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# COUNTABLE DENSE HOMOGENEOUS SPACES UNDER MARTIN'S AXIOM

BY

STEWART BALDWIN AND ROBERT E. BEAUDOIN

*Department of Mathematics, Auburn University, Auburn, AL 36849-5310, USA*

ABSTRACT

We show that Martin's axiom for countable partial orders implies the existence of a countable dense homogeneous Bernstein subset of the reals. Using Martin's axiom we derive a characterization of the countable dense homogeneous spaces among the separable metric spaces of cardinality less than $\mathfrak{c}$. Also, we show that Martin's axiom implies the existence of a subset of the Cantor set which is $\lambda$-dense homogeneous for every $\lambda < \mathfrak{c}$.

## §1. Introduction

A separable topological space $X$ is countable dense homogeneous (or CDH) provided that for any two countable dense subsets $A$ and $B$ of $X$ there is a homeomorphism $f: X \to X$ such that $f(A) = B$. This notion was introduced in [Be]. Fitzpatrick and Zhou [FZ] have used the continuum hypothesis to construct various kinds of CDH subspaces of $\mathbf{R}$. Our results are constructions of, and characterizations of, CDH spaces (usually subspaces of $\mathbf{R}$) under the assumption of Martin's axiom, or some weak form of Martin's axiom.

Let us use the notation $\mathrm{MA}_\kappa(\Gamma)$, where $\kappa$ is an infinite cardinal and $\Gamma$ is a class of partial orders, to denote the statement that for every partial order $P$ in $\Gamma$ and every family $\Delta$ of $\kappa$ or fewer dense subsets of $P$ there is a $\Delta$-generic filter on $P$. Similarly, let $\mathrm{MA}(\Gamma)$ be the statement that for every $P$ in $\Gamma$ and every family $\Delta$ of fewer than $\mathfrak{c}$ dense subsets of $P$, there is a $\Delta$-generic filter on $P$ (where $\mathfrak{c}$ denotes the cardinality of the continuum). Thus if $\Gamma_0$ is the class of all partial orders satisfying the countable chain condition, then $\mathrm{MA}_\kappa(\Gamma_0)$ and $\mathrm{MA}(\Gamma_0)$ are just the usual versions of Martin's axiom: $\mathrm{MA}_\kappa$ and MA.

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Varying $\Gamma$ gives axioms of differing strength. Taking $\Gamma_1$ to be the class of all stationarity-preserving partial orders (which properly includes all orders satisfying the c.c.c.), $MA_{\aleph_1}(\Gamma_1)$ is the strong axiom called Martin's maximum (because $\Gamma$ cannot be taken larger without yielding an axiom inconsistent with ZFC), and abbreviated MM, in [FMS]. Conversely, taking $\Gamma_2$ to be the class of all countable partial orders, $MA(\Gamma_2)$ is the weakening of Martin's axiom called MAC in [W]. We shall also consider the axiom $MA(\Gamma_3)$, where $\Gamma_3$ is the class of all $\sigma$-centered partial orders. (A partial order $P$ is $\sigma$-centered if $P = \bigcup \{P_n : n \in \omega\}$ where, for each $n$, $P_n$ is centered: every finite subset of $P_n$ has a lower bound in $P$.) Let us abbreviate $MA(\Gamma_3)$ by $MA\sigma$. $MA\sigma$ is intermediate in strength between MAC and MA.

The weakest possible axiom of this form not provable in ZFC is $MA_{\aleph_1}(\{P\})$, where $P$ is the order for adding one Cohen real. We christen this axiom Martin's minimum, to be abbreviated Mm. Observe that Mm implies $\neg$CH, by the standard argument (see e.g., [K], p. 54) showing that $MA_{\aleph_1}$ implies $\neg$CH. (So Mm is indeed not provable in ZFC.) Also, because every non-trivial countable partial order contains a dense subset isomorphic to a dense subset of $P$, MAC is equivalent to $MA(\{P\})$. So MAC follows from $Mm + \mathfrak{c} = \aleph_2$, but also from CH.

We shall be using these axioms to construct auto-homeomorphisms of $\mathbf{R}$ and of the Cantor set. If $f \in V$ is such an auto-homeomorphism and $V[G]$ is a generic extension of the universe, then we shall abuse notation by identifying $f$ with the unique extension of $f$ to an auto-homeomorphism in $V[G]$. We freely assume the existence of $V$-generic filters on various partial orders; the reader who is troubled by this is invited to replace $V[G]$ by a Boolean-valued model where necessary.

The results of Section 3 are due to the first author. The second author formulated the results in Section 2 (the proofs are joint). Proposition 4.3 is due to the second author, who also accepts the blame for the terminology Martin's minimum.

## §2. Construction of a CDH Bernstein set

In this section we shall use MAC to construct a CDH Bernstein subset of $\mathbf{R}$. (A subset $X$ of $\mathbf{R}$ is Bernstein if for any non-empty, perfect $P \subseteq \mathbf{R}$, both $P \cap X$ and $P - X$ are non-empty.) Actually, we shall do somewhat more than this. Let us say that a linear order $(X, <)$ is countable dense homogeneous provided that $X$ has a countable dense subset, and for any two countable dense subsets $A$

and $B$ of $X$ there is an order-preserving (i.e., $<$-preserving, not just $\leqq$-preserving) function $f$ from $X$ onto $X$ such that $f(A)=B$. Clearly such an $(X,<)$ is a CDH space under the order topology. Note, however, that if $X$ is a dense subset of $\mathbf{R}$ and is CDH as an ordering, then for any two countable dense subsets $A$ and $B$ of $X$ we can in fact find an order-preserving $f:\mathbf{R}\rightarrow\mathbf{R}$ with $f(A)=B$. For, every order-preserving function whose domain and range are both dense in $\mathbf{R}$ extends uniquely to an order-automorphism of $\mathbf{R}$. We shall find such a CDH suborder (not just subspace) $X$ of $\mathbf{R}$ which is also a homogeneous Bernstein set.

