SOME EMBEDDINGS OF INFINITE-DIMENSIONAL SPACES

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ABSTRACT

It is shown that if A is a weakly infinite-dimensional subset of a metric space R then a G_δ set B of R exists such that $A \subseteq B$ and B is weakly infinite-dimensional. A similar result holds for a set having strong transfinite inductive dimension. As a consequence each weakly infinite-dimensional metric space possesses a weakly infinite-dimensional complete metric extension. A similar result holds also for a space having strong transfinite inductive dimension.

1. In this paper all spaces are metrizable. The dimension function used is covering dimension dim.

A space R is called *weakly infinite-dimensional* (see [4], [5]) hereafter abbreviated *w.i.d.*, if for every countable family of pairs $(F_i, G_i)\, i = 1,2,\cdots$ of closed disjoint sets of R, there exist open sets U_i, $F_i \subseteq U_i \subseteq R - G_i$, such that $\cap_{i=1}^{k} \operatorname{Bd} U_i = \phi$ for some number k.

A space R has *strong transfinite dimension* -1, $\operatorname{Ind} R = -1$ if $R = \phi$. Let α be an ordinal number. If for every pair of closed disjoint sets F, G of R an open set U exists such that $F \subseteq U \subseteq R - G$ with $\operatorname{Ind} \operatorname{Bd} U < \alpha$ we say that R has *strong transfinite inductive dimension* $\leqq \alpha$, $\operatorname{Ind} R \leqq \alpha$. $\operatorname{Ind} R = \alpha$ if $\operatorname{Ind} R \leqq \alpha$, and $\operatorname{Ind} R \leqq \beta$ is not satisfied for any $\beta < \alpha$.

A *complete extension* of R is a pair (h, R^*), where R^* is a complete space, and h a homeomorphism from R into R^* such that $\overline{h(R)} = R^*$.

A theorem of Tumarkin ([2], p. 32) states that each finite-dimensional set of R is contained in a G_δ set of R with the same dimension.

In this paper it is shown that an analogue of this theorem holds for w.i.d. sets (Theorem 2). It is also shown that if A is a subset of R having strong transfinite

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inductive dimension then a G_δ set B of R exists, containing A, such that B has strong transfinite inductive dimension as well (Theorem 4), where a simple connection exists between Ind A and Ind B (Corollary 1). As a consequence we get that every w.i.d. space has a w.i.d. complete extension. A similar result holds for a space having strong transfinite inductive dimension. It should be noted that in [4] it is shown that a w.i.d. separable space always has a w.i.d compact extension.

2. In the following let A be a subset of a metric space R, satisfying $A = \bigcup_{n=0}^{\infty} P_n$. Denote by (I), (II), (III) the following properties of the subsets P_n:

(I) P_n are open in A and $\dim P_n \leqq n$, $n = 0,1,2,\cdots$.

(II) $P_n \subseteq P_{n+1}$ for $n = 0,1,2,\cdots$.

(III) $P_n = \cup \{u \mid u$ is open in A with $\dim u \leqq n\}$, $n = 0,1,2,\cdots$.

Note that (III) implies (I) and (II).

LEMMA 1. *Let* $A = \cup_{n=0}^{\infty} P_n$ *be a subset of* R *satisfying* (I). *Then there exists a* G_δ *set* B *in* R *which contains* A, *such that* $B = \cup_{n=0}^{\infty} Q_n$, *where* Q_n *are open in* B, $Q_n \cap A = P_n$ *and* $\dim Q_n = \dim P_n$.

PROOF. Given in [3] corollary 2.

LEMMA 2. *Let* $A = \cup_{n=0}^{\infty} P_n$ *be a subset of* R, *where* P_n *satisfy* (I) *and* (II). *Then the* G_δ *set* $B = \cup_{n=0}^{\infty} Q_n$ *of Lemma* 1 *can be chosen so that* $Q_n \subseteq Q_{n+1}$ *for* $n = 0,1,2,\cdots$.

