqq 2$; μ is the constant δ_0 when $p > 2$.

A p-stable element is said to be non-trivial if it is distinct from the "zero-element"

$$0 = (\delta_0, 0).$$

The following proposition is merely an adaptation to the Orlicz setting of the result of Aldous ([A], th. 3.10) or Krivine–Maurey ([KM], th. IV.2).

Proposition 1. *If $\sigma \in S_\psi \setminus i(L_\psi)$ then $\bar{K}_\psi(\sigma)$ contains a nontrivial p-stable element (for a certain $p \in [1, \infty[$).*

Proof. *1st Case.* Suppose that $\sigma \in \mathcal{O}_K$, that is $\sigma = (\delta_0, f), f \in \mathcal{O}_K$. Then by [LT1], th. 4a9 and 4a8, there exist reals $(\alpha_i^k)_{i=1,\dots,N_K,\,k=1,\dots}$ and $a \neq 0$ such that the functions $f_k: f_k(\lambda) = \Sigma_{i=1}^{N_k} f(\lambda\alpha_i^k)$ converge (in $\mathcal{O}_K$) to $f_\infty(\lambda) = a^p\lambda^p$. Then

$$\sigma_k = (\delta_0, f_k) = \mathop{*}_{i=1}^{N_k} \alpha_i^k \,.\, \sigma$$

belongs to $K_\psi(\sigma)$ and converges to $\sigma_\infty = (\delta_0, f_\infty)$ which is a p-stable element of S_ψ.

2nd Case. Suppose $\bar{K}_\psi(\sigma) \cap \mathcal{O}_K \neq \{0\}$. Then if $\tau \in \bar{K}_\psi(\sigma) \cap \mathcal{O}_K$, $\bar{K}_\psi(\tau)$ contains a p-stable element and is contained in $\bar{K}_\psi(\sigma)$.

3rd Case. Suppose $\bar{K}_\psi(\sigma) \cap \mathcal{O}_K = \{0\}$. If $\sigma = (\mu, f)$, then (by Aldous theorem) there exist reals $(\alpha_i^k)_{i=1,\ldots,N_k,\, k=1,\ldots}$ such that:

$$\mu_k := \mathop{*}_{i=1}^{N_k} s_{\alpha_i^k}\mu \xrightarrow[k\to\infty]{\text{w.m.}} \mu_\infty,$$

a random p-stable probability measure (note that

$$\begin{aligned} |\mu| := \mathbf{E}\int |t|\,d\mu_\omega(t) &\leqq 1 + \mathbf{E}\int_{t\geqq 1} |t|\,d\mu_\omega(t) \\ &\leqq 1 + \mathbf{E}\int_{t\geqq 1} \psi(\omega, |t|)\,d\mu_\omega(t) \\ &\leqq 1 + \Psi(\mu) < \infty, \end{aligned}$$

as we suppose $\psi(\omega, 1) = 1$) and moreover such that

$$|\mu_k| \xrightarrow[k\to\infty]{} 1.$$

Then $|\mu_\infty| \leqq 1$ but it is *a priori* not clear that $\mu_\infty \in N_\psi$.

Let $\|\mu_k\|_\psi$ be the real such that $\Psi(s_{1/\|\mu_k\|}\cdot\mu) = 1$ and $\alpha'_k = \alpha_k / \|\mu_k\|$,

$$\mu'_k = \mathop{*}_{i=1}^{N_k} s_{\alpha_i'^k}\cdot\mu.$$

Then $\Psi(\mu'_k) = 1$.

We claim that $\sup_k \Sigma_{i=1}^{N_k} f(\alpha_i'^k) < \infty$. If not, there would exist reals γ_k with

$$\gamma_k \xrightarrow[k\to\infty]{} \infty \quad\text{and}\quad \sum_{i=1}^{N_k} f\left(\frac{\alpha_i'^k}{\gamma_k}\right) \xrightarrow[k\to\infty]{} 1.$$

Set:

$$\alpha_i''^k = \frac{\alpha_i'^k}{\gamma_k}, \quad \mu''_k = \mathop{*}_{i=1}^{N_k} s_{\alpha_i''^k}\mu = s_{1/\gamma_k}\mu'_k; \quad f''_k = \sum_{i=1}^{N_k} s_{\alpha_i''^k} f; \quad \sigma''_k = (\mu''_k, f''_k).$$

Then $\Psi(\mu''_k) \xrightarrow[k\to\infty]{} 0$ and $(f''_k)_k$ is relatively compact in $\mathcal{O}_K$. Thus we would have $\bar{K}_\psi(\sigma) \cap \mathcal{O}_K \neq \{0\}$ (containing any limit point of (σ''_k)).