For any countable dense subsets $A$ and $B$ of $\mathbf{R}$, let $P(A,B)$ be the set of all finite order-preserving functions $p$ such that $\mathrm{dom}(p)\subseteq A$ and $\mathrm{ran}(p)\subseteq B$. Partially order $P(A,B)$ by reverse inclusion. Clearly $P(A,B)$ is countable, and if $G$ is a $V$-generic filter on $P(A,B)$ then $\bigcup G$ is an order-isomorphism from $A$ onto $B$ which has a unique extension to an order-isomorphism from $\mathbf{R}$ onto $\mathbf{R}$.

For any group $\mathscr{G}$ of auto-homeomorphisms of $\mathbf{R}$, and any symbol $\tilde{g}$ not contained in $\mathscr{G}$, let $\mathscr{G}[\tilde{g}]$ denote the group freely generated by $\mathscr{G}$ and $\tilde{g}$. (I.e., the only non-trivial relations in $\mathscr{G}[\tilde{g}]$ are the relations that hold in $\mathscr{G}$.) For any $f\in\mathscr{G}[\tilde{g}]$ and any auto-homeomorphism $g$ of $\mathbf{R}$ let $f^g$ be the auto-homeomorphism of $\mathbf{R}$ obtained by substituting $g$ for every occurrence of $\tilde{g}$ in $f$.

LEMMA 2.1. *Let $A$ and $B$ be countable dense subsets of $\mathbf{R}$, and let $X$ and $Y$ be subsets of $\mathbf{R}$ with $A\cup B\subseteq X$ and $X\cap Y=\varnothing$. Let $\mathscr{G}$ be a group of auto-homeomorphisms of $\mathbf{R}$ such that both $X$ and $Y$ are closed under every element of $\mathscr{G}$.*

(i) *Let $G$ be a $V$-generic filter (where $A, B, X, Y, \mathscr{G}\in V$) on $P(A,B)$, and let $g$ be the unique order-automorphism of $\mathbf{R}$ such that $g\supseteq\bigcup G$. Then for any element $f$ of the group generated by $\mathscr{G}\cup\{g\}$, $f(X)\cap Y=\varnothing$.*

(ii) *Let $\tilde{g}$ be a symbol not in $\mathscr{G}$, and let $f\in\mathscr{G}[\tilde{g}]$. For any $x\in X$ and $y\in Y$ let $D(x,y,f)$ be the set of all $p\in P(A,B)$ such that for every order-automorphism $g$ of $\mathbf{R}$ extending $p$, $f^g(x)\neq y$. Then $D(x,y,f)$ is dense in $P(A,B)$.*

PROOF. (i) Suppose the contrary, and work in $V[G]$. There must be a least $n$ such that for some $x\in X$ and $y\in Y$ there is an $f=h_nh_{n-1}\cdots h_2h_1$ so that $f(x)=y$, where each $h_i$ is either $g$, $g^{-1}$, or an element of $\mathscr{G}$. Let $x_0=x$, and for $1\leqq i\leqq n$ let $x_i=h_ih_{i-1}\cdots h_1(x)$. Note that $x_n=y$, and that $x_0, x_1, \ldots, x_n$ are distinct, for if $i<j$ and $x_i=x_j$, then $h_n\cdots h_{j+1}h_i\cdots h_1(x)=y$, contradicting minimality of $n$. Also, as $Y$ is closed under all functions in $\mathscr{G}$, we must have $h_n=g$ or $h_n=g^{-1}$, for otherwise $x_{n-1}=h_{n-1}h_{n-2}\cdots h_1(x)\in Y$, again contradicting minimality of $n$.

Let $p \in G$ decide $n$, $\{i : h_i = g\}$, $\{i : h_i = g^{-1}\}$, and the value in $\mathcal{G}$ of each $h_i$ not equal to $g$ or $g^{-1}$, and let $p \Vdash$ "$\mathbf{h}_n \mathbf{h}_{n-1} \cdots \mathbf{h}_1(x) = y$" (where $\mathbf{h}_i$ is of course a name for $h_i$). Since $x_n = y \in Y$ and $Y \cap (A \cup B) = \varnothing$, $(x_{n-1}, x_n) \notin p$ and $(x_n, x_{n-1}) \notin p$. Choose $s_n, t_n, s_{n-1}, t_{n-1}, \ldots, s_0, t_0$ (in that order), all elements of $A \cup B$, satisfying:

(a) $s_i < x_i < t_i$, $0 \leqq i \leqq n$.

(b) If $i \neq j$ then $[s_i, t_i] \cap [s_j, t_j] = \varnothing$.

(c) $h_{i+1}([s_i, t_i]) \subseteq (s_{i+1}, t_{i+1})$, $0 \leqq i < n$.

(d) If $h_{i+1} = g$ then $s_i, t_i \in A$.

(e) If $h_{i+1} = g^{-1}$ then $s_i, t_i \in B$.

(f) If $h_n = g$ then $p \cap ([s_{n-1}, t_{n-1}] \times [s_n, t_n]) = \varnothing$.

