ISRAEL JOURNAL OF MATHEMATICS 81 (1993), 369–379

ON THE WEIGHT-SPECTRUM OF A COMPACT SPACE

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ABSTRACT

The weight-spectrum $Sp(w, X)$ of a space X is the set of weights of all infinite closed subspaces of X. We prove that if $\kappa > \omega$ is regular and X is compact T_2 with $w(X) \geq \kappa$ then some λ with $\kappa \leq \lambda \leq 2^{<\kappa}$ is in $Sp(w, X)$. Under CH this implies that the weight spectrum of a compact space can not omit ω_1, and thus solves problem 22 of [M]. Also, it is consistent with $2^\omega = c$ being anything it can be that every countable closed set T of cardinals less than c with $\omega \in T$ satisfies $Sp(w, X) = T$ for some separable compact LOTS X. This shows the independence from ZFC of a conjecture made in [AT].

Given a cardinal function φ and a space X we define the φ-spectrum of X, in symbols $\mathrm{Sp}(\varphi, X)$, as the set of φ-values taken on all infinite closed subspaces of X, i.e.

$$\mathrm{Sp}(\varphi, X) = \{\varphi(F): F = \overline{F} \subset X \quad \text{and} \quad |F| \geq \omega\}.$$

The aim of this note is to study this in the case of $\varphi = w$, the weight function, especially for compact T_2 spaces X. Let us note that for the cardinality function $\varphi(X) = |X|$ this problem has been earlier considered, using different terminology, e.g. in [J2], [JN] or [JW].

Our interest in this problem was motivated by a problem of Arhangel'skii and Tkačuk from [AT], which in our notation asked whether a compact T_2 space X

* Research supported by OTKA grant no. 1908.
Received April 13, 1992 and in revised form August 17, 1992

is necessarily metrizable if $cf(\kappa) = \omega$ holds for each $\kappa \in \mathrm{Sp}(w, X)$? They showed in [AT] that the answer to this question is "yes" if $c = 2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$ hold. It follows from our results below that actually $c < \omega_\omega$ is sufficient to yield the "yes" answer, but it is consistent with ZFC (and $c > \omega_\omega$, of course) to have counterexamples.

In what follows, for κ and λ given cardinal numbers the (closed) interval $[\kappa, \lambda]$ denotes the set of **cardinals** μ satisfying $\kappa \le \mu \le \lambda$.

THEOREM 1: *Let X be a space and κ a cardinal with $\omega \le \kappa < w(X)$. Then*
(i) *if X is T_2 then $\mathrm{Sp}(w, X) \cap [\kappa, \exp_3(\kappa)] \ne \emptyset$;*
(ii) *if X is T_3 then $\mathrm{Sp}(w, X) \cap [\kappa, 2^\kappa] \ne \emptyset$.*

Proof: By the main result of [HJ2] (see also 6.8 of [J1]) there is a subspace $Y \subset X$ with $|Y| \le \kappa$ such that $w(Y) \ge \kappa$. Now, if we let $F = \overline{Y}$, then we clearly have $w(F) \in [\kappa, 2^\kappa]$ if X is T_3 and $w(F) \in [\kappa, \exp_3(\kappa)]$ if X is just T_2. ∎

Since a T_3 space X is metrizable if $w(X) = \omega$, already from Theorem 1 we get the following significant strengthening of the above quoted result of [AT].

COROLLARY 2: *If $2^{\omega_1} < \omega_\omega$ then a T_3 space X is metrizable if $cf(\kappa) = \omega$ holds for each $\kappa \in \mathrm{Sp}(w, X)$.*

Let us now turn to the study of weight-spectra of compact T_2 spaces. It is well-known that the Stone-Čech compactification $\beta\omega$ of the countable discrete space ω satisfies

$$\mathrm{Sp}(w, \beta\omega) = \{c\},$$

showing that the following result, at least for $\kappa = \omega_1$, is best possible.

THEOREM 3: *Let κ be an uncountable regular cardinal and X be a compact T_2 space with $w(X) \ge \kappa$. Then there is a closed subspace $F \subset X$ such that $w(F) \in [\kappa, 2^{<\kappa}]$ and*

$$|F| \le \sum\{\exp_2(\lambda)\colon \lambda < \kappa\}.$$

Proof: Let us first consider the case in which $\hat{t}(X) > \kappa$, i.e. there is a point $p \in X$ and a set $A \subset X$ with $p \in \overline{A}$ but $p \notin \overline{B}$ for each set $B \subset A$ with $|B| < \kappa$ (i.e. $a(p, A) \ge \kappa$). Then (see e.g. [J1], 3.12) there is a free sequence in X of length κ, hence by Theorem 1.2 of [JSz], there is also a free sequence $S = \{x_\alpha\colon \alpha \in \kappa\}$ that converges to some point $x \in X$.