We claim now that $\mathrm{Sup}_k \| \mu_k \|_\psi < \infty$. If not, then $|\mu_k'| \xrightarrow[k\to\infty]{} 0$, and thus $\mu_k' \xrightarrow{\text{w.m.}} 0$. Up to extraction, we could suppose:

$$\Psi_{\mu_k'} \xrightarrow[k\to\infty]{} g_\infty; \qquad f_k' = \sum_{i=1}^{N_k} s_{\alpha_i'^k} f \xrightarrow[k\to\infty]{} f_\infty' \in \mathcal{O}_K,$$

thus $\sigma_k'(\mu_k', f_k') \to (0, f_\infty' + g_\infty) \in \mathcal{O}_K$, and again $\bar{K}_\psi(\sigma) \cap \mathcal{O}_K \neq \{0\}$.

Now, by local compactness of S_ψ, the sequence $\sigma_k = (\mu_k, f_k)$ is relatively compact in S_ψ; clearly any of its limit points is of the form (μ_∞, h_∞), for some $h_\infty \in \mathcal{O}_K$ (μ_∞ being the preceding p-stable random probability). Note that $\mu_\infty \neq \delta_0$.

By the 1st step, $\bar{K}_\psi(0, h_\infty)$ contains a q-stable element $(0, k)$. Suppose $k = \lim_{k\to\infty} \Sigma_{i=1}^{M_k} s_{\beta_i^k} h_\infty$. If $b_k = \Sigma_{i=1}^{M_k} |\beta_i^k|^p \xrightarrow[k\to\infty]{} \infty$, then

$$\mathop{*}_{i=1}^{M_k} \frac{\beta_i^k}{b_k} \sigma_\infty \to (\mu_\infty, 0)$$

which is a non-trivial p-stable element in $\bar{K}_\psi(\sigma)$. If not, suppose

$$\left(\sum_{i=1}^{M_k} |\beta_i^k|^p \right)^{1/p} \xrightarrow[k\to\infty]{} b_\infty;$$

then:

$$* \beta_i^k \sigma_\infty \xrightarrow[k\to\infty]{} (s_{b_\infty} \mu_\infty, k) = \theta_\infty.$$

By considering $(1/n^{1/p}) *_{i=1}^n \theta_\infty$, we see that $\bar{K}_\psi(\sigma) \cap \mathcal{O}_K \neq \{0\}$ implies $p \geqq q$; if $p > q$ then, again, $\bar{K}_\psi(\sigma)$ contains $(\mu_\infty, 0)$; if $p = q$ it contains $(s_{b_\infty} \mu_\infty, k)$ which is a p-stable element. ■

Corollary 2. *Let $(x_n)_{n=1}^\infty$ be a l_p-sequence in the Musielak–Orlicz space $L_\psi(\Omega, \mathcal{A}, \mathbf{P})$. There exist a real $a \geqq 0$, and (if $p \leqq 2$) a random p-stable symmetric probability distribution $\mu = (\mu_\omega)_\omega$, and a sequence of normalized disjoint blocks $y_k = \Sigma_{i=1}^{N_k} \alpha_i^k x_{n_i}$ such that:*

(i) *For each $x \in L_\psi$,*

$$\lim_{k\to\infty} \Psi(\lambda(x + y_k)) = \mathbf{E} \int \psi(\omega, \lambda |x(\omega) + t|) d\mu_\omega(t) + a^p \lambda^p.$$

(ii) *μ is the limit conditional distribution of $(x_n)_n$ (in the terminology of* [BeR]).

(iii) $a^p \lambda^p = \lim_{M\to\infty} \lim_{k\to\infty} \Psi(\lambda \cdot y_k \cdot \mathbf{1}_{|y_k| > M})$.

The second condition is equivalent to:

$$\forall t \in \mathbf{R}, \qquad \exp(it\, x_n(\,.\,)) \xrightarrow[n\to\infty]{\sigma(L_\infty, L_1)} \hat{\mu}_{(\,.\,)}(t).$$

PROOF. We apply Proposition 1 to any limit point of $(i(x_n))_n$ in S_ψ. We obtain disjoint blocks $(y_k)_k$ with $i(y_k) \xrightarrow[k\to\infty]{} \sigma$, a q-stable element of S_ψ. But then as in the L_1 case ([A]), the subspace $\overline{\text{span}}(y_k)_k$ contains l_q, thus $q = p$. The points (i) to (iii) follow then immediately from [Ga]. ∎

II. The main result

We state now our main result:

THEOREM 3. *Let $1 \leqq p < \infty$, $p \neq 2$, and X be a Banach space; L_φ an Orlicz space which does not contain c_0.*

If $l_p(X)$ is C-finitely representable in L_φ then X is $K.C$-isomorphically embeddable in L_p (where $K = K(p)$).

In proving this theorem, we can suppose that φ is moderate on $\mathbf{R}_+$ and that X is separable. In fact we may suppose that $l_p(X)$ C-embeds in a Musielak–Orlicz space $L_\psi(\Omega, \mathscr{A}, \mathbf{P})$.

1. *Representation of X in S_ψ*

If $\sigma \in S_\psi$, $\sigma = (\mu, f)$, let us denote $\|\sigma\|$ the real such that

$$\mathbf{E}\int \psi\left(\omega, \frac{|t|}{\|\sigma\|}\right) d\mu_\omega(t) + f\left(\frac{1}{\|\sigma\|}\right) = 1.$$

Note that if $(x_n)_n \subset L_\psi$, $i(x_n) \xrightarrow[n\to\infty]{} \sigma$ then $\|x_n\| \xrightarrow[n\to\infty]{} \|\sigma\|$.

