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SEQUENTIALLY EQUIDISTANT POINTS IN METRIC SPACES

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ABSTRACT

A sequence $(z_0, z_1, z_2, \ldots, z_n, z_{n+1})$ of points from $p = z_0$ to $q = z_{n+1}$ in a metric space X is said to be *sequentially equidistant* if $d(z_{i-1}, z_i) = d(z_i, z_{i+1})$ for $1 \leqq i \leqq n$. If there is path in X from p to q (or if a certain weaker condition holds), then such a sequence exists, with all points distinct, for every choice of n, while if X is compact and connected, then such a sequence exists at least for $n = 2$. An example is given of a dense connected subspace S of $\mathbf{R}^m$, $m \geqq 2$, and an uncountable dense subset E disjoint from S for which there is no sequentially equidistant sequence of distinct points ($n \geqq 2$) in $S \cup E$ between any two points of E. Techniques of dimension theory are utilized in the construction of these examples, as well as in the proofs of some of the positive results.

Fix distinct points p and q in a connected metric space (X, d). The Intermediate Value Theorem shows that there exists a point z in X which is equidistant from p and q. Generalizing this, we define a sequence $(z_0, z_1, z_2, \ldots, z_n, z_{n+1})$ from $p = z_0$ to $q = z_{n+1}$ to be *sequentially equidistant* if $d(z_{i-1}, z_i) = d(z_i, z_{i+1})$ for $1 \leqq i \leqq n$, and we ask when such a sequence exists. If n is even, then there is a trivial solution $p = z_2 = z_4 = \cdots = z_n$, $z_1 = z_3 = \cdots = z_{n-1} = q$. Our main interest is the case of sequences of distinct points. In path-connected spaces, such sequences always exist.

THEOREM 1.1. *Let p and q be distinct points in a metric space (X, d). If there is a path in X from p to q, then for any n there is a sequentially equidistant sequence $(z_0, z_1, \ldots, z_n, z_{n+1})$ of distinct points in X from $p = z_0$ to $q = z_{n+1}$.*

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Theorem 1.1 has long been known to geometers [M1], [S], [B]. Applying Theorem 1.1 to a sequence of path-connected approximations to a compact connected space yields easily

COROLLARY 1.2. *Let p and q be two distinct points in a compact connected metric space X. Then there exists a sequentially equidistant sequence* (z_0, z_1, z_2, z_3) *of distinct points in X from* $p = z_0$ *to* $q = z_3$.

We do not know whether Corollary 1.2 is true for longer sequences, even for the case $n = 3$. A means for eliminating possible degeneracies that may arise when attempting to generalize the proof of Corollary 1.2 eludes us. However, Theorem 1.1 is true under a weaker hypothesis than path-connectedness of X, and in §2, we extend it to a larger class of spaces, which we call the *approximately arc-connected* spaces. This class contains many spaces which are not path-connected, such as all pseudo-arcs (in fact, all chainable continua), but does not contain all 1-dimensional continua. All path-connected spaces are approximately arc-connected, so Theorem 1.1 follows from the more general case, but we include a slightly modernized version of Schoenberg's proof for the path-connected case because it is short, transparent, and introduces the main ideas used in §2.

In contrast to these positive results, sequentially equidistant points may fail to exist for noncompact spaces. Sequences of not-necessarily-distinct points (chains) in metric spaces were studied by Klee [K], who constructed examples of connected subsets of the plane which do not contain chains satisfying certain conditions. Our modification of his construction, given as Proposition 4.2, yields as a special case the following examples.

COROLLARY 4.6. *For each* $m \geqq 2$, *there exist a connected dense subset* $S \subseteq \mathbf{R}^m$ *and a dense subset* $E \subseteq \mathbf{R}^m$ *disjoint from S and of cardinality* $2^{\aleph_0}$ *such that for no pair of points e and e′ in E is there a sequentially equidistant sequence of* 4 *or more distinct points in* $S \cup E$ *starting at e and ending at e′.*

We show that many of our examples must have dimension 1 — in particular, all spaces $S \cup E$ as in Corollary 4.6 — but the dimension theory of the more general examples in §4 is not fully understood. A more extensive introduction to the content of §4 is given at its beginning.

We thank Professor R. Freese of St. Louis University for bringing this topic to our attention, and we acknowledge the work of V. Klee [K] as an inspiration and source of ideas exploited in our §4.

1. Proofs of Theorem 1.1 and Corollary 1.2

Theorem 1.1 appears in [M1], and the first satisfactory proof appears to have been given by Schoenberg [S]. We present here a modernized version, to be generalized in §2.

PROOF OF THEOREM 1.1. Choose a path $\gamma : I \to X$ from $\gamma(0) = p$ to $\gamma(1) = q$. By Theorems 3-30 and 3-15 of [H–Y], we may assume that γ is an imbedding.

Define $\sigma \subseteq \mathbf{R}^n$ to be the set $\{(t_1, t_2, \ldots, t_n) \mid 0 \leqq t_1 \leqq t_2 \leqq \cdots \leqq t_n \leqq 1\}$. For $1 \leqq i \leqq n+1$, let $v_i \in \mathbf{R}^n$ be the point $(0, 0, \ldots, 0, 1, 1, \ldots, 1)$ having $i-1$ initial zeros (so $v_1 = (1, 1, \ldots, 1)$ and $v_{n+1} = (0, 0, \ldots, 0)$); then σ is the n-simplex spanned by the v_i (because

$$(t_1, t_2, \ldots, t_n) = t_1 v_1 + (t_2 - t_1)v_2 + \cdots + (t_n - t_{n-1})v_n + (1 - t_n)v_{n+1}).$$