Now we distinguish two cases again. If there is an $\alpha \in \kappa$ such that (with $S_\alpha = \{x_\beta : \beta \in \alpha\}$) we have $w(\overline{S}_\alpha) \geq \kappa$, then clearly $F = \overline{S}_\alpha$ is as required since $w(\overline{S}_\alpha) \leq 2^{|\alpha|}$ and $|\overline{S}_\alpha| \leq \exp_2(|\alpha|)$.

If, on the other hand, we have $w(\overline{S}_\alpha) = \lambda_\alpha < \kappa$ for each $\alpha \in \kappa$, then $F = \overline{S}$ will work. Indeed, then we have

$$F = \overline{S} = \{x\} \cup \bigcup\{\overline{S}_\alpha : \alpha \in \kappa\},$$

since S converges to x, and our assumption that $w(\overline{S}_\alpha) < \kappa$ implies that $nw(F) = w(F) \leq \kappa$. (Note that $w(\overline{S}_\alpha) \leq \kappa$ would suffice for this.) But S is free, hence discrete, thus $w(F) \geq w(S) = |S| = \kappa$, i.e. we actually got $w(F) = \kappa$ and $|F| \leq \sum_{\alpha\in\kappa} 2^{\lambda_\alpha} \leq \sum_{\lambda<\kappa} 2^\lambda = 2^{<\kappa}$.

Next, if $\hat{t}(X) \leq \kappa$, i.e. whenever $p \in \overline{A}$ there is a $B \subset A$ with $|B| < \kappa$ and $p \in \overline{B}$, then use [HJ2] again to find a set $S \subset X$ with $|S| = \kappa$ and $w(S) \geq \kappa$. Note that by $\hat{t}(X) \leq \kappa$ and the regularity of κ we now have

$$\overline{S} = \cup\{\overline{S}_\alpha : \alpha \in \kappa\},$$

by putting $S = \{x_\alpha : \alpha \in \kappa\}$ and $S_\alpha = \{x_\beta : \beta \in \alpha\}$ like above. Also, from here we proceed like there: if there is some $\alpha \in \kappa$ with $w(\overline{S}_\alpha) \geq \kappa$ then $F = \overline{S}_\alpha$ works, and otherwise $F = \overline{S}$ will satisfy the requirements, even with $w(F) = \kappa$ and $|F| \leq 2^{<\kappa}$. ∎

We now list several immediate corollaries of Theorem 3 and its proof.

Corollary 4: *If κ is a (strongly) inaccessible cardinal and X is compact T_2 with $w(X) \geq \kappa$ then there is a closed $F \subset X$ with $w(F) = |F| = \kappa$. In particular, inaccessible cardinals may not be omitted by compact T_2 spaces.*

Corollary 5: *If $\kappa > \omega$ is regular and not inaccessible and X is compact T_2 with $w(X) \geq \kappa$, then there is a closed subspace $F \subset X$ and a cardinal $\lambda < \kappa$ such that $\kappa \leq w(F) \leq 2^\lambda$ and $\kappa \leq |F| \leq \exp_2(\lambda)$.*

Of course, with $\kappa = \omega_1$ this immediately implies that the answer to the problem of [AT] is yes if $c < \omega_\omega$.

Corollary 6: *If $\kappa > \omega$ is regular and X is compact T_2 such that $w(\overline{S}) < \kappa$ whenever $S \subset X$ and $|S| < \kappa$, then there is a closed $F \subset X$ with $w(F) = \kappa$ and $\kappa \leq |F| \leq 2^{<\kappa}$. In particular, if X is compact and ω-monolithic with $w(X) > \omega$ then there is a closed $F \subset X$ with $w(F) = \omega_1$ and $|F| \leq c$.*

Finally, we summarize what Theorem 3 yields under GCH.