(g) If $h_n = g^{-1}$ then $p \cap ([s_n, t_n] \times [s_{n-1}, t_{n-1}]) = \varnothing$.

Since $p$ is finite, the $x_i$'s are distinct, each $h_i$ is a homeomorphism and both $A$ and $B$ are dense, this is just a matter of choosing $s_i$ and $t_i$ sufficiently close to $x_i$.

Now $h_n = g$ or $h_n = g^{-1}$, both are order-preserving, and $s_{n-1} < x_{n-1} < t_{n-1}$, so $h_n(s_{n-1}) < x_n < h_n(t_{n-1})$. Choose $z$ so that $h_n(s_{n-1}) < z < x_n$, and if $h_n = g$ then $z \in B$, whereas if $h_n = g^{-1}$ then $z \in A$.

Now let

$$p' = p \cup \{(s_i, h_{i+1}(s_i)) : i < n, h_{i+1} = g\}$$
$$\cup \{(h_{i+1}(s_i), s_i) : i < n, h_{i+1} = g^{-1}\}$$
$$\cup \{(t_i, h_{i+1}(t_i)) : i < n - 1, h_{i+1} = g\}$$
$$\cup \{(h_{i+1}(t_i), t_i) : i < n - 1, h_{i+1} = g^{-1}\}.$$

As $g(A) = B$, $p' \subseteq A \times B$. As $p'$ is finite, $p' \in V$. As $p' \subseteq g$, $p'$ is order-preserving, and so $p' \in P(A, B)$. Clearly $p' \leqq p$ and

$$p' \Vdash \text{“}\mathbf{h}_{n-1}\mathbf{h}_{n-2} \cdots \mathbf{h}_1(x) \in [s_{n-1}, t_{n-1}]\text{”}.$$

If $h_n = g$ let $q = p' \cup \{(t_{n-1}, z)\}$, whereas if $h_n = g^{-1}$ let $q = p' \cup \{(z, t_{n-1})\}$. Then $q \subseteq A \times B$. We claim $q$ is order-preserving. First suppose $h_n = g$. Take any $(u, v) \in p'$; it suffices to show that $u < t_{n-1}$ iff $v < z$. If $u \notin [s_{n-1}, t_{n-1}]$ then $u < t_{n-1}$ implies $u < s_{n-1}$, which implies $v < h_n(s_{n-1})$, and so $v < z$, whereas $t_{n-1} < u$ implies $h_n(t_{n-1}) < v$, which implies $z < v$. On the other hand, if $u \in [s_{n-1}, t_{n-1}]$ then $v = h_n(u) \in h_n([s_{n-1}, t_{n-1}]) \subseteq (s_n, t_n)$, so $(u, v) \in p' \cap ([s_{n-1}, t_{n-1}] \times [s_n, t_n])$. By (f), $(u, v) \in p' - p$. By (b) and the definition of $p'$, $(u, v) = (s_{n-1}, h_n(s_{n-1}))$. Thus $u < t_{n-1}$ and $v < z$. This proves that $q$ is order-preserving if $h_n = g$; the proof when $h_n = g^{-1}$ is similar.

So $q \in P(A, B)$ (though of course $q \notin G$), and $q \leqq p' \leqq p$. But

$q \Vdash$ "$\mathbf{h}_n \mathbf{h}_{n-1} \cdots \mathbf{h}_1(x) \in [s_n, z]$", so $q \Vdash$ "$\mathbf{h}_n \mathbf{h}_{n-1} \cdots \mathbf{h}_1(x) < y$", contradicting $p \Vdash$ "$\mathbf{h}_n \mathbf{h}_{n-1} \cdots \mathbf{h}_1(x) = y$".

(ii) Fix $x$, $y$, $f$, and an arbitrary condition $p \in P(A, B)$. Let $\mathbf{g}$ be the canonical name for the order-automorphism added by $P(A, B)$. By part (i) $\{q : q \Vdash$ "$f^{\mathbf{g}}(x) \neq y$"$\}$ is dense in $P(A, B)$, so there is a $q \leqq p$ such that $q \Vdash$ "$f^{\mathbf{g}}(x) \neq y$". Picking a $V$-generic filter $G$ on $P(A, B)$ with $q \in G$, and working in $V[G]$, a construction like the construction of $p'$ from $p$ in part (i) yields (in $V$) a $q' \leqq q$ such that $q' \in D(x, y, f)$. Q.E.D.

LEMMA 2.2. *Assume* MAC. *Let $A$ and $B$ be countable dense subsets of* $\mathbf{R}$, *let $X$ and $Y$ be subsets of* $\mathbf{R}$ *of cardinality less than* $\mathfrak{c}$ *with $A \cup B \subseteq X$ and $X \cap Y = \varnothing$, and let $\mathcal{G}$ be a group of auto-homeomorphisms of* $\mathbf{R}$, *of cardinality less than* $\mathfrak{c}$, *such that both $X$ and $Y$ are closed under every element of $\mathcal{G}$. Then there is an order-automorphism $g$ of* $\mathbf{R}$ *such that $g(A) = B$ and if $f$ is any element of the group generated by $\mathcal{G} \cup \{g\}$, then $f(X) \cap Y = \varnothing$. Furthermore, for any given $a \in A$ and $b \in B$ there is such a $g$ also satisfying $g(a) = b$.*