Let $\mathbf{R}^{n+1}_+$ be the set of points $(x_1, x_2, \ldots, x_{n+1})$ in $\mathbf{R}^{n+1}$ with all coordinates nonnegative. Define $\Psi : \sigma \to \mathbf{R}^{n+1}_+$ by the formula

$$\Psi(t_1, t_2, \ldots, t_n) = (d(p, \gamma(t_1)), d(\gamma(t_1), \gamma(t_2)), \ldots,$$
$$d(\gamma(t_{n-1}), \gamma(t_n)), d(\gamma(t_n), q)).$$

The faces of σ are its intersections with the $(n-1)$-planes $t_1 = 0$, $t_1 = t_2$, $t_2 = t_3, \ldots, t_{n-1} = t_n$, and $t_n = 1$. The image of the face $t_1 = 0$ lies in the n-plane $x_1 = 0$ in $\mathbf{R}^{n+1}_+$, the image of each face $t_i = t_{i+1}$ lies in the n-plane $x_{i+1} = 0$, and the image of the face $t_n = 1$ lies in the n-plane $x_{n+1} = 0$. Moreover, since $p \neq q$, the image of Ψ does not contain the origin.

Let Δ be the set of points $\mathbf{R}^{n+1}_+$ with all coordinates equal, and let $\psi : \partial\sigma \to \mathbf{R}^{n+1}_+ - \Delta$ denote the restriction of Ψ. We claim that ψ represents a nonzero element of the homotopy group $\pi_{n-1}(\mathbf{R}^{n+1}_+ - \Delta)$. To see this, let σ_0 denote the n-simplex spanned by the standard unit vectors $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ in $\mathbf{R}^{n+1}_+$ (with 1 in the ith place), and define a retraction $r : \mathbf{R}^{n+1}_+ - \{0\} \to \sigma_0$ by the formula $r(x_1, x_2, \ldots, x_{n+1}) = (1/h)(x_1, x_2, \ldots, x_{n+1})$ where $h = \Sigma_{i=1}^{n+1} x_i$. Then, ψ is linearly homotopic in $\mathbf{R}^{n+1}_+ - \Delta$ to $r \circ \psi : \partial\sigma \to \partial\sigma_0$. Now $r \circ \psi$ takes the $(n-1)$-dimensional faces of σ bijectively to the $(n-1)$-dimensional faces of σ_0. But any degree-zero map between $(n-1)$-spheres must take some pair of antipodal points to the same point (see for example exercise 7 on p. 124 of [K2]), hence $r \circ \psi$ has nonzero degree. Since $\partial\sigma_0$ is a deformation retract of $\mathbf{R}^{n+1}_+ - \Delta$, the claim is proved. Because Ψ represents a nullhomotopy of ψ, there must be a point $s = (s_1, s_2, \ldots, s_n)$ in the interior of σ with $\Psi(s) \in \Delta$, and by definition of Ψ the sequence $(p, \gamma(s_1), \gamma(s_2), \ldots, \gamma(s_n), q)$ is sequentially equidistant. Since s is in the interior of σ, we have $0 < s_1 < s_2 < \cdots < s_n < 1$, hence the images $\gamma(0) = p$, $\gamma(s_1), \gamma(s_2), \ldots, \gamma(s_n), \gamma(1) = q$ under the imbedding γ are distinct. This completes the proof of Theorem 1.1.

For use in the proof of Corollary 1.2, we note that the sequence of points obtained in the proof of Theorem 1.1 has increasing parameter on the imbedded path from p to q.

PROOF OF COROLLARY 1.2. By rescaling, we may assume that $d(p, q) = 1$. By the Eilenberg–Wojdyslawski Theorem (p. 81 of [H]), there is an isometric imbedding of (X, d) into a Banach space L; the ε-balls in L are path connected. For each positive integer m, define

$$X_m = \bigcup_{x \in X} B(x, 1/m),$$

$$X'_m = X_m \cup B(p, \tfrac{1}{4}) \cup B(q, \tfrac{1}{4}).$$

Each X'_m is path-connected, and $\bigcap_{m=1}^{\infty} X'_m = X \cup B(p, \frac{1}{4}) \cup B(q, \frac{1}{4})$.

In each X'_m, choose a path γ_m from p to q which is an imbedding; moreover choose γ_m so that the preimages of $B(p, \frac{1}{4})$ and $B(q, \frac{1}{4})$ are connected. By Theorem 1.1, on each γ_m there is a pair of points $z_{1,m}, z_{2,m}$ so that the sequence $(p, z_{1,m}, z_{2,m}, q)$ is sequentially equidistant; as remarked after the proof of Theorem 1.1, they may be chosen to have increasing parameter on γ_m. The distance from p to $z_{1,m}$ is at least $\frac{1}{3}$, so $z_{1,m}$ cannot lie in $B(p, \frac{1}{4})$. Since the parameter of $z_{2,m}$ on γ_m is greater than that of $z_{1,m}$ and γ_m does not reenter $B(p, \frac{1}{4})$ after leaving it, $z_{2,m}$ cannot lie in $B(p, \frac{1}{4})$. Similarly, $z_{1,m}$ and $z_{2,m}$ cannot lie in $B(q, \frac{1}{4})$. In particular, $z_{1,m}$ and $z_{2,m}$ lie within distance $1/m$ of X.

Since X is compact, we may choose convergent subsequences and assume that the $z_{i,m}$ converge to z_i in X for $i = 1, 2$. The sequence (p, z_1, z_2, q) is sequentially equidistant. Since no $z_{i,m}$ is within distance $\frac{1}{4}$ of p or q, these four points are distinct. This completes the proof of Corollary 1.2.

2. A generalization

In this section, we generalize Theorem 1.1 to a larger class of metric spaces. By a *compactum* we mean a compact metric space. A *continuum* is a connected compactum, and an *ε-map* is a map such that the preimage of each point has diameter less than ε. A map $f: K \to D$, where D is an m-cell, is *essential* if the restriction of f to $f^{-1}(\partial D)$ does not extend to a map from K to ∂D.