COROLLARY 7 (GCH): *For any compact T_2 space X its weight-spectrum* $\mathrm{Sp}(w, X)$ *contains every uncountable regular cardinal* $\kappa < w(X)$*, while its cardinality spectrum contains* κ *if it is inaccessible and either* κ *or* κ^+ *otherwise.*

The strength of these results is illustrated by the following remarks. Note that the weight of a compact space is equal to the character of the diagonal in its square (see [J1], 3.32), hence a compact space of weight ω_1 doesn't have a small diagonal. (Recall that a space is said to have a small diagonal iff for every uncountable set in the complement of its diagonal there is a neighbourhood of the diagonal that misses uncountably many points of that set.) Thus it follows immediately from Corollary 5 that under CH every compact space with a small diagonal is metrizable. Moreover, from Corollary 6 we see in ZFC that every ω-monolithic compact space with a small diagonal is metrizable. Using Stone's duality, it is another immediate consequence of Corollary 5 that under CH every uncountable Boolean algebra has a homomorphic image of size ω_1, hence the answer to problem 22 in [M] is negative.

Note that by [vD] no singular cardinal of countable cofinality may belong to the weight or cardinality spectrum of a compact F-space. It remains an open question whether, under GCH, singular cardinals of uncountable cofinality could be omitted by the weight or cardinality spectrum of a compact T_2 space?

It is immediate from Corollary 6 that a counterexample to the problem of [AT] can not be ω-monolithic. But this means that if there is a counterexample then it must have a **separable** closed subspace that is also a counterexample. Note also that the weight of the latter then must be less than c. In view of this we are now going to study weight spectra of compact spaces of weight below c. It will turn out that we can obtain (consistent) examples of separable compact linearly ordered topological spaces that yield a great variety of such weight spectra, in particular they yield strong counterexamples to the problem of [AT]. The following lemma is the key to the construction of these examples.

LEMMA 8: *Let* κ *be a fixed infinite cardinal. Then the following two statements* (i) *and* (ii) *are equivalent.*

(i) *There is a (separable) compact LOTS, say* X*, such that* $\mathrm{Sp}(w, X) = \{\omega, \kappa\}$.

(ii) *There is a subset* $S \subset \mathbb{R}$ *with* $|S| = \kappa$ *such that for any closed set* $F \subset \mathbb{R}$ *we have* $|F \cap S| \le \omega$ *or* $|F \cap S| = \kappa$.

Proof: (i) $\to$ (ii). Since (ii) is obviously true if $\kappa \le \omega_1$, we may assume that

$\kappa > \omega_1$. As it was pointed out above, Corollary 6 implies that our compact LOTS X may be assumed to be separable. Let $\prec$ be the order relation on X defining its topology. It is well-known (see e.g. [HJ1]) that

$$w(X) = d(X) + u(X),$$

where $U(X) = \{x \in X\colon x \text{ has a } \prec\text{-successor } x^+\}$ and $u(X) = |U(X)|$. We also set $L(X) = \{x^+\colon x \in U(X)\}$. Let us note that, since X is separable, for any subset $H \subset X$ there can be at most countably many points $x \in H$ such that x has an immediate $\prec$-successor in H which is not its immediate successor in X. Hence if the subspace topology of H is determined by $\prec$ restricted to H, in particular if H is closed, we have $|U(H)| = |U(X) \cap H|$ if $|U(H)| > \omega$, consequently we have

$$(*) \qquad w(H) = |H \cap U(X)| + \omega$$

whenever $H \subset X$ is closed. Here we were using also the well-known fact that a separable LOTS is also hereditarily separable.

Let us consider the following equivalence relation $\sim$ on X: For $x, y \in X$ we set $x \sim y$ iff the interval $[x, y]$ (or $[y, x]$) between them is a countable set. Clearly, every equivalence class of $\sim$ is a countable closed interval hence it is also obvious that an equivalence class is not a singleton if and only if it meets $U(X)$. Since $d(X) = \omega < \kappa$ implies

$$w(X) = u(X) = \kappa,$$

we obtain immediately that the set T of non-singleton equivalence classes has cardinality κ, too.

Let $\tilde{X} = X/\sim$ be the quotient LOTS determined by the canonical quotient ordering. Let $\pi\colon X \to \tilde{X}$ be the canonical quotient map that sends each $x \in X$ to its equivalence class. Since $U(\tilde{X})$ is clearly empty, we have that $\tilde{X}$ is densely ordered and

$$w(\tilde{X}) = d(\tilde{X}) = d(X) = \omega,$$

hence $\tilde{X}$ can be embedded as a closed interval in $\mathbb{R}$.