PROOF. By Lemma 1 a G_δ set B exists, containing A, such that $B = \cup_{n=0}^{\infty} \hat{Q}_n$, $\hat{Q}_n$ open in B, $\hat{Q}_n \cap A = P_n$ and $\dim \hat{Q}_n = \dim P_n$. Let $Q_n = \cup_{i=0}^{n} \hat{Q}_i$ for $n = 0,1,2,\cdots$. It follows that $B = \cup_{n=0}^{\infty} Q_n$ where Q_n are open in B, $\dim Q_n = \dim \hat{Q}_n = \dim P_n$, $Q_n \cap A = \cup_{i=0}^{n} (\hat{Q}_i \cap A) = \cup_{i=0}^{n} P_i = P_n$ and $Q_n \subseteq Q_{n+1}$ for $n = 0,1,2,\cdots$.

LEMMA 3. *Let* $A = \cup_{n=0}^{\infty} P_n$ *be a subset of* R *where* P_n *satisfy* (III). *Then a* G_δ *set* $B = \cup_{n=0}^{\infty} Q_n$ *exists such that (i)* B *contains* A *(ii)* Q_n *are open in* B, *with* $\dim Q_n \leqq n$ *and* $Q_n \subseteq Q_{n+1}$ *for* $n = 0,1,2,\cdots$, *(iii) If* $p \in B - Q_n$ *then each open neighborhood* u_p^* *of* p *satisfies* $\dim (u_p^* \cap A) \geqq n+1$.

PROOF. The subsets P_n are open in A and satisfy (I) and (II). By Lemma 2 a G_δ set $B^* = \cup_{n=0}^{\infty} Q_n^*$ containing A exists such that Q_n^* are open in B^* and $Q_n^* \subseteq Q_{n+1}^*$ for $n = 0,1,2,\cdots$.

Also:

(1) $$\cdots Q_n^* \cap A = P_n \text{ and } \dim Q_n^* = \dim P_n.$$

Define:

(2) $$M_j = (Q_{j+1}^* - Q_j^*) - \overline{(P_{j+1} - P_j)}^R.$$

By (1):

(3) $$M_j \cap A \subseteq (Q_{j+1}^* - Q_j^*) \cap A - (P_{j+1} - P_j) \cap A = \emptyset$$

Since each Q_j^* is open in B^*, M_j is an F_σ in B^* ([1] p. 26).

Hence $M = \cup_{j=0}^{\infty} M_j$ is an F_σ set in B^* and $M \cap A = \emptyset$. Now let $B = B^* - M$, $Q_n = Q_n^* - M$. One easily gets that B is a G_δ set in R and $A \subseteq B$. Also $B = \cup_{n=0}^{\infty} Q_n$, where Q_n are open in B, $Q_n \cap A = P_n$ and $\dim Q_n = \dim P_n$. From $Q_n^* \subseteq Q_{n+1}^*$ it is clear that $Q_n \subseteq Q_{n+1}$, so (i), (ii) follow. By $Q_n^* \subseteq Q_{n+1}^*$, (1) and (2) we get for $n \neq j$:

(4) $$(Q_{n+1}^* - Q_n^*) \cap M_j \subseteq (Q_{n+1}^* - Q_n^*) \cap (Q_{j+1}^* - Q_j^*) = \emptyset.$$

From the definition of M and by (2) (3) and (4):

(5) $$\begin{aligned} Q_{n+1} - Q_n &= (Q_{n+1}^* - Q_n^*) - \bigcup_{j=0}^{\infty} M_j = (Q_{n+1}^* - Q_n^*) - M_n \\ &= (Q_{n+1}^* - Q_n^*) \cap \overline{(P_{n+1} - P_n)}^R. \end{aligned}$$

By (5) it therefore follows that if $p \in Q_{n+1} - Q_n$ then each open neighborhood u_p^* of p satisfies:

(6) $$u_p^* \cap (P_{n+1} - P_n) \neq \emptyset.$$

Suppose now that $\dim(u_p^* \cap A) \leqq n$. From the definition of P_n (see (III)) it is clear that $(u_p^* \cap A) \subseteq P_n$, contradicting (6). Thus $\dim (u_p^* \cap A) \geqq n + 1$. Since $B = \cup_{n=o}^{\infty} Q_n$ where $Q_n \subseteq Q_{n+1}$, it is clear that for $p \in B - Q_n$, and for each open neighborhood u_p^* of p, $\dim (u_p^* \cap A) \geqq n + 1$, which proves (iii).

3. Following Sklyarenko [4], we introduce:

Definition 1. [4]. A sequence of points $\{x_i \,|\, {}_{i=1,2,\ldots}\}$ in a space R is called *scattering* if it has no accumulation point in R.

Definition 2. [4]. A countable family of open sets $\{P_n \,|\, {}_{n=0,1,\ldots}\}$ in a space R is called *convergent*, if for every scattering sequence in R $\{x_i \,|\, {}_{i=1,2\ldots}\}$ a number n_0 exists, such that the set $\{x_i \,|\, {}_{i=1,2,\ldots}\} - P_{n_0}$ is finite.

In [4] Sklyarenko proved the following:

THEOREM 1. *A space R is w.i.d. if and only if it possesses a convergent family of open sets $\{P_n|_{n=0,1,\dots}\}$ with $\dim P_n \leqq n$, such that the subset $K = R - \cup_{n=0}^{\infty} P_n$ is compact and w.i.d..*

REMARK 1. The subsets P_n of Theorem 1 can be assumed to be $P_n = \cup\{u|u$ is open in R with $\dim u \leqq n\}$ ([2] p. 178).

We can now prove:

THEOREM 2. *Let A be a w.i.d. subset of a metric space R. Then a G_δ set B of R exists such that $A \subseteq B$, and B is w.i.d.*

PROOF. By Theorem 1 and Remark 1 the family of open subsets $P_n = \cup\{u|\, u$ is open in A with $\dim u \leqq n\}$ converges in A so that $K = A - \cup_{n=0} P_n$ is compact and w.i.d. $\cup_{n=0}^{\infty} P_n$ is therefore contained in the open set $R - K$ and satisfies the assumptions of Lemma 3. Hence a subset B^* exists such that: (i) B^* is a G_δ set of $R - K$, and thus a G_δ set of R, containing $\cup_{n=0}^{\infty} P_n$. (ii) $B^* = \cup_{n=0}^{\infty} Q_n$ where Q_n is open in B with $\dim Q_n = \dim P_n$, $Q_n \subseteq Q_{n+1}$, (iii) for each $p \in B^* - Q_n$ and for each open neighborhood u_p^* of p:

$$\dim(u_p^* \cap A) \geqq n + 1. \tag{7}$$

Put $B = K \cup B^* = K \cup \cup_{n=0}^{\infty} Q_n$. It is clear that B contains A, and that B is a G_δ set of R (being the union of a compact set and a G_δ set). It remains to prove that B is w.i.d. By their definitions K and $B^* = \cup_{n=0}^{\infty} Q_n$ are disjoint and since K is compact B^* is open in B. The subsets Q_n are thus open in B and finite-dimensional. It suffices to show that the family $\{Q_n|_{n=0,1,\dots}\}$ converges in B.