We say that a metric space X containing distinct points p and q is *approximately arc-connected with respect to the points p and q* if, for each $\varepsilon > 0$, there exists a continuum $Y \subset X$ such that each of p and q lies within distance ε from Y, and Y admits an ε-map onto the closed interval $\mathrm{I} = [0, 1]$. Then, X is *approximately arc-connected* if it is approximately arc-connected with respect to each pair of distinct points.

THEOREM 2.1. *Let p and q be distinct points in a metric space (X, d). If X is approximately arc-connected with respect to p and q, then for any n there is a sequentially equidistant sequence $(z_0, z_1, \ldots, z_n, z_{n+1})$ of distinct points in X from $p = z_0$ to $q = z_{n+1}$.*

The proof of Theorem 2.1 occupies the rest of this section. By rescaling, we shall assume that $d(p, q) = 1$. We begin with a lemma which gives a convenient reformulation of the definition of approximately arc-connected.

LEMMA 2.2. *A metric space X is approximately arc-connected with respect to p and q if and only if for each $\varepsilon > 0$ there exist a continuum $Y \subseteq X$ and an ε-map $\pi : Y \cup \{p, q\} \to \mathrm{I}$ such that $\pi(p) = 0$, $\pi(q) = 1$, and $\pi(Y) = \mathrm{I}$.*

PROOF. Sufficiency is clear. For the necessity, fix $\varepsilon > 0$ and choose $\delta > 0$ so that $d(p, q) \geqq 4\delta$ and $\delta < \varepsilon/2$. By definition, there is a continuum $Y_0 \subseteq X$, with p and q in a δ-neighborhood of Y_0, admitting a δ-map $\theta : Y_0 \to \mathrm{I}$. Since Y_0 is connected, θ is essential.

Choose points $p_0, q_0 \in Y_0$ such that $d(p_0, p) < \delta$ and $d(q_0, q) < \delta$. If $p \in Y_0$, take $p_0 = p$, and if $q \in Y_0$, take $q_0 = q$. Put $a = \theta(p_0)$ and $b = \theta(q_0)$. Then $a \neq b$ because $\mathrm{diam}(\theta^{-1}(a) \cup \{p\}) < 2\delta$, $\mathrm{diam}(\theta^{-1}(b) \cup \{q\}) < 2\delta$, and $d(p, q) \geqq 4\delta$. We may assume $a < b$. Extend $\theta : Y_0 \to \mathrm{I}$ to $\theta : Y_0 \cup \{p, q\} \to \mathrm{I}$ by setting $\theta(p) = a$ and $\theta(q) = b$.

Now the restriction $\mu = \theta \mid \theta^{-1}([a, b]) : \theta^{-1}([a, b]) \to [a, b]$ is essential by Theorem 1.9 of [K1]. Let $\alpha : [a, b] \to \mathrm{I}$ be an order-preserving homeomorphism. By Theorem 3.1 of [K1], $\alpha \circ \mu : \theta^{-1}([a, b]) \to \mathrm{I}$ is essential. Using Proposition 2.2 of [K1], we obtain a component Y of $\theta^{-1}[a, b]$ intersecting both $(\alpha \circ \mu)^{-1}(0)$ and $(\alpha \circ \mu)^{-1}(1)$. It is easily verified that the restriction of $\alpha \circ \mu$ to $Y \cup \{p, q\}$ satisfies the conditions stated in Lemma 2.2. This completes the proof.

Now fix n and let $\sigma, \sigma_0, v_i, e_i, \mathbf{R}_+^{n+1}, \Delta$, and r be as in the proof of Theorem 1.1. Choose a continuum Y and an ε-map $\pi : Y \cup \{p, q\} \to \mathrm{I}$ satisfying the properties stated in Lemma 2.2. Consider the product $\Pi\, \pi : (Y \cup \{p, q\})^n \to \mathrm{I}^n$. By

Theorem 2.3 of [K1] (which requires compactness of Y), $\Pi\,\pi$ is essential. Let Z be the inverse image of σ, and let $f\colon Z \to \sigma$ be the restriction of $\Pi\,\pi$. By Theorem 1.9 of [K1], f is also essential.

Define $\Psi : Z \to \mathbf{R}_+^{n+1}$ by

$$\Psi(z_1, z_2, \ldots, z_n) = (d(p, z_1), d(z_1, z_2), \ldots, d(z_{n-1}, z_n), d(z_n, q)).$$

Consider a k-dimensional face $\langle v_{i_0}, v_{i_1}, \ldots, v_{i_k}\rangle$ of σ, where $i_0 < i_1 < \cdots < i_k$.

Claim. *By choosing ε sufficiently small, the image of $f^{-1}(\langle v_{i_0}, v_{i_1}, \ldots, v_{i_k}\rangle)$ under $r \circ \Psi$ can be made to lie in an arbitrary neighborhood of the face $\langle e_{i_0}, e_{i_1}, \ldots, e_{i_k}\rangle$ of σ_0.*

Proof. If $f(z_1, z_2, \ldots, z_n) \in \langle v_{i_0}, v_{i_1}, \ldots, v_{i_k}\rangle$, then $f(z_1, z_2, \ldots, z_n) = \Sigma_{j=0}^{k} s_j v_{i_j}$. So in coordinates in $\mathbf{R}^n$, writing $c_l = \Sigma_{j=0}^{l} s_j$, we have

$$f(z_1, z_2, \ldots, z_n) = (0, 0, \ldots, 0, c_0, c_0, \ldots, c_0, c_1, \ldots, c_1, c_2, \ldots, c_{k-1}, 1, 1, \ldots, 1)$$

where the initial zeros occupy the first $i_0 - 1$ places, the c_0's occupy the next $i_1 - i_0$ places, and in general the c_l's occupy $i_{l+1} - i_l$ places. Writing z_0 for p and z_{n+1} for q, we have $\pi(z_{j-1}) = \pi(z_j)$ except when j is one of $i_0, i_1, \ldots, i_k$. Since π is an ε-map, it follows that the distance from z_{j-1} to z_j is less than ε except for those values of j. Therefore the jth coordinate of $\Psi(z_1, z_2, \ldots, z_n)$ is less than ε except when j is one of $i_0, i_1, \ldots, i_k$. Since the coordinates of $\Psi(z_1, \ldots, z_n)$ add up to at least 1 (the distance from p to q), the map r does not increase the size of coordinates. Thus the coefficient of e_j in $r \circ \Psi(z_1, z_2, \ldots, z_n)$ is less than ε unless j is one of $i_0, i_1, \ldots, i_k$. This proves the claim.