So let $i\colon \tilde{X} \to \mathbb{R}$ be an embedding map and set $S = i^{\rightarrow}[T]$. Clearly, we have $|S| = |T| = \kappa$. Now let $F \subset \mathbb{R}$ be any closed set in $\mathbb{R}$ with $|F \cap S| > \omega$. Then $\hat{F} = (i \circ \pi)^{\leftarrow}[F]$ is a closed set in X with $|\hat{F} \cap (\cup T)| > \omega$. But every point in $\cup T$ is either in $U(X)$ or $L(X)$, which clearly implies that $w(\hat{F}) > \omega$, hence actually $w(\hat{F}) = \kappa$. From the latter and $(*)$, however, we conclude that

$$|\hat{F} \cap \cup T| = \kappa,$$

consequently we must have $\hat{F} \cap t \neq \emptyset$ for κ many $t \in T$, hence by $(i \circ \pi)^{\rightarrow}[\hat{F}] = F \cap S$ we have $|F \cap S| = \kappa$. Thus S is as required by (ii).

(ii) $\rightarrow$ (i). We may assume without any loss of generality that $S \subset [0,1]$, the unit interval. Informally, our space X is then obtained by "splitting" each $x \in S$ in two. Formally, this means that the underlying set of X is

$$([0,1] \times \{0\}) \cup (S \times \{1\})$$

and the ordering $\prec$ is the lexicographic ordering on this set. To simplify notation, we shall replace $\langle x, 0\rangle$ with x and $\langle x, 1\rangle$ with x^+ in what follows. Clearly, X is a separable compact LOTS with $U(X) = S$.

Let $\pi\colon X \to [0,1]$ be the canonical map that sends both x and x^+ to x. Then π is a continuous and thus also closed map. Let $F \subset X$ be any closed subspace, then $\tilde{F} = \pi^{\rightarrow}[F]$ is closed in $\mathbb{R}$ hence $|\tilde{F} \cap S| \leq \omega$ or $|\tilde{F} \cap S| = \kappa$. Applying $(*)$ from the previous part of the proof we get that $w(F) = \omega$ in the first case and $w(F) = \kappa$ in the second, and this completes the proof of the lemma. ∎

Of course, if $\kappa = c$ then (ii) (and thus (i)) is simply true with $S = \mathbb{R}$. The compact LOTS obtained with this choice of S is just the "double-arrow" space of Alexandrov. The interesting question is what happens for cardinals κ satisfying $\omega_1 < \kappa < c$? According to our next result, the validity of (ii) (hence (i)) in this case is both consistent with and independent of ZFC!

THEOREM 9:

(1) *Let $\mathcal{C}_\lambda = Fn(\lambda, 2)$ be the standard notion of forcing that adds λ Cohen reals to a ground model V. Then, in $V^{\mathcal{C}_\lambda}$, (ii) of Lemma 8 holds for each $\kappa \leq \lambda$.*

(2) *If κ is bigger than ω_1 and $MA_\kappa(\sigma -$ centered) holds then (ii) fails for κ.*

Proof: (1) Since $\kappa < \lambda$ implies $\mathcal{C}_\lambda \cong \mathcal{C}_\kappa \times \mathcal{C}_{\lambda \setminus \kappa} \cong \mathcal{C}_\kappa \times \mathcal{C}_\lambda$, it clearly suffices to show that (ii) for κ holds in $V^{\mathcal{C}_\kappa}$ for any given κ.

Now, if $r\colon \kappa \to 2$ is the Cohen generic map in $V^{\mathcal{C}_\kappa}$, then for any $\alpha \in \kappa$ let $r_\alpha\colon \omega \to 2$ be defined by

$$r_\alpha(n) = r(\omega \cdot \alpha + n),$$

so r_α is the αth Cohen real added to V. Now, we set $S = \{r_\alpha\colon \alpha \in \kappa\}$, note that actually S is contained in the Cantor set $\mathbb{C}$.