The idea of the following key lemma is essentially Maurey's one ([M]).

LEMMA 4. *Let T be an embedding of $l_p(X)$ in L_ψ.*

There is a map: $X \to S_\psi$, $x \to \tau(x)$ such that:

(i) *for all $x \in X$, $\tau(x)$ is a p-stable element of S_ψ;*

(ii) *$\tau(x) = \lim_{k\to\infty} T(b_k \otimes x)$, where $(b_k)_{k=1}^\infty$ is a sequence of disjoint normalized blocks on the l_p basis (fixed independently of x);*

(iii) *$\forall x_1, \ldots, x_n \in X$:*

$$\|T^{-1}\|^{-1}\left(\sum_i \|x_i\|^p\right)^{1/p} \leqq \left\| \mathop{*}_i \tau(x_i)\right\| \leqq \|T\| \left(\sum_i \|x_i\|^p\right)^{1/p}.$$

PROOF. Denote by $(e_n)_{n=1}^{\infty}$ the natural l_p basis. Then for each $x \in X$, the sequence $(T(e_n \otimes x))_{n=1}^{\infty}$ in L_ψ is equivalent to the l_p basis. Up to extraction we can suppose that

$$\forall x \in X, \qquad i(T(e_n \otimes x)) \xrightarrow[n \to \infty]{} \sigma(x) \in S_\psi$$

(use separability of X and a diagonal argument to obtain this convergence for the same subsequence of the l_p-basis).

Fix $x_0 \in X$. By Proposition 1, a suitable sequence of combinations

$$\sigma_k = \mathop{*}_{i=1}^{N_k} \alpha_i^k \sigma(x_0), \qquad \text{where } \sum_i |\alpha_i^k|^p = 1,$$

converges to a p-stable $\tau_0(x_0) \in S_\psi$. Again, by extracting and using a diagonal argument, we can suppose that $*_{i=1}^{N_k} \alpha_i^k \sigma(x)$ converges (for each $x \in X$) to a $\tau_0(x) \in S_\psi$, which is however *a priori* not p-stable for $x \neq x_0$. Note that:

$$\tau_0(x) = \lim_{k \to \infty} \lim_{n_1 \to \infty} \cdots \lim_{n_{N_k} \to \infty} i\left(\sum_{i=1}^{N_k} \alpha_i^k T(e_{n_i} \otimes x_0)\right),$$

i.e. $\tau_0(x)$ is a limit of a countable family $i(T(b_\alpha^0 \otimes x))$, where the b_α^0 are normalized blocks on the l_p basis, which may easily be taken disjoint.

Fix $x_1 \in X$. We can now choose $(\alpha_i'^k)_{i=1,\dots,N_k'}$ such that $\Sigma_i |\alpha_i'^k|^p = 1$, and:

$$\forall x \in X, \qquad \mathop{*}_{i=1}^{N_k'} \alpha_i'^k \tau_0(x) \to \tau_1(x) \in S_\psi,$$

and that $\tau_1(x_1)$ is a p-stable element. Note that $\tau_1(x_0) = \tau_0(x_0)$.

Iterating this procedure for a dense sequence $(x_l)_{l=1}^{\infty}$ in X, and using again a diagonal argument we obtain (i) and (ii) of Lemma 4; (iii) is an easy consequence of (i) and (ii). ∎

2. *The case $p > 2$*

In this case Lemma 4 provides a map $X \to \mathbf{R}_+$, $x \to a(x)$ such that:

$$(1) \qquad \|T^{-1}\|^{-1} \|x\| \leqq a(x) \leqq \|T\| \, \|x\| \qquad (\forall x \in X)$$

and a sequence $(b_k)_k$ of normalized disjoint blocks on the l_p basis such that:

$$\forall \lambda > 0, \quad \forall x \in X: \qquad \lim_{k \to \infty} \Psi(\lambda T(b_k \otimes x)) = a(x)^p \lambda^p.$$

This formula can be interpreted in any ultrapower $L_\psi = L_\psi^{\mathbf{N}}/\mathcal{U}$ by:

$$\tilde{\Psi}(\lambda v(x)) = a(x)^p \lambda^p \tag{2}$$

where $\tilde{\Psi}$ is the modular on the Musielak–Orlicz space $L_{\tilde{\psi}} = \hat{L}_\psi$ and $v: X \to L_{\tilde{\psi}}$ is the linear operator such that, for each $x \in X$, $v(x)$ is represented by $(T(b_k \otimes x))_{k=1}^{\infty}$.