Suppose, to the contrary, that a scattering sequence $\{x_i|_{i=1,2\dots}\}$ exists in B (hence in B^*) such that $\{x_i| i = 1,2,\cdots\} - Q_n$ is infinite for each n. A scattering subsequence $\{x_n| n = 0,1,2,\cdots\}$ can be selected satisfying: $x_n \in B^* - Q_n$ for $n = 0,1,2\cdots$. By lemma 1 [4], a locally finite family of open (in B) neighborhoods $\{O_{x_n}| n = 0,1,2,\cdots\}$ exists such that:

$$O_{x_i}^B \cap O_{x_j}^B = \phi \text{ for } i \neq j. \tag{8}$$

Denoting $D_n = O_{x_n} \cap A$ we get by (7) and by the way the points x_n were selected that $\dim D_n \geqq n + 1$. Since $\{O_{x_n}|_{n=0,1,\dots}\}$ is locally finite in B, $\{D_n|_{n=0,1,\dots}\}$ is locally finite in A. Since $\bar{D}_n^A \subseteq O_{x_n}^B \cap A$ it follows by (8) that $\bar{D}_i^A \cap \bar{D}_j^A = \phi$ for $i \neq j$. The sets $\{\bar{D}_n^A| n = 0,1,\cdots\}$ form a locally finite family of closed pairwise disjoint sets with $\dim \bar{D}_n^A \geqq n$. By lemma 2 [4], $D = \cup_{n=0}^{\infty} \bar{D}_n^A$ is not w.i.d. On the other hand, the local finitness of $\{D_n|_{n=0,1\dots}\}$ implies that

$$D = \bigcup_{n=0}^{\infty} \bar{D}_n^{A} = \overline{\bigcup_{n=0}^{\infty} D_n}^{A}$$

D is thus a closed subset of the w.i.d. set A, and therefore w.i.d. as well, which is a contradiction. Hence $\{Q_n \mid_{n=0,1\ldots}\}$ converges in $B = K \cup \bigcup_{n=0}^{\infty} Q_n$ and by Theorem 1, B is w.i.d.

THEOREM 3. *Let R be a w.i.d. metric space, then R has a w.i.d. complete metric extension.*

PROOF. Let (h, R^*) denote a complete metric extension of R. By Theorem 2, R is topologically contained in a w.i.d. G_δ set B of R^*. Being a G_δ in the complete space R^*, B is homeomorphic to a complete space.

THEOREM 4. *Suppose that $A \subseteq R$, and that A has strong transfinite inductive dimension. Then a G_δ set B of R exists such that $A \subseteq B$, and B has strong transfinite inductive dimension.*

PROOF. As shown by Smirnov [5] (see also [2] p. 177) A is w.i.d., and so by Theorem 1 has a decomposition $A = K \cup \bigcup_{n=0}^{\infty} P_n$, where K is compact and clearly has strong transfinite inductive dimension. By the proof of Theorem 2 a w.i.d. G_δ set B of R exists such that $A \subseteq B$ and $B = K \cup \bigcup_{n=0}^{\infty} Q_n$. Another theorem of Smirnov ([2] p. 182) then asserts that B itself has strong transfinite dimension.

Using the mapping $\beta(\alpha)$ introduced by Smirnov ([2] p. 181) we get:

COROLLARY 1. *Let A, B be as in Theorem 4 and let* $\operatorname{Ind} K = \operatorname{Ind}(A - \bigcup_{n=0}^{\infty} P_n) = \alpha$. *By Theorem 3 [5],* $\operatorname{Ind} A \leq \beta(\alpha)$. *The G_δ set B in Theorem 4 satisfies* $\operatorname{Ind} B \leqq \beta(\alpha)$ *as well.*

By Theorem 4 we also get:

THEOREM 5. *Let R be a metric space having strong transfinite inductive dimension. Then R has a complete metric extension having strong transfinite inductive dimension.*

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References

1. C. Kuratowski, *Topologie* 1, 1958.

2. J.I. Nagata, *Modern Dimension Theory*, 1965.

3. Z. Shmuely, *On strongly countable-dimensional sets*, Duke Math. J. **38** (1971), 169-173.

4. E.G. Sklyarenko, *On dimensional properties of infinite dimensional spaces*, Amer. Math. Soc. Transl. **21** (1962), 35–50.

5. Yu. M. Smirnov, *Some remarks on transfinite-dimension*, Soviet Math. Dokl. **2** (1961), 1572–1575.

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