PROOF. Pick a new symbol $\tilde{g}$ not in $\mathcal{G}$, and form $\mathcal{G}[\tilde{g}]$. By Lemma 2.1(ii), $\Delta_0 = \{D(x, y, f): x \in X,\ y \in Y,\ f \in \mathcal{G}[\tilde{g}]\}$ is a family of dense subsets of $P(A, B)$. For each $a \in A$ and $b \in B$ let $E(a, b)$ be the set of all $p \in P(A, B)$ such that $a \in \mathrm{dom}(p)$ and $b \in \mathrm{ran}(p)$. Since $\Delta_1 = \{E(a, b) : a \in A, b \in B\}$ is countable and $\Delta_0$ has cardinality less than $\mathfrak{c}$, we can apply MAC to get a $(\Delta_0 \cup \Delta_1)$-generic filter $G$ on $P(A, B)$. Then the unique extension $g$ of $\bigcup G$ to an order-automorphism of $\mathbf{R}$ is clearly as desired. Given a fixed $a \in A$ and $b \in B$, to get $g(a) = b$, just replace $P(A, B)$ by $\{p \in P(A, B) : p(a) = b\}$. Q.E.D.

THEOREM 2.3. *Assume* MAC. *Then there is a homogeneous* CDH *Bernstein suborder of* $\mathbf{R}$.

PROOF. Note that if $X$, $Y$, and $\mathcal{G}$ are as in Lemma 2.2, $a, b \in X$, and $X$ is dense in $\mathbf{R}$, then we can apply the lemma to any countable dense subsets $A$ and $B$ of $X$ with $a \in A$ and $b \in B$ to get a $g$ so that $g(a) = b$ and for any $f$ in the group generated by $\mathcal{G} \cup \{g\}$, $f(X) \cap Y = \varnothing$. Hence using Lemma 2.2 we can inductively construct sequences $\langle X_\alpha : \alpha < \mathfrak{c} \rangle$ and $\langle Y_\alpha : \alpha < \mathfrak{c} \rangle$ of dense sets of reals, and a sequence $\langle \mathcal{G}_\alpha : \alpha < \mathfrak{c} \rangle$ of groups of order-automorphisms of $\mathbf{R}$, all three sequences increasing under inclusion, so that:

(a) $|X_\alpha|, |Y_\alpha|, |\mathcal{G}_\alpha| < \mathfrak{c}$.

(b) $X_\alpha \cap Y_\alpha = \varnothing$.

(c) For every $g \in \mathcal{G}_\alpha$, $g(X_\alpha) = X_\alpha$ and $g(Y_\alpha) = Y_\alpha$.

(d) For every non-empty perfect $P \subseteq \mathbf{R}$ there is an $\alpha < \mathfrak{c}$ so that $X_\alpha \cap P \neq \varnothing$ and $Y_\alpha \cap P \neq \varnothing$.

(e) For all countable dense $A, B \subseteq \mathbf{R}$ if there is an $\alpha < \mathfrak{c}$ such that $A, B \subseteq X_\alpha$ then there is a $\beta < \mathfrak{c}$ such that for some $g \in \mathscr{G}_\beta$, $g(A) = B$.

(f) For all $a, b \in \mathbf{R}$ if there is an $\alpha < \mathfrak{c}$ so that $a, b \in X_\alpha$ then there is a $\beta < \mathfrak{c}$ so that for some $g \in \mathscr{G}_\beta$, $g(a) = b$.

Let $X = \bigcup \{X_\alpha : \alpha < \mathfrak{c}\}$ and $\mathscr{G} = \bigcup \{\mathscr{G}_\alpha : \alpha < \mathfrak{c}\}$. By (c), for every $g \in \mathscr{G}$, $g(X) = X$. By (b) and (d) $X$ is Bernstein, and by (e) and (f) $X$ is CDH and homogeneous. Q.E.D.

## §3. CDH subspaces of the Cantor set

Using the techniques of §2 one can also show that MAC implies there is a CDH Bernstein subspace of the Cantor set. However, even stronger results can be obtained for the Cantor set under the assumption of MA$\sigma$, and these results provide a complete characterization (under MA$\sigma$) of all CDH spaces among the separable metric spaces of cardinality less than $\mathfrak{c}$. (Of course, a characterization of such spaces is only of interest when CH fails.) Besides requiring MA$\sigma$ rather than MAC, we pay for the extra strength of the results of this section in another way. The auto-homeomorphisms of the Cantor set that we construct will not, in general, be order-preserving. That is, we will be able to construct CDH subspaces of the Cantor set by these methods, but not CDH suborders.

The referee informs us that the following lemma is folklore. Results quite similar to Lemmas 3.1 and 3.2 have also been obtained by Steprāns and Watson [SW] for metrizable manifolds of dimension greater than one, rather than the Cantor set.

Lemma 3.1. *Assume* MA$\sigma$. *Let* $\mathbf{C}$ *denote the Cantor set. Suppose* $\kappa < \mathfrak{c}$ *and for each* $\alpha < \kappa$ *suppose* $A_\alpha$ *and* $B_\alpha$ *are countable dense subsets of* $\mathbf{C}$, *so that* $\alpha < \beta$ *implies* $A_\alpha \cap A_\beta = \varnothing$ *and* $B_\alpha \cap B_\beta = \varnothing$. *Then there is a homeomorphism* $f : \mathbf{C} \to \mathbf{C}$ *such that* $f(A_\alpha) = B_\alpha$ *for every* $\alpha < \kappa$. *Furthermore, for any* $a \in A_0$ *and* $b \in B_0$ *we can find such an* $f$ *also satisfying* $f(a) = b$.