Let h be the linear homeomorphism from σ_0 to σ determined by the vertex map that sends e_i to v_i. Let $g = h \circ r \circ \Psi : Z \to \sigma$ and let g_0 and f_0 denote the restrictions of g and f, respectively, to $f^{-1}(\partial\sigma)$ (taken as maps to σ). For sufficiently small ε, the claim shows that g_0 carries the preimage of each face of σ to within a small distance of that face. If H denotes the linear homotopy (which exists since σ is convex) from g to f, it follows that the restriction H_0 of H to $f^{-1}(\partial\sigma)$ gives a homotopy from g_0 to f_0 which has image in a small neighborhood of $\partial\sigma$.

Let b be the barycenter of σ, and let R be a self-map of σ that retracts a neighborhood of $\partial\sigma$ containing the image of H_0 to $\partial\sigma$, but is the identity on a neighborhood of b (and maps no other points to this neighborhood). Now $R \circ H_0$ gives a homotopy in $\partial\sigma$ from $R \circ g_0$ to f_0. Since f is essential, and the homotopy $R \circ H$ maps $f^{-1}(\partial\sigma)$ to $\partial\sigma$ at each level, Proposition 1.1 of [K1] shows that $R \circ g$ is essential and hence also surjective. It follows that the image of Ψ must intersect Δ. (Otherwise, the image $\text{im}(r \circ \Psi)$ would not contain the barycenter b_0 of σ_0. In turn, $\text{im}(g) = \text{im}(h \circ r \circ \Psi)$ would not contain the barycenter $b = h(b_0)$ of σ, and finally, $b = R(b)$ would not be in the image of $R \circ g$.)

Let $z = (z_1, \ldots, z_n)$ be a point with $\Psi(z) \in \Delta$. By definition of Ψ, the sequence $(p, z_1, \ldots, z_n, q)$ is sequentially equidistant. Now $f(z)$ is not in $\partial\sigma$, since we have shown that $r \circ \Psi$ maps $f^{-1}(\partial\sigma)$ close to $\partial\sigma_0$. Therefore the points $\pi(p) = 0$, $\pi(z_1)$, $\pi(z_2), \ldots, \pi(z_n)$, $\pi(q) = 1$ are distinct points of I, hence $p, z_1, z_2, \ldots, z_n, q$ are distinct. This completes the proof of Theorem 2.1.

Not all continua are approximately arc-connected. We present an example of a planar 1-dimensional continuum X containing points p and q such that X is not approximately arc-connected with respect to p and q.

EXAMPLE 2.3. Denote by J the union of the half-open intervals $[-1, 0)$ and $(0, 1]$. Let $k : J \to \mathbf{R}^2$ be the map defined by

$$k(t) = \left((1+t)\cos\left(\frac{1}{t}\right), (1+t)\sin\left(\frac{1}{t}\right)\right).$$

Let X_0 denote the unit circle in $\mathbf{R}^2$ and let X_1 denote the image of k, so that $X = X_0 \cup X_1$ is the closure of X_1. The Borsuk–Ulam theorem (see Theorem 12, p. 109 of [K2]) implies that for no $\varepsilon < 2$ does there exist an ε-map from X_0 into $[0, 1]$. But for sufficiently small ε, if $Y \subseteq X$ is a subcontinuum with $k(-1)$ and $k(1)$ in the ε-neighborhood of Y, then it is clear that $X_0 \subseteq Y$. Hence for sufficiently small ε there cannot be an ε-map from Y to $[0, 1]$.

3. Connected subsets of $\mathbf{R}^m$

In this section, we give some criteria for a subset of $\mathbf{R}^m$ to be connected.

LEMMA 3.1. *Let K be a closed subset of S^m, $m \geqq 2$, which separates two points a and b of S^m. Then some component of K separates a and b in S^m.*

PROOF. Let U be the component of $S^m - K$ containing a and let M be the closure of U. Then M is closed and connected, and $M \subseteq U \cup K$. Hence there is a component C of $S^m - M$ containing b. Let C_0 be the component of $S^m - K$ such that $b \in C_0 \subseteq C$.

By Property II (The Brouwer Property) on p. 47 of [W], the boundary $\text{bd}(C)$

is a closed and connected subset of S^m. Clearly, $\mathrm{bd}(C) \subseteq K$, so $\mathrm{bd}(C)$ is contained in a component L of K. Now $\mathrm{bd}(C)$ separates U and C_0, and $\mathrm{bd}(C) \subseteq L$, while $L \cap U$ and $L \cap C_0$ are empty. Therefore $S^m - L$ has at least two components F_1 and F_2, where $U \subseteq F_1$ and $C_0 \subseteq F_2$. This completes the proof.

Recall from [H–W] that a compact m-dimensional metric space, $m \geqq 1$, is called an *m-dimensional Cantor manifold* if it cannot be disconnected by a subset of dimension $\leqq m-2$. In particular, an m-cell is such a space.

The following is an easy consequence of Theorem VI 11 of [H–W].

LEMMA 3.2. *If C is a compact minimal separator of two points in S^{m+1}, then C is an m-dimensional Cantor manifold.*

LEMMA 3.3. *If D is a compact minimal separator of a pair of opposite faces A and B of* I^{m+1}, *then D is an m-dimensional Cantor manifold.*

PROOF. Regard S^{m+1} as the double of I^{m+1} along $\partial\mathrm{I}^{m+1}$. Then the double D^* of D has as its set of components the doubled components of D (i.e. the subspaces resulting from doubling the components of D along their intersections with $\partial\mathrm{I}^{m+1}$).