We use then the following:

Lemma 5. *Let L_ψ be a Musielak–Orlicz space with modular Ψ.*

Let $\mathcal{G}$ be the order ideal in L_ψ formed by those elements g for which

$$\sup_{\beta > 0} \frac{1}{\beta^p} \Psi(\beta g) < \infty;$$

Let $\mathcal{G}_c$ be the subset of $\mathcal{G}$ formed by the elements g having constant ratio $(1/\beta^p)\Psi(\beta g)$.

There exists a lattice homomorphism h from $\mathcal{G}$ into a L_p-space such that for any $g \in \mathcal{G}_c$, and $\beta > 0$:

$$\| h(g) \|_{L_p}^p = \frac{1}{\beta^p} \Psi(\beta g).$$

Proof of Lemma 5. For each $g \in \mathcal{G}$, set:

$$\theta(g) = \lim_{\Lambda, \mathcal{U}} \frac{1}{2 \log \Lambda} \int_{1/\Lambda}^{\Lambda} \Psi(\lambda g) \frac{d\lambda}{\lambda^{p+1}}$$

where $\mathcal{U}$ is any nontrivial ultrafilter finer than the filter of neighborhoods of $+\infty$ in $\mathbf{R}_+$.

θ is nondecreasing and additive on disjoint elements of $\mathcal{G}$. Moreover it is homogeneous (of degree p). For, if $\rho \geqq 1$:

$$\begin{aligned}
\theta(\rho g) &= \lim_{\Lambda, \mathcal{U}} \frac{1}{2 \log \Lambda} \int_{1/\Lambda}^{\Lambda} \Psi(\lambda \rho g) \frac{d\lambda}{\lambda^{p+1}} \\
&= \rho^p \lim_{\Lambda, \mathcal{U}} \frac{1}{2 \log \Lambda} \int_{\rho/\Lambda}^{\rho\Lambda} \Psi(\lambda g) \frac{d\lambda}{\lambda^{p+1}} \\
&= \rho^p \theta(g) - \lim_{\Lambda, \mathcal{U}} \frac{1}{2 \log \Lambda} \int_{\Lambda}^{\rho\Lambda} \Psi(\lambda g) \frac{d\lambda}{\lambda^{p+1}} - \cdots
\end{aligned}$$

$$\cdots - \lim_{\Lambda,\mathscr{U}} \frac{1}{2\log\Lambda} \int_{1/\Lambda}^{\rho/\Lambda} \Psi(\lambda g)\, \frac{d\lambda}{\lambda^{p+1}}\,.$$

In this last expression, each integral is less than

$$\frac{\log\rho}{2\log\Lambda} \cdot \sup_{\lambda>0} \frac{\Psi(\lambda g)}{\lambda^p}$$

which converges to 0 as $\Lambda \to \infty$.

Hence the homogeneity of θ. Note that:

$$\theta(g) = \frac{1}{\beta^p}\, \Psi(\beta g) \qquad \text{for each } g \in \mathscr{G}_c \quad \text{and} \quad \beta > 0.$$

It is now a standard exercise to show that $g \to \theta(g)^{1/p}$ is a seminorm on $\mathscr{G}$ (see e.g. [K], Proof of lemma 2.1).

Let $\mathscr{N} = \{g \in \mathscr{G} : \theta(g) = 0\}$ and $E = \mathscr{G}/\mathscr{N}$: this is a normed vector lattice; its norm is a L_p norm (i.e. $|e| \wedge |f| = 0 \Rightarrow \| e + f \|^p = \| e \|^p,\ \forall e, f \in E$); its bidual E^{**} is therefore an L_p space ([LT2], th. l.b.2). ■

Coming back to the relation (2) we see that there exists $h : v(X) \to L_p(v)$ such that:

$$\forall x \in X \qquad a(x) = \| hv(x) \|^p_{L^p(v)}\,.$$

By (1), hv is a $\| T \| \, \| T^{-1} \|$-embedding of X in $L_p(v)$, which proves the conclusion of Theorem 3 in this case.

3. *The case* $1 \leqq p < 2$

Consider now a new Musielak–Orlicz function on Ω, defined by:

$$\bar\psi(\omega, \lambda) = \mathbf{E}_{\omega'}\, \psi(\omega, \lambda \,|\, Y(\omega')|)$$

where Y is a p-stable symmetric random variable (of Fourier transform $\mathbf{E} e^{itY} = e^{-|t|^p}$).

Extend $\bar\psi$ to the space $\bar\Omega = \Omega \cup \{\bar\omega\}$ by setting $\bar\psi(\bar\omega, t) = |t|^p$, and $\mathbf{P}$ to a measure on $\bar\Omega$ by giving the weight 1 to the point $\bar\omega$.