Proof. Let $A = \bigcup \{A_\alpha : \alpha < \kappa\}$ and $B = \bigcup \{B_\alpha : \alpha < \kappa\}$. Let $Z$ be the collection of all finite partitions of $\mathbf{C}$ into non-empty clopen sets. Note that $Z$ is countable, since there are only countably many clopen subsets of $\mathbf{C}$. Let $P$ be the set of all quadruples $(g, h, J, K)$ satisfying:

(1) $g$ is a finite, one-to-one function with $\mathrm{dom}(g) \subseteq A$ and $\mathrm{ran}(g) \subseteq B$.

(2) If $x \in \mathrm{dom}(g) \cap A_\alpha$ then $g(x) \in B_\alpha$.

(3) $J, K \in Z$.

(4) $h$ is a one-to-one function from $J$ onto $K$.

(5) For every $x \in \mathrm{dom}(g)$ and every $Y \in J$, $x \in Y$ iff $g(x) \in h(Y)$.

Order $P$ by putting $(g, h, J, K) \leqq (g', h', J', K')$ iff

(a) $g' \subseteq g$, and

(b) for every $Y \in J$ there is a $Y' \in J'$ such that $Y \subseteq Y'$ and $h(Y) \subseteq h'(Y')$.

We claim that $P$ is $\sigma$-centered. Let $Q = \{g : \exists h, J, K\, (g, h, J, K) \in P\}$, and for each $\alpha < \kappa$ let $Q_\alpha$ be the set of all finite, one-one partial functions from $A_\alpha$ to $B_\alpha$. Order both $Q$ and $Q_\alpha$ by reverse inclusion. Then $Q_\alpha$, being countable, is isomorphic to a dense subset of the order for adding one Cohen real, and $Q$ is the finite-support product of the $Q_\alpha$. So $Q$ is isomorphic to a dense subset of the order for adding $\kappa$ Cohen reals, and as $\kappa < \mathfrak{c}$ this order is $\sigma$-centered. Write $Q$ as a disjoint union $Q = \bigcup \{Q'_n : n \in \omega\}$ where each $Q'_n$ is centered. As there are only countably many triples $(h, J, K)$ such that for some $g$, $(g, h, J, K) \in P$, one can write $P$ as a countable union of sets $P_n$ such that $(g_0, h_0, J_0, K_0)$, $(g_1, h_1, J_1, K_1) \in P_n$ implies $h_0 = h_1$, $J_0 = J_1$, $K_0 = K_1$, and for some $m$ both $g_0$ and $g_1$ belong to $Q'_m$. Then each $P_n$ will be centered in $P$.

(Let us remark that the obvious partial order adding an order-preserving $f$ with $f(A_\alpha) = B_\alpha$ for each $\alpha$, namely the set of all finite order-preserving $g$ with $g(A_\alpha) \subseteq B_\alpha$ and $g^{-1}(B_\alpha) \subseteq A_\alpha$ for each $\alpha$, fails to satisfy the countable chain condition.)

For each $n \in \omega$ let $D_0(n)$ be the set of all $(g, h, J, K) \in P$ such that every element of $J \cup K$ has diameter less than $2^{-n}$. For each $a \in A$ let $D_1(a)$ be the set of all $(g, h, J, K) \in P$ such that $a \in \mathrm{dom}(g)$. For each $b \in B$ let $D_2(b)$ be the set of all $(g, h, J, K) \in P$ such that $b \in \mathrm{ran}(g)$. Clearly all of these sets are dense in $P$, so by MA$\sigma$ there is a filter $G$ on $P$ which meets each of them. Then $f_0 = \bigcup \{g : \exists h, J, K\ (g, h, J, K) \in G\}$ is a homeomorphism from $A$ onto $B$ which extends uniquely to the desired homeomorphism $f : \mathbf{C} \to \mathbf{C}$. (We may define $f$ as follows. For any $x \in \mathbf{C}$,

$$\{h(Y) : \exists g, h, J, K\, (g, h, J, K) \in G \wedge x \in Y \in J\}$$

is a family of compact sets of arbitrarily small diameters, and has the finite intersection property. Let $f(x)$ be the unique element of the intersection of this family.)

Given a fixed $a \in A_0$ and $b \in B_0$, to get $f(a) = b$ just replace $P$ by $\{(g, h, J, K) \in P : g(a) = b\}$. Q.E.D.

Recall that a subset $A$ of a space $X$ is $\lambda$-dense in $X$ ($\lambda$ a cardinal) iff every non-empty open subset of $X$ contains exactly $\lambda$ points of $A$.

LEMMA 3.2. *Assume* MA$\sigma$. *Let* $\mathbf{C}$ *be the Cantor set, suppose* $\kappa < \mathfrak{c}$, *and for* $\alpha < \kappa$ *let* $\lambda_\alpha$ *be an infinite cardinal less than* $\mathfrak{c}$. *Suppose that for each* $\alpha < \kappa$, $A_\alpha$ *and* $B_\alpha$ *are* $\lambda_\alpha$*-dense in* $\mathbf{C}$, *and that* $\alpha < \beta$ *implies* $A_\alpha \cap A_\beta = \emptyset$ *and* $B_\alpha \cap B_\beta = \emptyset$. *Then there is a homeomorphism* $f: \mathbf{C} \to \mathbf{C}$ *such that* $f(A_\alpha) = B_\alpha$ *for every* $\alpha < \kappa$. *Furthermore, for any* $a \in A_0$ *and* $b \in B_0$ *we can find such an* $f$ *also satisfying* $f(a) = b$.