Let $a \in A$ and $b \in B$. Then D^* separates a and b in S^{m+1}. By Lemma 3.1, some component of D^*, say the double of the component K of D, separates a and b in S^{m+1}. It is clear that K separates a and b in I^{m+1}. Since A and B are connected and $K \cap A$ and $K \cap B$ are empty, it follows that K separates A and B in I^{m+1}. Since D was minimal, we must have $D = K$ and hence D is connected.

Since the double D^* of D is a compact minimal separator of two points in S^{m+1}, Lemma 3.2 shows it is an m-dimensional Cantor manifold. If a closed subset of D separates D, then its double separates D^*. This proves that D is an m-dimensional Cantor manifold, completing the proof of Lemma 3.3.

For $m \geqq 1$, regard $\mathbf{R}^m$ as $\mathbf{R} \times \mathbf{R}^{m-1}$. Each subspace of the form $\{x\} \times \mathbf{R}^{m-1}$ will be called a *vertical flat*.

PROPOSITION 3.4. *Let S be a subset of* $\mathbf{R}^m$, $m \geqq 1$. *Suppose that S intersects each vertical flat, and each $(m-1)$-dimensional continuum in* $\mathbf{R}^m$ *that is a Cantor manifold which is not contained in some vertical flat. Then S is connected and dense in* $\mathbf{R}^m$.

PROOF. The density is obvious. For $m = 1$, S must equal $\mathbf{R}$, so we assume $m \geqq 2$. Suppose for contradiction that S is not connected, and let A, B form a separation of S. Let U and V be disjoint open subsets of $\mathbf{R}^m$ with $A \subseteq U$ and

$B \subseteq V$. Denote $\mathbf{R}^m - (U \cup V)$ by C, and for each vertical flat F, let $C_F = C \cap F$. We claim that $F - C_F$ is entirely contained either in U or in V. For if not, let $u \in (F - C_F) \cap U$ and $v \in (F - C_F) \cap V$. Choose a PL arc $\alpha \subseteq F$ running from u to v. Choose a regular neighborhood N' of α in F of the form $\mathrm{I}^{m-2} \times \mathrm{I}$ in such a way that $N'_0 = \mathrm{I}^{m-2} \times \{0\} \subseteq U$ and $N'_1 = \mathrm{I}^{m-2} \times \{1\} \subseteq V$. Let $\mathrm{I}_\varepsilon = [x - \varepsilon, x + \varepsilon]$, where $F = \{x\} \times \mathbf{R}^{m-1}$. For sufficiently small ε, $N = \mathrm{I}_\varepsilon \times N'$ is a regular neighborhood of α in $\mathbf{R}^m$ such that $N_0 = \mathrm{I}_\varepsilon \times N'_0 \subseteq U$ and $N_1 = \mathrm{I}_\varepsilon \times N'_1 \subseteq V$. No vertical flat separates N_0 from N_1 in N. Now $C \cap N$ must separate N_0 from N_1 in N. By Lemma 3.3, a minimal closed connected subset of K which separates them must be an $(m-1)$-dimensional Cantor manifold. It cannot lie in a vertical flat, so must contain a point of S, a contradiction. We conclude that $F - C_F$ is entirely contained either in U or in V. Since this is true for all vertical flats, the images of U and V under the projection from $\mathbf{R} \times \mathbf{R}^{m-1}$ to $\mathbf{R}$ are disjoint open subsets of $\mathbf{R}$. Therefore there is a point p that is not in their image, so the vertical flat $\{p\} \times \mathbf{R}^{m-1}$ lies in C. This is a contradiction, since every vertical flat contains a point of S. This completes the proof.

It seems worth noting the following special case of Proposition 3.4.

COROLLARY 3.5. *Let S be a subset of $\mathbf{R}^m$. If S intersects each $(m-1)$-dimensional continuum in $\mathbf{R}^m$, or even each $(m-1)$-dimensional Cantor manifold, then S is connected and dense in $\mathbf{R}^m$.*

4. Spaces without equidistant sequences

The examples we construct in this section are the union $X = S \cup E$ of two disjoint dense subsets of $\mathbf{R}^m$. The set S and hence also X will be connected, and E will have cardinality $2^{\aleph_0}$. We may choose X to contain no sequentially equidistant sequence of four or more distinct points with both endpoints in E. More generally, X can be constructed so as to contain no sequences of distinct points (actually, no sequences in which each three successive points are distinct) with endpoints in E for which the successive distances are in an arbitrarily selected set of fewer than $2^{\aleph_0}$ ratios (see the paragraph before Proposition 4.2 for precise definitions).

The constructions are based on an inductive selection procedure developed in Proposition 4.2. At each step, one must avoid selecting a point which would create a "bad" sequence; the sets of points that must be avoided for S and E are described in Lemma 4.3. On the other hand, one wants be sure to select enough points to make sure that S satisfies the connectedness criterion given in

Proposition 3.4. This is posible because of a simple yet remarkable property of algebraic sets, given in Lemma 4.1. An additional complication in Proposition 4.2 is that for a fixed k with $1 \leqq k < m$, the set S is to be selected so that it contains exactly one point from each subset of the form $\{x\} \times \mathbf{R}^{m-k} \subseteq \mathbf{R}^k \times \mathbf{R}^{m-k} = \mathbf{R}^m$. This shows that S can be taken to be the graph of a function from $\mathbf{R}^k$ to $\mathbf{R}^{m-k}$.

The sets X have interesting dimension-theoretic properties. Being connected, they must have dimension at least 1, and the constructions may always be performed to make certain that X has dimension equal to 1. In fact we show that if even one of the avoided ratios involves sequences of 4 points or of 5 points, as opposed to longer sequences, then the dimension of X cannot exceed 1; in particular, this applies to the examples containing no sequentially equidistant sequences. We also show that the construction we use to prove Proposition 4.2 always yields X of dimension at most 2. But the general question of whether an example which excludes all sequences with distances in a given ratio must have dimension 1 remains open.