Recall that an application $f: X \to L$, where L is a vector lattice, is said to be of negative type iff:

$$\forall n, \quad \forall (x_i)_{i=1,\dots,n} \subset X, \quad \forall (c_i)_{i=1,\dots,n} \subset \mathbf{R}^n,$$

$$\sum_{i=1}^{n} c_i = 0 \Rightarrow \sum_{i,j=1}^{n} c_i c_j f(x_i - x_j) \leqq 0.$$

Lemma 6. *If $l_p(X)$ embeds in L_ψ, then there exists an application $\bar{A} : X \to L_{\bar{\psi}}^+(\bar{\Omega})$ such that:*

(i) *$\bar{A}$ is homogeneous of degree* 1,

(ii) *$\bar{A}^p : X \to L_0^+$ is of negative type,*

(iii) *for all $x_1, \ldots, x_n$ in X:*

$$\| T^{-1} \|^{-1} \left(\sum_{i=1}^{n} \| x_i \|^p \right)^{1/p} \leqq \left\| \left(\sum_{i=1}^{n} \bar{A}(x_i)^p \right)^{1/p} \right\|_{\bar{\psi}} \leqq \| T \| \left(\sum_{i=1}^{n} \| x_i \|^p \right)^{1/p}.$$

Proof. Lemma 4 provides now two maps:

$$a : X \to \mathbf{R}_+, \qquad A : X \to L_0$$

such that, μ^x being the random probability distribution of Fourier transform $\hat{\mu}_\omega^x(t) = e^{-A(x;\omega)^p |t|^p}$, we have:

$$\forall \lambda > 0, \qquad \lim_{k \to \infty} \Psi(\lambda T(b_k \otimes x)) = \mathbf{E}_\omega \int \psi(\omega, |t|) d\mu_\omega^x(t) + a(x)^p \lambda^p.$$

Let $\bar{A}(x, \omega) = A(x, \omega)$ and $\bar{A}(x, \bar{\omega}) = a(x)$. We have clearly:

$$\forall \lambda > 0, \quad \forall x \in X: \ \lim_{k \to \infty} \Psi(\lambda T(b_k \otimes x)) = \bar{\Psi}(\lambda \bar{A}(x)).$$

As $\tau(x) = (\mu^x, a(x)\lambda^p)$ we have $*_i \tau(x_i) = (*_i \mu^{x_i}, (\Sigma_i a(x_i)) . \lambda^p)$. Noticing that $*_i \mu^{x_i}$ has Fourier transform $\exp(-(\Sigma_i A(x_i)^p) |t|^p)$ we obtain more generally, for all $x_1, \ldots, x_n \in X$:

$$\forall \lambda > 0, \ \lim_{k_1 \to \infty} \cdots \lim_{k_n \to \infty} \Psi\left(\lambda \sum_{j=1}^{n} T(b_{k_j} \otimes x_j) \right) = \bar{\Psi}\left(\left(\sum_{j=1}^{n} \bar{A}(x_j)^p \right)^{1/p} \right)$$

which implies in particular the assertion (iii) of Lemma 6.

Assertion (i) is a consequence of the fact that $\tau(\alpha . x) = \alpha . \tau(x)$. To check assertion (ii) we note that (see (ii) of Corollary 2):

$$\forall t, \qquad e^{iT(b_k \otimes x)t} \xrightarrow[k \to \infty]{\sigma(L_\infty, L_1)} e^{-A(x)^p t^p}$$

thus the map $x \to e^{-A(x)^p}$, $X \to L_\infty$ is positive definite (as the w^* limit of positive definite functions); i.e. the map $x \to A(x)^p$ is of negative type. On the other hand (see (iii) of Corollary 2) we have:

$$\lim_{M\to\infty}\lim_{k\to\infty}(\lambda T(b_k\otimes x))\mathbf{1}_{|T(b_k\otimes x)|>M}=a(x)^p\lambda^p.$$

As in subsection 2, let us introduce an ultrapower $L_\psi=L_\psi^{\mathbf{N}}/\mathscr{U}=L_{\tilde\psi}$ and the linear operator $v:X\to L_{\tilde\psi}$, $x\to(T(b_k\otimes x))_k$. Consider in $L_{\tilde\psi}$ the band defined by sequences $(f_k)_k$ of functions in L_ψ whose support tends to 0 measure. Let P be the associated band projection. Then the preceding relation is interpreted as:

$$\tilde\Psi(\lambda Pv(x))=a(x)^p\lambda^p\qquad(\forall x\in X,\ \forall\lambda\in\mathbf{R}_+).$$

Now the same proof as in subsection 2 provides an operator $h:Pv(X)\to L_p(\nu)$ such that $a(x)=\|h\,Pv(x)\|_{L_p(\nu)}$ $(\forall x\in X)$; thus $x\to a(x)^p$ is a function of negative type (see [BDCK]). ■

Note that in the preceding we could obtain a very degenerate Orlicz function $\bar\psi_\omega$, i.e. $\bar\psi(\omega,t)=+\infty$ $(\forall t>0)$. However it does not happen for ω in the essential union S of the supports of $\bar A(x)$, $x\in X$. On S the Musielak–Orlicz function $\bar\psi$ is, up to equivalence, p-concave. For we have the following:

Lemma 7. *Let $1\leqq p<2$ and Y be a p-stable random variable normalized in $L_{p,\infty}$. If ψ is a moderate Orlicz function then:*

$$\bar\psi(\lambda)=\mathbf{E}\psi(\lambda Y)\sim\lambda^p\int_\lambda^\infty\psi'(u)\,\frac{du}{u^p}$$

(*with absolute equivalence constants*).