Now, every closed set $F \subset \mathbb{C}$ has a code that is a countable subset of V, hence (see e.g. [K]) there is a countable set $A \subset \kappa$ in V such that the code of F belongs to $V^{\mathcal{C}_A}$. This shows that we may actually assume that the code of F is in V. But in this case $F \cap S = \emptyset$ unless the interior of F is non-empty, while in the latter case it is obvious to see that $F \cap S$ is forced to have size κ. Indeed, $|F \cap S| < \kappa$ would imply the existence of a subset $A \subset \kappa$ with $\{\alpha: r_\alpha \in F\} \subset A$, $|A| < \kappa$ and $A \in V$. But if $p \in \mathcal{C}_\kappa$ forces this and $\beta \in \kappa$ is such that

$$[\omega \cdot \beta, \omega \cdot (\beta + 1)) \cap (A \cup \text{ Dom } (p)) = \emptyset,$$

then taking an $s \in^{<\omega} 2$ with $[s] \subset F$ and setting $\text{Dom}(\hat{s}) = \omega \cdot \beta +$ Dom (s) with

$$\hat{s}(\omega \cdot \beta + i) = s(i),$$

we see that $q = p \cup \hat{s}$ is a condition that forces $s_\beta \in F \cap S$, a contradiction.

(2) Now, it is well-known (see e.g. [W]) that $MA_\kappa(\sigma -$ centered) implies that any set $S \subset \mathbb{R}$ with $|S| = \kappa$ is a Q-set, i.e. every subset of S is a relative F_σ. But if $A \subset S$ has cardinality ω_1 and $A = \cup\{F_n \cap S: n \in \omega\}$ where each F_n is closed in $\mathbb{R}$, then there must be an $n \in \omega$ such that $|F_n \cap S| = \omega_1$, showing that S does not have the property required by (ii). ∎

As an immediate consequence of part 1) we get of course the failure of the [AT] conjecture in models of the form $V^{\mathcal{C}_\lambda}$ with $\lambda \geq \omega_\omega$. In fact, we get much more by gluing several copies of spaces of type (i) from Lemma 8 together.

THEOREM 10: *Suppose that (i) of Lemma 8 holds for each $\kappa \leqq c$. Then for every countable closed set of cardinals $T \subset c$ with $\omega \in T$ there is a separable compact LOTS, say X, such that* $\text{Sp}(w, X) = T$.

Proof: By assumption, for each $\kappa \in T$ there is a separable compact LOTS, say $X(\kappa)$, such that $\text{Sp}(w, X(\kappa)) = \{\omega, \kappa\}$. Let X be the LOTS whose topology is determined by the lexicographic ordering on

$$\bigcup\{\{\kappa\} \times X(\kappa): \kappa \in T\}.$$

Since T is a countable closed, and therefore compact subset of c, it is clear that X is a separable compact LOTS. It is also obvious that $T \subset \text{Sp}(w, X)$.

To see that the converse inclusion also holds, let F be any non-empty closed subset of X. For each $\kappa \in T$ then $F_\kappa = F \cap (\{\kappa\} \times X(\kappa))$ is homeomorphic to a closed set in $X(\kappa)$, hence

$$w(F_\kappa) \in \{\omega, \kappa\}.$$

Let us set

$$Q = \{\kappa \in T \colon w(F_\kappa) \neq \omega\}.$$

If $Q = \emptyset$ then $w(F) = \omega$ by the addition theorem for compact spaces (i.e. because F has a countable network). If Q has a largest element, then clearly $w(F) = \max\{w(F_\kappa) \colon \kappa \in Q\} \in T$, using the addition theorem again.

Finally, if Q does not have a largest member, let λ be the largest limit point of Q in c, again $\lambda \in T$ because T is closed. Since no element of Q is greater than λ, we have $w(F) = \lambda$, for again F has a network of size λ and for any $\mu < \lambda$ there is a $\kappa \in Q$ with $\mu < \kappa = w(F_\kappa) \leq w(F)$. ∎

Putting together what we have proven in Theorems 9 and 10 we get the following result.

COROLLARY 11: *It is consistent with the continuum being anything it can be, that for every countable closed set of cardinals below c there is a separable compact LOTS whose weight-spectrum consists exactly of ω and the elements of that set.*

Note that, as opposed to $\beta\omega$, the weight-spectrum of a compact LOTS cannot omit ω.

Now Corollary 11 gives us a large variety of situations in which the [AT] problem has lots of different counterexamples. However these cannot answer the naturally raised question whether a counterexample already exists just from the (by Corollary 5) necessary condition that $c > \omega_\omega$. Indeed, our next result implies that under $MA(\sigma - \text{centered})$ there can be no counterexample to the [AT] problem which is a LOTS.