It will be convenient to use "small inductive dimension," the dimension theory in [H–W]. Since all our spaces will be separable and metrizable, this agrees with all other definitions of topological dimension.

We call a subset $K \subseteq \mathbf{R}^m$ *algebraic* if it is the set of common zeros of a collection of nonzero polynomials. In most of our applications, the algebraic sets will be hyperplanes or round spheres of dimension $m-1$ or $m-2$. Algebraic sets have the following property.

LEMMA 4.1. *Every open subset of* $\mathbf{R}^m$ *contains an arc that intersects each algebraic subset of* $\mathbf{R}^m$ *in at most finitely many points.*

PROOF. We will construct an arc that intersects every algebraic subset of $\mathbf{R}^m$ in at most finitely many points. Translating any segment of this into the given open subset will then prove the lemma.

It is enough to find an arc that intersects the set of zeros of each nonzero polynomial in at most finitely many points. Choose real numbers $k_1, k_2, \ldots, k_m$ which are linearly independent over the rationals, and let $\alpha(t) = (e^{k_1 t}, e^{k_2 t}, \ldots, e^{k_m t})$. Let

$$f(x_1, x_2, \ldots, x_m) = \sum_{j=1}^{N} c_j x_1^{n_{1,j}} x_2^{n_{2,j}} \cdots x_m^{n_{m,j}}$$

be any nonzero polynomial in m variables, written so that all $c_j \neq 0$ and no two of the m-tuples $(n_{1,j}, \ldots, n_{m,j})$ are equal. Now $f \circ \alpha$ maps $\mathbf{R}$ to $\mathbf{R}$, and the preimage of zero is the set of t such that $\alpha(t)$ lies in the zero set of f. Write

$$f \circ \alpha(t) = \sum_{j=1}^{N} c_j e^{l_j t}.$$

Since the l_j are distinct linear combinations of the k_i with integral coefficients, they are distinct. We may reorder so that $l_1 < l_2 < \cdots < l_N$. Now

$$\frac{e^{-l_N t}}{c_N} f \circ \alpha(t) = 1 + \sum_{j=1}^{N-1} \frac{c_j}{c_N} e^{(l_j - l_N)t}$$

so the set of zeros of $f \circ \alpha$ is bounded above. Similarly, it is bounded below. Since $f \circ \alpha$ is nonzero and analytic, its zeros form a discrete closed set, so there are only finitely many of them. Therefore the image of α intersects any algebraic set in $\mathbf{R}^m$ in at most finitely many points. This completes the proof of Lemma 4.1.

A sequence $z = (z_0, z_1, z_2, \ldots, z_n, z_{n+1})$ of points in a space X, with $n \geqq 2$, is said to be *nondegenerate* if $z_i \neq z_{i+1}$ for $0 \leqq i \leqq n$ and $z_i \neq z_{i+2}$ for $0 \leqq i < n$. In particular, any sequence of four or more distinct points is nondegenerate. Given a nondegenerate sequence $z = (z_0, z_1, z_2, \ldots, z_n, z_{n+1})$ in a metric space with metric d, let $\delta_i = d(z_{i-1}, z_i)$ and let $\delta_0 = \Sigma_{i=1}^{n+1} \delta_i$. Following Klee [K] we will define the *ratio sequence* of z to be the point $r(z)$ in the standard n-simplex σ_n which has barycentric coordinates $(\delta_1/\delta_0, \delta_2/\delta_0, \ldots, \delta_{n+1}/\delta_0)$. Since no two consecutive points are equal, $r(z)$ lies in the interior of σ_n.

PROPOSITION 4.2. *Let $m \geqq 2$ and let $\mathscr{R} = \{r_\nu\}$ be a set of cardinality less than $2^{\aleph_0}$, with each r_ν a point in the interior of the standard simplex of dimension $n_\nu \geqq 2$. Let k be an integer with $1 \leqq k \leqq m - 1$, and let $\mathscr{K}$ be the collection of all subsets of $\mathbf{R}^m = \mathbf{R}^k \times \mathbf{R}^{m-k}$ of the form $\{x\} \times \mathbf{R}^{m-k}$. Then there exist disjoint sets S and E in $\mathbf{R}^m$ with the following properties.*

(a) *If $k \geqq 2$ then S intersects each $(m-1)$-dimensional continuum in $\mathbf{R}^m$. If $k = 1$, then S intersects each $(m-1)$-dimensional continuum in $\mathbf{R}^m$ that is a Cantor manifold not contained in some element of $\mathscr{K}$.*

(b) *S intersects each element of $\mathscr{K}$ in exactly one point.*

(c) *E has cardinality $2^{\aleph_0}$ and E is dense in $\mathbf{R}^m$.*

(d) *If $X = S \cup E$, then for no nondegenerate sequence $z = (z_0, z_1, z_2, \ldots, z_{n+1})$ in X with $z_0, z_{n+1} \in E$ is the ratio sequence $r(z)$ an element of $\mathscr{R}$.*

Moreover, X may be chosen so that $\dim(X) \leqq 1$.

PROOF. We begin by fixing some convenient notation. Let S and E be subspaces of $\mathbf{R}^m$, for some $m \geqq 2$. Define $\mathrm{seq}(S, E)$ to be the set of all nondegenerate (finite) sequences in $S \cup E$ whose first and last points are in E. A pair (S, E) is said to be *admissible* if it satisfies the following:

(1) the cardinality of $S \cup E$ is less than $2^{\aleph_0}$,

(2) $S \cap E$ is empty,

(3) for no $z \in \mathrm{seq}(S, E)$ is $r(z) \in \mathscr{R}$.