Proof of Lemma 7. We split:

$$\mathbf{E}\psi(\lambda Y)=\mathbf{E}\psi(\lambda Y\mathbf{1}_{|Y|\leqq 1})+\mathbf{E}\psi(\lambda Y\mathbf{1}_{|Y|>1}).\tag{3}$$

The first term is smaller than $\psi(\lambda)$. For the second we have:

$$\mathbf{E}\psi(\lambda Y\mathbf{1}_{|Y|>1})=\psi(\lambda)\mathbf{P}(|Y|>1)+\zeta(\lambda)$$

with

$$\zeta(\lambda)=\int_{\psi(\lambda)}^\infty\mathbf{P}[\psi(\lambda Y)\geqq u]du=\int_{\psi(\lambda)}^\infty\mathbf{P}[\lambda Y\geqq\psi^{-1}(u)]du$$

$$=\int_\lambda^\infty\mathbf{P}[\lambda Y\geqq t]\psi'(t)dt\sim\int_\lambda^\infty\frac{\lambda^p}{t^p}\,\psi'(t)dt,$$

the last equivalence resulting from standard asymptotical estimation of p-stable distribution.

On the other hand, as $t\psi'(t) \geqq \psi(t)$ we have:

$$\lambda^p \int_\lambda^\infty \psi'(t) \frac{dt}{t^p} \geqq \lambda^p \int_\lambda^\infty \psi(t) \frac{dt}{t^{p+1}} = \int_1^\infty \psi(\lambda s) \frac{ds}{s^{p+1}} \geqq \psi(\lambda) \int_1^\infty \frac{ds}{s^{p+1}} .$$

Thus $\zeta(\lambda) \gtrsim \psi(\lambda)$ (up to a constant factor). ■

So if $\bar{\psi}(\lambda) < \infty$ then $\bar{\psi}(\lambda)/\lambda^p$ is equivalent to a decreasing function; it is then well known that $\bar{\psi}$ is equivalent to a (not necessarily normalized) p-concave Orlicz function, with absolute equivalence constants (see [BDC]). So our preceding Musielak–Orlicz space $L_{\bar{\psi}}$ is in fact p-concave.

End of the Proof of Theorem 3. We apply to the (nonlinear, but homogeneous) operator $\bar{A} : X \to L_{\bar{\psi}}(\bar{\Omega})$ the same argument as in the proof of Krivine's factorization theorem ([LT 2], th. 1.d.11 or [Kr]) to obtain an L_1 norm on the lattice $\mathscr{T}$ generated by the elements $\bar{A}(x)^p$, $x \in X$ in $L_{\bar{\psi}}$, such that:

$$\forall x \in X \qquad \| \bar{A}(x)^p \|_1 \leqq \| T \|^p \| x \|^p,$$

$$\forall \xi \in \mathscr{T} \qquad \| \xi \|_1 \geqq \frac{1}{c_p(\bar{\psi})^p} \| \, |\xi|^{1/p} \|_{\bar{\psi}}^p$$

where $c_p(\bar{\psi})$ is the p-concavity constant of the lattice $L_{\bar{\psi}}$. Thus

$$\| \bar{A}(x)^p \|_1 \geqq \frac{1}{c_p(\bar{\psi})^p \| T^{-1} \|^p} \| x \|^p \qquad (\forall x \in X).$$

As the map $x \to \bar{A}(x)^p$ is of negative type ($X \to L_{\bar{\psi}}$), the same is true for the map $x \to \| \bar{A}(x)^p \|_1$. Thus $x \to \| x \|^p$ is C_1^p equivalent to a negative type function ($C_1 = C \, . \, c_p(\bar{\psi})$). By the isomorphic version of a theorem of Bretagnolle, Dacunha-Castelle and Krivine (Lemma 8 below) X is C_1-isomorphic to a subspace of L_p. ■

For the sake of completeness, we state the following lemma, which is a slight modification of th. 6.1 of [AMM].

Lemma 8. *Let X be a normed space; suppose that the map $x \to \| x \|^p$ is C^p equivalent to a negative type function $x \to f(x)$. Then X C-embeds in L_p.*

Proof. Note that $x \to e^{-f(x)}$ is positive definite and that, for all $q < p$:

$$\| x \|^q \underset{C^q}{\sim} K_q \int_0^\infty (1 - e^{-f(tx)}) \frac{dt}{t^{q+1}} .$$

By [AMM], lemma 4.2, there is a continuous linear operator U: $X \to L_0(\Omega', \mathcal{A}', \mathbf{P}')$ such that:

$$e^{-f(tx)} = \int_{\Omega'} \exp(itUx(\omega'))d\mathbf{P}'(\omega')$$

and we obtain:

$$\int_0^\infty (1 - e^{-f(tx)}) \frac{dt}{t^{q+1}} = K'_q \int_\Omega | Ux(\omega)|^q d\mathbf{P}'(\omega')$$

Thus X is $C^{q/p}$ embeddable in L_q, for each $q < p$. ■

III. The case $p = 2$

In this case the preceding Musielak–Orlicz functions ψ and $\bar{\psi}$ are equivalent. For we can suppose $\psi(t)/t^q$ decreasing (for a $q < \infty$) and thus (G being a L_2-normalized gaussian random variable)

$$C'_q \psi(\lambda) \leqq \bar{\psi}(\lambda) = \mathbf{E}_{\omega'} \psi(\lambda G(\omega')) \leqq C_q \psi(\lambda)$$

with $C_q = \mathbf{E}(|G| \vee |G|^q)$ and $C'_q = \mathbf{E}(|G| \wedge |G|^q)$.