PROOF. For each $\alpha < \kappa$, if $\lambda_\alpha$ is uncountable then the $\lambda_\alpha$-denseness of $A_\alpha$ and $B_\alpha$ implies that $A_\alpha$ and $B_\alpha$ can each be partitioned into $\lambda_\alpha$-many countable dense subsets. Now apply Lemma 3.1 to all of the resulting sets. Q.E.D.

We shall call a topological space $Y$ *compressed* if every non-empty open subset of $Y$ has the same cardinality as $Y$. If $X$ is any space and $x \in X$ has a compressed neighborhood, we call $x$ a point of compression of $X$. Naturally, a space $X$ is called locally compressed if every point of $X$ is a point of compression.

Using these ideas one can characterize the CDH separable metric spaces of small cardinality, assuming MA$\sigma$.

LEMMA 3.3. *If $X$ is a second-countable* CDH *space then $X$ is locally compressed, and every infinite open subset of $X$ is uncountable.*

PROOF. The set of all points of compression of $X$ is dense in $X$, so if $A$ is a countable dense set of points of compression of $X$, and $x \in X$ were not a point of compression, then $A$ could not be mapped onto $A \cup \{x\}$ by an auto-homeomorphism of $X$. So $X$ is locally compressed. Hence $X$ has a countable basis $B$ of compressed open sets. If $X$ has a countably infinite open subset, then $B$ must have a countably infinite member. Let $U$ be the union of all countably infinite members of $B$. $U$ is countably infinite. Also, $x \in U$ iff $x$ has a countably infinite compressed neighborhood, so any auto-homeomorphism of $X$ carries $U$ onto $U$. Let $C$ be a countable dense subset of $X - U$, and pick any $u \in U$. No auto-homeomorphism of $X$ can map $(U - \{u\}) \cup C$ onto $U \cup C$. But $X$ is CDH, a contradiction. Q.E.D.

THEOREM 3.4. *Assume* MA$\sigma$. *Let $X$ be a separable metric space of cardinality less than* $\mathfrak{c}$. *Then $X$ is* CDH *iff $X$ is locally compressed and every infinite open subset of $X$ is uncountable.*

PROOF. Lemma 3.3 establishes one direction of the biconditional; for the other assume $X$ is locally compressed, and every infinite open subset of $X$ is uncountable. For each $\lambda < \mathfrak{c}$ let $X_\lambda$ be the set of all points of $X$ having a compressed neighborhood of cardinality $\lambda$. Then $\{X_\lambda : \lambda < \mathfrak{c},\ X_\lambda \neq \varnothing\}$ is a partition of $X$ into clopen subsets.

By assumption $X_{\aleph_0} = \varnothing$, and as $X$ is Hausdorff $X_n = \varnothing$ for all finite $n \neq 1$. $X_1$, being the set of all isolated points of $X$, is discrete and so (if non-empty) is CDH. Now fix $\lambda > \aleph_0$ such that $X_\lambda \neq \varnothing$. As $X_\lambda$ is a separable metric space of cardinality less than $\mathfrak{c}$ it is clear that $X_\lambda$ is zero-dimensional, and so embeddable in the Cantor set $\mathbf{C}$. Hence we may assume that $X_\lambda \subseteq \mathbf{C}$. Let $Y$ be the closure of $X_\lambda$ in $\mathbf{C}$. If $A$ and $B$ are countable dense subsets of $X_\lambda$ then $X_\lambda - A$ and $X_\lambda - B$ are $\lambda$-dense in $Y$. But $Y$ is a non-empty perfect subset of $\mathbf{C}$, and so is homeomorphic to $\mathbf{C}$, whence by Lemma 3.2 there is a homeomorphism $f : Y \to Y$ such that $f(A) = B$ and $f(X_\lambda - A) = X_\lambda - B$. Hence $f(X_\lambda) = X_\lambda$, which shows $X_\lambda$ is CDH.

As the non-empty $X_\lambda$'s form a clopen partition of $X$, and each is CDH, it follows easily that $X$ is CDH. Q.E.D.

Let us say that a space $X$ is $\lambda$-dense homogeneous if for any two $\lambda$-dense subsets $A$ and $B$ of $X$ there is a homeomorphism $f : X \to X$ such that $f(A) = B$. We can use Lemma 3.2 to prove a generalization of Theorem 4.2 of [FZ], weakening the hypotheses given there from CH to MA$\sigma$, and strengthening the conclusion from CDH to $\lambda$-dense homogeneous for any $\lambda < \mathfrak{c}$.

THEOREM 3.5. *Assume* MA$\sigma$. *Then there is a homogeneous Bernstein subset of the Cantor set which is $\lambda$-dense homogeneous for every $\lambda < \mathfrak{c}$.*

PROOF. Since MA $\sigma$ implies that $2^{<\mathfrak{c}} = \mathfrak{c}$, there are only $\mathfrak{c}$-many pairs of $\lambda$-dense subsets of $\mathbf{C}$, where $\lambda$ ranges over all cardinals less than $\mathfrak{c}$. Hence we can use Lemma 3.2 to inductively construct increasing sequences $\langle X_\alpha : \alpha < \mathfrak{c}\rangle$, $\langle Y_\alpha : \alpha < \mathfrak{c}\rangle$, and $\langle \mathscr{G}_\alpha : \alpha < \mathfrak{c}\rangle$, such that $X_\alpha$, $Y_\alpha \subseteq \mathbf{C}$, and $\mathscr{G}_\alpha$ is a group of auto-homeomorphisms of $\mathbf{C}$, satisfying:

(a) $|\mathscr{G}_\alpha| \leqq |X_\alpha| = |Y_\alpha| = |\alpha| + \aleph_0$.