The sets S and E of the theorem will be constructed simultaneously by transfinite induction. The inductive step is based on the following lemma:

LEMMA 4.3. *Suppose (S, E) is admissible.*

(a) *The set of all x in $\mathbf{R}^m$ such that $(S \cup \{x\}, E)$ is not admissible is contained in the union X_{m-2} of a collection of fewer than $2^{\aleph_0}$ round $(m-2)$-spheres in $\mathbf{R}^m$.*

(b) *The set of all x in $\mathbf{R}^m$ such that $(S, E \cup \{x\})$ is not admissible is contained in the union X_{m-1} of a collection of fewer than $2^{\aleph_0}$ round $(m-1)$-spheres in $\mathbf{R}^m$.*

(In (a) and (b) and all future statements about spheres, the spheres may have radius 0.)

PROOF. We will adopt some notation from [K]. For distinct points p and q in $\mathbf{R}^m$, and a positive number s, define

$$\Gamma(p, q; s) = \{x \in \mathbf{R}^m \mid d(p, x) = sd(x, q)\}.$$

It is easily seen that if $s \neq 1$, then $\Gamma(p, q; s)$ is a round $(m-1)$-sphere whose center lies on the straight line containing p and q, but is not equal to either p or q. We note for later use that if $s < 1$, then $\Gamma(p, q; s)$ bounds an m-ball in $\mathbf{R}^m$ which contains p in its interior, and the diameter of this ball approaches 0 as s approaches 0. If $s = 1$, then $\Gamma(p, q; s)$ is an $(m-1)$-dimensional hyperplane, while if $s > 1$, then $\Gamma(p, q; s)$ is an $(m-1)$-sphere enclosing q. Now define

$$\mathscr{Q} = \{0\} \cup \left\{ \frac{s_i}{s_{i+1}} \,\middle|\, (s_1, s_2, \ldots, s_{n+1}) \in \mathscr{R} \right\}$$

$$\cup \left\{ \frac{s_{i+1}}{s_i} \,\middle|\, (s_1, s_2, \ldots, s_{n+1}) \in \mathscr{R} \right\},$$

$$\mathscr{D} = \{td(p, q) \mid t \in \mathscr{Q} \text{ and } p, q \in S \cup E\},$$

$$\Sigma_{m-1} = \{Z \mid Z \text{ is an } (m-1)\text{-sphere in } \mathbf{R}^m \text{ with center in } S \cup E \text{ and radius in } \mathscr{D}\},$$

$$\Sigma'_{m-1} = \{\Gamma(p, q; s) \mid p, q \in S \cup E, s \in \mathscr{Q} - \{0\}\},$$

$$\Sigma_{m-2} = \{S_1 \cap S_2 \mid S_1 \in \Sigma_{m-1}, S_2 \in \Sigma'_{m-1}, S_1 \neq S_2\}.$$

Define X_{m-1} to be the union of $S \cup E$ and all elements of Σ_{m-1} and define X_{m-2} to be the union of $S \cup E$ and all elements of Σ_{m-2}.

SUBLEMMA 4.4. *Let $(z_0, z_1, \ldots, z_{n+1})$ be a nondegenerate sequence with ratio sequence $(s_1, s_2, \ldots, s_{n+1}) \in \sigma_n$.* Then

(a) *For each i with $0 \leqq i \leqq n-1$, z_i lies on the sphere with center z_{i+1} and radius $(s_{i+1}/s_{i+2})d(z_{i+1}, z_{i+2})$.*

(b) *For each i with $2 \leqq i \leqq n+1$, z_i lies on the sphere with center z_{i-1} and radius $(s_i/s_{i-1})d(z_{i-2}, z_{i-1})$.*

(c) *For each i with $1 \leqq i \leqq n$, z_i lies on the sphere $\Gamma(z_{i-1}, z_{i+1}; s_i/s_{i+1})$.*

PROOF. These are immediate from the fact that

$$\frac{d(z_{j-1}, z_j)}{d(z_j, z_{j+1})} = \frac{s_j}{s_{j+1}},$$

where j is respectively $i+1$, $i-1$, and i in the three cases.

We can now prove Lemma 4.3. For (b), suppose that $(S, E \cup \{x\})$ is not admissible. Obviously, part (1) of the definition of admissibility cannot fail. If (2) fails, then $x \in X_{m-1}$ since $S \cup E \subseteq X_{m-1}$. Suppose that condition (3) fails. Then there exists a sequence $z = (z_0, z_1, \ldots, z_{n+1})$ in $\operatorname{seq}(S, E \cup \{x\})$, having ratio sequence $r(z) \in \mathscr{R}$. Since (S, E) was admissible, x must appear at least once in this sequence, and then Sublemma 4.4(a) or 4.4(b) implies that $x \in X_{m-1}$. This proves (b). For (a), suppose that $(S \cup \{x\}, E)$ is not admissible. Again, conditions (1) and (2) cannot fail. If (3) fails, then there exists $z \in \operatorname{seq}(S \cup \{x\}, E)$ with ratio sequence $r(z) \in \mathscr{R}$. Since (S, E) was admissible, x must appear at least once in this sequence, say $x = z_i$. Since z_0 and z_{n+1} are in E, both z_{i-1} and z_{i+1} are defined. Since $n \geqq 2$, at least one of z_{i-2} or z_{i+2} is defined. By Sublemma 4.4(a) or (b), x must lie in a sphere in Σ_{m-1} centered at z_{i-1} or z_{i+1}. By Sublemma 4.4(c), x must lie on the sphere (or hyperplane) $\Gamma(z_{i-1}, z_{i+1}; s_i/s_{i+1})$ in Σ'_{m-1}. Since $\Gamma(z_{i-1}, z_{i+1}; s_i/s_{i+1})$ in Σ'_{m-1} cannot be a sphere centered at z_{i-1} or z_{i+1}, x lies in an element of Σ_{m-2} and hence lies in X_{m-2}. This completes the proof of Lemma 4.3.