The proof of §II, subsection 3 works if ψ is 2-concave; then L_ψ is a subspace of L_1 (by [BDC]) and this case was already known ([M]). It works as well if ψ is 2-convex. We obtain therefore:

Proposition 9. *If $l_2(X)$ is C-finitely representable into a 2-convex Orlicz space L_φ, then X is $K \cdot C_2(\varphi) \cdot C$ isomorphic to an Hilbert space ($C_2(\varphi)$ being the 2-convexity constant of L_φ).*

We leave as open the question if this result can be extended to general Orlicz spaces. We will only show that X is necessarily of type 2^- and cotype 2^+ (as a consequence of Corollary 12 below).

However we can settle the problem when X is supposed to have l.u.st (in the sense of [DPR]).

Theorem 10. *If X is a Banach space with local unconditional structure such that $l_2(X)$ is C-finitely representable into an Orlicz space L_φ (not containing c_0) then X is (isomorphic to) an Hilbert space.*

PROOF. We will only sketch the proof, which is very similar to that of Theorem 3.

As in [K] we may suppose w.l.o.g. that X has an unconditional basis $(f_n)_{n=1}^{\infty}$. If $l_2(X)$ embeds in L_ψ, then (as a consequence of Maurey–Khintchine inequalities, see [LT2], th. 1.d.6) it embeds as a sublattice in $L_\psi(l_2)$. Let $Y = X_{1/2}$ be the (*a priori* quasi-normed) lattice defined by:

$$\| \Sigma a_n f_n \|_Y = \| \Sigma | a_n |^{1/2} f_n \|_X^2 .$$

Let Y_+ be the positive cone of Y (with respect to $(f_n)_{n=1}^{\infty}$) and $\zeta = \psi_{1/2}$ the Musielak–Orlicz function defined by:

$$\zeta(t) = \psi(\sqrt{t}).$$

We have then clearly an embedding $S: l_1^+(Y_+) \hookrightarrow L_\zeta^+$ which is positively linear (i.e. $S(\alpha u + \beta v) = \alpha S(u) + \beta S(v)$ for all positive reals α, β and elements u, v of $l_1^+(Y_+)$) and verifies:

$$A \| T^{-1} \|^{-1} \| u \| \leqq \| Su \| \leqq B \| T \| \; \| u \|$$

(T being the given embedding of $l_2(X)$ in L_ψ and constants A, B depending only on the q-concavity of L_ψ, for some $q < \infty$).

To ζ we associate the space S_ζ^+ of the pairs (μ, f), where μ is a positive random probability distribution verifying $Z(\mu) := \mathbf{E} \int \zeta(|t|) d\mu_\omega(t) < \infty$, and f a generalized $\frac{1}{2}$-convex Orlicz function satisfying Δ_2 conditions with fixed constant K.

Using an adapted version of Aldous theorem (Proposition 11 below) and proceeding as in §II, subsection 1, we see that S induces an application $Y_+ \to S_\zeta^+$ which satisfies:

(i) for all $y \in Y_+$, $\sigma(y)$ is a 1-stable element of S_ζ^+ (i.e. of the form $(\delta_A, a\lambda)$),

(ii) $\sigma(y) = \lim_{k\to\infty} S(b_k \otimes y)$ where $(b_k)_k$ is a normalized sequence of disjoint blocks in l_+^1,

(iii) for all $y_1, \ldots, y_n \in Y_+$,

$$A \| T^{-1} \|^{-1} \sum_{i=1}^{n} \| y_i \| \leqq \left\| \mathop{\ast}_{i=1}^{n} \sigma(y_i) \right\| \leqq B \| T \| \sum_{i=1}^{n} \| y_i \| .$$

So we obtain applications $A: Y \to L_\zeta^+$ and $a: Y \to \mathbf{R}_+$ such that:

$$\| y \| \sim \| \sigma(y) \| \sim \| A(y) \|_\zeta + a(y).$$

As in §II, subsection 2 we have $a(y) = \| u(y) \|_{L_1}$ for a certain positively linear operator $u : Y_+ \to L_1^+(\nu)$. Thus $a : Y_+ \to \mathbf{R}_+$ is a positively linear map. On the other hand we have:

$$\forall y \in Y_+ \qquad S(b_k \otimes y) \xrightarrow[k\to\infty]{L_0} A(y).$$

This is a consequence of the coincidence of the w.m. and s.m. topologies at degenerate random measures (see [A], lemma 2.14). Thus the map $y \to A(y)$ is positively linear.