THEOREM 12: *Suppose that $MA(\sigma - \text{centered})$ holds. Then*

(a) *for every separable compact LOTS, say X, if $w(X) < c$ then*

$$\mathrm{Sp}(w, X) = [\omega, w(X)];$$

(b) *if X is any compact LOTS whose weight is uncountable but less than c then*

$$\omega_1 \in \mathrm{Sp}(w, X).$$

Proof: (a) To prove this one needs the following generalization of Lemma 8:

LEMMA 8': *Given a set of infinite cardinals T, the following two statements are equivalent:*

(i) *There is a separable compact LOTS X such that* $\mathrm{Sp}(w, X) = T$.

(ii) *There is a set of reals $S \subset \mathbb{R}$ such that $T = \{\kappa: \exists F \subset \mathbb{R}$ closed with $|F \cap \mathbb{R}| = \kappa\}$, i.e. T is the cardinality spectrum of S.*

Since the proof of this is exactly the same as that of Lemma 8, we omit it.

Now, to finish the proof of a) note that $MA(\sigma-$ centered) implies that any $S \subset \mathbb{R}$ with $|S| < c$ is a Q-set, hence if $\kappa < |S|$ and $cf(\kappa) > \omega$ then any subset $S' \subset S$ with $|S'| = \kappa$, being an F_σ, contains a relatively closed set of size κ. If, on the other hand, $cf(\kappa) = \omega$ then choose regular cardinals $\kappa_n < \kappa$ for $n \in \omega$ such that

$$\kappa = \Sigma\{\kappa_n: n \in \omega\}.$$

Clearly, it is possible to find distinct points $x_n \in \mathbb{R}$ such that every neighborhood of x_n intersects S in a set of size at least κ_n and the sequence x_n converges to some $x \in \mathbb{R}$. We can place disjoint closed intervals I_n about each x_n such that the length of I_n is less than $\frac{1}{n}$ and $|I_n \cap S| \geq \kappa_n$ for $n \in \omega$, and choose sets $F_n \subset I_n \cap S$ with $|F_n| = \kappa_n$ and F_n relatively closed in S, by the above. But $\cup\{F_n: n \in \omega\} = F$ may have no other limit point in S not already in F than x, hence $(F \cup \{x\}) \cap S$ is a relatively closed set in S of size κ.

(b) If X is ω-monolithic then this follows from Corollary 6. Otherwise X has a separable closed subspace whose weight is uncountable and less than c, hence we may apply a) to this closed subspace. ∎

Since the weight-spectrum of the double arrow space is clearly $\{\omega, c\}$, the assumptions about the weight of the space being less than c cannot be dropped from Theorem 12.

However, a number of questions concerning Theorems 10 and 12 remain open. Thus we do not know whether one could realize arbitrary uncountable closed sets of cardinals below c as weight spectra of (separable?) (ordered?) compact spaces? Also, we don't know whether part b) is also valid for ω_2 instead of ω_1.

To conclude, let us formulate an easy result which, however, could be quite useful in finding applications of the positive results on the weight spectra of normal spaces.

THEOREM 13: *If X is a normal space and $\kappa \in \mathrm{Sp}(w, X)$ then there is a Tychonov continuous image Y of X with $w(Y) = \kappa$ as well. (In particular, if X is compact then so is Y).*

Proof: Let F be a closed subspace of X with $w(F) = \kappa$ and let $j\colon F \to I^\kappa$ be an embedding of F into the Tychonov cube of weight κ. By the Tietze extension theorem, there is a map $f\colon X \to I^\kappa$ which extends j. Now, if Y is the image of X under f then clearly

$$\kappa = w(F) = w(j^{\rightarrow}[F]) \leq w(Y) \leq w(I^\kappa) = \kappa,$$

hence $w(Y) = \kappa$ and Y is as required. ∎

From Corollary 6 and this result we immediately get, for instance, the next result.

COROLLARY 14: *Every ω-monolithic compact space of uncountable weight has a (necessarily ω-monolithic) continuous image of weight ω_1.*

Since the continuous image of a CCC space is also CCC, it follows immediately that if there is a CCC non-metrizable ω-monolithic compact space then there is also one of weight ω_1. However, a compact CCC space of weight ω_1 is separable under MA_{ω_1} by [HJ3], hence the next result from [ASh] is obtained immediately.

COROLLARY 15: *(Arhangel'skii and Shapirovskii) Under MA_{ω_1} every CCC compact ω-monolithic space is metrizable.*

Corollary 14 also easily implies that the problem whether a compact ω-monolithic space is metrizable if ω_1 is its caliber, also raised in [ASh], can also be reduced to such spaces of weight ω_1. The latter, however, remains unsolved.

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