(b) $X_\alpha \cap Y_\alpha = \varnothing$.

(c) For every $g \in \mathscr{G}_\alpha$, $g(X_\alpha) = X_\alpha$ and $g(Y_\alpha) = Y_\alpha$.

(d) For every $\alpha < \mathfrak{c}$, $X_{\alpha+1} - X_\alpha$ and $Y_{\alpha+1} - Y_\alpha$ are $(|\alpha| + \aleph_0)$-dense in $\mathbf{C}$.

(e) For every non-empty perfect $P \subseteq \mathbf{C}$ there is an $\alpha < \mathfrak{c}$ so that $X_\alpha \cap P \neq \varnothing$ and $Y_\alpha \cap P \neq \varnothing$.

(f) For every $\lambda < \mathfrak{c}$ and every pair of $\lambda$-dense sets $A, B \subseteq \mathbf{C}$, if there is an

$\alpha < \mathfrak{c}$ so that $A, B \subseteq X_\alpha$, then there is a $\beta < \mathfrak{c}$ so that, for some $g \in \mathcal{G}_\beta$, $g(A) = B$.

(g) For all $a, b \in \mathbf{C}$, if there is an $\alpha < \mathfrak{c}$ so that $a, b \in X_\alpha$ then there is a $\beta < \mathfrak{c}$ so that, for some $g \in \mathcal{G}_\beta$, $g(a) = b$.

Note that (a) and (d) imply that, if $\omega \leqq \alpha < \gamma < \mathfrak{c}$, then both $X_\gamma - X_\alpha$ and $Y_\gamma$ are $|\gamma|$-dense in $\mathbf{C}$. Hence given $A, B \subseteq X_\alpha$, both $\lambda$-dense homogeneous for some $\lambda < \mathfrak{c}$, we can indeed use Lemma 3.2 to find a homeomorphism $g : \mathbf{C} \to \mathbf{C}$ carrying $A$, $B$, $X_\gamma - (A \cup B)$, and $Y_\gamma$ onto $B$, $A$, $X_\gamma - (A \cup B)$, and $Y_\gamma$ respectively. So we can satisfy (f) with $\beta = \gamma + 1$ by putting $X_\beta = X_\gamma$, $Y_\beta = Y_\gamma$, and letting $\mathcal{G}_\beta$ be the group generated by $\mathcal{G}_\gamma \cup \{g\}$. Similarly, (a) and (d) also imply that we can carry out the induction in such a way that (g) is satisfied.

Let $X = \bigcup \{X_\alpha : \alpha < \mathfrak{c}\}$ and $\mathcal{G} = \bigcup \{\mathcal{G}_\alpha : \alpha < \mathfrak{c}\}$. By (c), $g(X) = X$ for every $g \in \mathcal{G}$. By (b) and (e), $X$ is Bernstein. If $A, B \subseteq X$ are both $\lambda$-dense for some $\lambda < \mathfrak{c}$, then (as $2^{<\mathfrak{c}} = \mathfrak{c}$ implies $\mathfrak{c}$ is regular) there is an $\alpha < \mathfrak{c}$ so that $A, B \subseteq X_\alpha$. So by (f) $X$ is $\lambda$-dense homogeneous for all $\lambda < \mathfrak{c}$. (Note that, being Bernstein, $X$ must be $\mathfrak{c}$-dense in $\mathbf{C}$, and so for each $\lambda < \mathfrak{c}$ $X$ has a $\lambda$-dense subset.) Finally, by (g) $X$ is homogeneous. Q.E.D.

A similar argument using Steprāns' and Watson's results (in [SW]) on compact manifolds of dimension greater than one intead of Lemma 3.2 shows that MA$\sigma$ implies that any (homogeneous) such manifold has a (homogeneous) Bernstein subspace which is $\lambda$-dense homogeneous for every $\lambda < \mathfrak{c}$. In particular, MA$\sigma$ implies that the $n$-sphere has such a subspace if $n \geqq 2$, and hence (by removing a point not in the subspace) so does $\mathbf{R}^n$. This generalizes a theorem from [FZ]: CH implies there is a CDH Bernstein subset of $\mathbf{R}^2$. Alternatively, one can prove an analogue of Lemma 2.1(ii) for Steprāns' and Watson's partial order adding an auto-homeomorphism of a compact $n$-manifold ($n \geqq 2$) which carries one given countable dense subset to another. As this partial order has a countable dense subset it follows (similarly to Theorem 2.3) that MAC implies any (homogeneous) compact manifold of dimension greater than one has a (homogeneous) CDH Bernstein subspace. Hence MAC suffices to show the $n$-sphere and $\mathbf{R}^n$ ($n \geqq 2$) have homogeneous CDH Bernstein subspaces.

## §4. Some open questions and remarks

We conclude with two open questions.

Question 4.1. Can one prove from ZFC the existence of a CDH Bernstein subspace of $\mathbf{R}$?

QUESTION 4.2. Can one prove from ZFC the existence of a CDH subspace of **R** of cardinality $\aleph_1$?