Finally the point (iii) before can be reformulated as:

$$\forall y_1, \ldots, y_n \in Y,$$
$$A \| T^{-1} \|^{-1} \sum_i \| y_i \| \leqq \left\| \sum_i A(y_i) \right\|_\zeta + \sum_i a(y_i) \leqq B \| T \| \sum_i \| y_i \|$$

where the central term can be written as:

$$\left\| A\left(\sum_i y_i\right) \right\|_\zeta + a\left(\sum_i y_i\right).$$

Now if $y = \Sigma_i \alpha_i f_i \in Y_+$, setting $y_i = \alpha_i f_i$ we obtain:

$$\| y \|_{Y_+} \sim \| A(y) \|_\zeta + a(y) \sim \Sigma |\alpha_i|.$$

Thus $Y_+ \sim l_1^+$ and therefore $X \sim l_2$. ∎

In the preceding proof we made use of the following proposition. By positive probability distribution we mean a probability on $\mathbf{R}_+$. We note

$$|\lambda|_{1/2} = \int_0^\infty |t|^{1/2} d\lambda(t).$$

Proposition 11. *Let $\mathscr{C}$ be a class of random positive probability distributions such that:*

(i) $\forall \mu \in \mathscr{C}$, $\mathbf{E}|\mu|_{1/2} < \infty$,

(ii) *$\mathscr{C}$ is closed under operations of scaling and convolution*,

(iii) *$\mathscr{C}$ is w.m. closed*,

(iv) *if $(\mu_n)_n \subseteq \mathscr{C}$ and $\mu_n \xrightarrow[n\to\infty]{w.m.} \mu$ then $\mathbf{E}|\mu_n|_{1/2} \xrightarrow[n\to\infty]{} \mathbf{E}|\mu|_{1/2}$.*

Then $\mathscr{C}$ contains a p-stable positive random probability distribution for some $\frac{1}{2} < p \leqq 1$.

Proof. To each probability μ on $\mathbf{R}_+$ we associate the probability $\tilde{\mu}$ on $\mathbf{R}$, whose Fourier transform is:

$$\hat{\tilde{\mu}}(t) = \mathscr{L}\mu(t^2) \tag{4}$$

where $\mathscr{L}\mu$ is the Laplace transform of μ.

Recall that if μ is the probability distribution of a random variable X, then $\tilde{\mu}$ is the probability distribution of $\sqrt{2}\,.\,X^{1/2}\otimes G$, G being a standard gaussian variable.

To $\mathscr{C}$ is associated a class $\tilde{\mathscr{C}}$ of random measures on $\mathbf{R}$, which is easily seen to be a C-class in Aldous' terminology.

Thus $\tilde{\mathscr{C}}$ contains a q-stable random measure $\lambda = \tilde{\mu}_0$; using (4) it is clear that μ_0 is a $q/2$-stable positive random probability distribution, belonging to $\mathscr{C}$. ■

COROLLARY 12. *If $l_p(l_q)$ is finitely (crudely) representable in an Orlicz space (not containing c_0) then $p \leqq q \leqq 2$ or (if $p > 2$) $q \in \{2, p\}$.*

We will now make use of the following fact, due to J. L. Krivine and B. Maurey (see [R] for a proof).

FACT. *If E is a stable infinite dimensional Banach space which contains l_q^n uniformly, then $(\bigoplus_{n=1}^{\infty} l_q^n)_{l_p}$ embeds in E (for some $1 \leqq p < \infty$).*

We refer to [KM] for the definition of stable Banach spaces and recall that Orlicz spaces not containing c_0 are stable ([Ga]).

COROLLARY 13. *Let $q > 2$. If a subspace E of an Orlicz space (not containing c_0) contains l_q^n uniformly, then E contains l_q.*

For by Corollary 12, if $l_p(l_q)$, $q > 2$ is finitely representable in an Orlicz space, then $p = q$. ■

COROLLARY 14. *Let E be an infinite dimensional subspace of an Orlicz space (not containing c_0); set:*

$$p(E) = \sup\{\, p : E \text{ is of type } p\} \quad \text{and} \quad q(E) = \inf\{q : E \text{ is of cotype } q\}.$$

Then E contains almost isometrically l_p for $p \in \{p(E), q(E)\} \setminus \{2\}$ (and l_2 if $p(E) = q(E) = 2$).

PROOF. By Krivine–Maurey–Pisier's theorem ([MS], th. 13.2) E contains $l_{p(E)}^n$ and $l_{q(E)}^n$ uniformly. Thus E contains $(\bigoplus_{n=1}^{\infty} l_{p(E)}^n)_{l_p}$ and $(\bigoplus_{n=1}^{\infty} l_{q(E)}^n)_{l_q}$, and by Corollary 12 we have $p \leqq p(E)$ if $p(E) < 2$, and $q = q(E)$ if $q(E) > 2$. ■

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