The questions corresponding to 4.1 and 4.2 but regarding suborders rather than subspaces of **R** are also of interest. The second author conjectures that if ZF is consistent then so is ZFC plus the statement that every uncountable CDH suborder of **R** contains a perfect subset. By Theorem 2.3 (and the observation that CH implies MAC), the truth of this conjecture would imply that the answer to both questions for suborders is no (unless, of course, ZF is inconsistent). Similarly, if it is consistent with ZFC that every uncountable CDH subspace of **R** contains a perfect subset, then the answers to both of 4.1 and 4.2 are no. The first author conjectures that the answer to 4.1 is yes. Note that it is possible (though we think it would be quite surprizing) that both conjectures are true, in which case there would be a model of ZFC in which there is a Bernstein CDH subspace of **R** which is not CDH as a suborder of **R**.

The referee points out that PFA implies that every $\aleph_1$-dense set of reals is a CDH suborder of **R**. (This follows, e.g., from the proof in [Ba] that PFA implies all $\aleph_1$-dense sets of reals are order-isomorphic.)

Note, in connection with 4.1, that a natural way to attempt to show that a consequence $\varphi$ of MA $+ \neg$CH does not follow from ZFC alone is to try to show that $\neg\varphi$ follows from $\diamond$. For statements $\varphi$ which are consequences of MA, the natural analogue is to see if $\clubsuit$ (introduced in [O]) implies $\neg\varphi$. Here $\clubsuit$ is the following weakening of $\diamond$:

> There is a sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$ so that, for each limit $\alpha$, $A_\alpha$ is a cofinal subset of $\alpha$, and for every cofinal $A \subseteq \omega_1$ there is a limit $\alpha$ for which $A_\alpha \subseteq A$.

It is well-known that $\mathrm{MA}_{\aleph_1}$ implies $\neg\clubsuit$. Indeed, $\mathrm{MA}\sigma_{\aleph_1}$ implies $\neg\clubsuit$. (Let $P$ be the set of all finite partial functions from $\omega_1$ to 2, and given $\langle A_\alpha : \alpha < \omega_1 \rangle$ let $A \subseteq \omega_1$ have as its characteristic function the union of a sufficiently generic filter on $P$; one can easily arrange that $A_\alpha \subseteq A$ for no limit $\alpha$.) Yet Shelah has shown in [S] that if ZFC is consistent then so is ZFC $+\clubsuit+\neg$CH. Thus it is possible to show that certain statements are independent of ZFC by showing that they follow from MA $+\neg$CH, while their negations follow from $\clubsuit + \neg$CH. One cannot play this game with the results of §2, however, or with any other consequences of MAC. Recall that MAC follows from Mm $+ \mathfrak{c} = \aleph_2$. It's not hard to see that Mm holds in Shelah's model of $\clubsuit + \mathfrak{c} = \aleph_2$.

PROPOSITION 4.3. *If ZF is consistent so is* ZFC $+\clubsuit+$ Mm $+\mathfrak{c}=\aleph_2$.

PROOF. Let $V$ be a model of ZFC + CH + $\diamondsuit(E)$, where $E=\{\alpha<\omega_2 : \mathrm{cf}\alpha=\omega\}$. For each $\mu\leqq\omega_3$ let $P_\mu$ be the set of all countable functions from $\mu$ to 2, ordered by reverse inclusion, and let $G$ be a $V$-generic filter on $P_{\omega_3}$. Note that in $V[G]$, $2^{\aleph_1}\geqq\aleph_3$. Let $Q$ be the Levy collapse adding a function from $\omega$ onto $\omega_1$ with finite conditions. Suppose $H$ is a $V[G]$-generic filter on $Q$. Then clearly $V[G,H]$ is a model of $\mathfrak{c}=\aleph_2$ $(=\aleph_3^V)$, and Shelah has proved in [S] that $V[G,H]\models\clubsuit$.

It remains to show that $V[G,H]\models$ Mm. So suppose $P$ is the order for adding one Cohen real, and $\langle D_\sigma:\sigma<\omega_1^{V[G,H]}\rangle$ is a sequence of dense subsets of $P$ in $V[G,H]$. Since $P_{\omega_3}$ satisfies the $\aleph_2$-c.c., and $|Q|=\aleph_1$, $P_{\omega_3}*Q$ satisfies the $\aleph_2$-c.c. Hence for each $\sigma$ there is a $\nu<\omega_3^V$ such that $D_\sigma\in V[G\cap P_\nu,H]$. As $\omega_1^{V[G,H]}<\omega_3^V$, there is a $\mu<\omega_3^V$ such that $\{D_\sigma:\sigma<\omega_1^{V[G,H]}\}\subseteq V[G\cap P_\mu,H]$. Let $\eta=\mu+\omega_1^V$, let $P_{\mu\eta}$ be the set of all countable functions from $\eta-\mu$ to 2 in $V$, and let $P_{\eta\omega_3}$ be the set of all countable functions from $\omega_3^V-\eta$ to 2 in $V$. Then $P_{\omega_3}\cong P_\mu*P_{\mu\eta}*P_{\eta\omega_3}$, and as $Q\in V$,

$$P_{\omega_3}*Q\cong P_\mu*Q*P_{\mu\eta}*P_{\eta\omega_3}.$$

But in $V[G\cap P_\mu,H]$, $P_{\mu\eta}$ is countable, and so forcing with $P_{\mu\eta}$ adds a Cohen real. Thus in $V[G,H]$ there is a $V[G\cap P_\mu,H]$-generic filter on $P$, and we are done. Q.E.D.

An argument similar to the above shows that if ZF is consistent then so is ZFC $+\clubsuit+$ MAC $+\mathfrak{c}=\kappa$, for any uncountable value of $\kappa$ with $\mathrm{cf}\,\kappa\geqq\omega_1$.

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