We will now begin the construction of the sets S and E satisfying Proposition 4.2. If $k \geqq 2$, let $\mathscr{C}$ denote the set of all $(m-1)$-dimensional continua M in $\mathbf{R}^m$, while if $k=1$, let $\mathscr{C}$ denote the set of all $(m-1)$-dimensional continua in $\mathbf{R}^m$ that are Cantor manifolds not contained in any element of $\mathscr{K}$. Let $\mathscr{B}$ denote the set of all closed m-balls in $\mathbf{R}^m$ of positive radius. Let $\mathbf{R}$ be well-ordered so that for each $\alpha \in \mathbf{R}$, the cardinality of $\{\beta \in \mathbf{R} \mid \beta < \alpha\}$ is less than $2^{\aleph_0}$. Additionally, assume that 0 is the minimal element. Choose one-to-one correspondences of $\mathbf{R}$ with $\mathscr{B}$, $\mathscr{C}$, and $\mathscr{K}$, denoting by B_α, C_α, and K_α the respective elements of $\mathscr{B}$, $\mathscr{C}$, and $\mathscr{K}$ corresponding to $\alpha \in \mathbf{R}$.

Choose any point $s \in B_0$. If $s \in K_0$, put $s = s'$. Otherwise, choose any $s' \in K_0$. Let $S_0 = \{s, s'\}$. Now, use Lemma 4.3(b) to choose $e \in B_0 - \{s, s'\}$ so that if $E_0 = \{e\}$ then (S_0, E_0) is an admissible pair (for example, on any straight line segment in B_0 are points not in the set X_{m-1} of Lemma 4.3(b)).

Suppose inductively that for each $\beta < \alpha$ there has been constructed a pair (S_β, E_β) which is admissible and additionally satisfies the following for all $\gamma \leqq \beta$:

(4) $S_\gamma \subseteq S_\beta$ and $E_\gamma \subseteq E_\beta$, with proper containment if $\gamma < \beta$,

(5) $S_\gamma \cap C_\gamma$ is nonempty,

(6) $E_\gamma \cap B_\gamma$ is nonempty,

and moreover

(7) $K_\gamma \cap S_\beta$ consists of exactly one point.

We shall construct an admissible pair (S_α, E_α) satisfying (4), (5), (6), and (7) for $\gamma \leqq \beta \leqq \alpha$. Let $S_\alpha^* = \bigcup_{\beta < \alpha} S_\beta$ and $E_\alpha^* = \bigcup_{\beta < \alpha} E_\beta$. Then it is clear that (S_α^*, E_α^*) is admissible, and meets each K_γ with $\gamma < \alpha$ in exactly one point.

We wish to select a point $x \in C_\alpha$ with the following properties:

(a) $x \notin \bigcup_{\beta < \alpha} K_\beta$, and

(b) $(S_\alpha^* \cup \{x\}, E_\alpha^*)$ is admissible.

Suppose first that $k \geqq 2$. Since C_α is $(m-1)$-dimensional, by [M] or [S2] there exists an $(m-1)$-dimensional coordinate plane P in $\mathbf{R}^m$ such that the projection of C_α to P has dimension $m-1$. By Theorem IV3 of [H–W], there must be a nonempty open (in P) set U in this projection. Consider the projection of the set X_{m-2} from Lemma 4.3(a) to P. The image of the projection of any $(m-2)$-sphere in $\mathbf{R}^m$ to P is either an $(m-2)$-dimensional ellipse, or a flat $(m-2)$-dimensional cell (or a point, if the radius is zero). Therefore the projection of X_{m-2} is contained in a union of fewer than $2^{\aleph_0}$ algebraic sets and points. Each K_γ for $\gamma < \alpha$ is a hyperplane of dimension $m-k$; since $k \geqq 2$ its projection to P is an alebraic subset (if k were 1, its image might be all of P). Apply Lemma 4.1 to get an arc in the open set U. This arc must contain a point

x^* which is not in the projected image of $X_{m-2} \cup (\bigcup_{\gamma<\alpha} K_\gamma)$. Any point x in C_α that projects to x^* in P will satisfy (a) and (b). If $S_\alpha^* \cup \{x\}$ contains a point of K_α, put $x' = x$. Otherwise, observe that the intersection of X_{m-2} with K_α consists of a collection of fewer than $2^{\aleph_0}$ algebraic subsets that are proper subsets of K_α; therefore, an application of Lemma 4.1 in K_α leads to a point x' in K_α which is not in the set X_{m-2} of Lemma 4.3(a), so that $(S_\alpha^* \cup \{x, x'\}, E_\alpha^*)$ is admissible. This pair now satisfies (5) and (7). Now, use Lemma 4.3(b) to choose $w \in B_\alpha - (S_\alpha^* \cup \{x, x'\} \cup E_\alpha^*)$ so that $(S_\alpha^* \cup \{x, x'\}, E_\alpha^* \cup \{w\})$ is admissible (for example, choose w on a straight line segment in B_α). This pair now satisfies (5) and (7) as well, so the inductive construction is complete for $k \geqq 2$.

Suppose now that $k = 1$. Then C_α is an $(m-1)$-dimensional Cantor manifold not contained in any set of the form $\{x\} \times \mathbf{R}^{m-1}$. Consequently, its projection to the first factor of $\mathbf{R}^m$ is connected and contains more than one point, so is 1-dimensional. We can now apply the following result, a special case of Theorem 4.9 of [S1].

THEOREM. *Let W be a compact $(m-1)$-dimensional Cantor manifold in $\mathbf{R} \times \mathbf{R}^{m-1}$. If the projection of W to the first factor is 1-dimensional, then there is an $(m-2)$-dimensional coordinate hyperplane $P' \subseteq \mathbf{R}^{m-1}$ such that the projection of W to $\mathbf{R} \times P'$ is $(m-1)$-dimensional.*

Since the projection of C_α to this coordinate plane $P = \mathbf{R} \times P'$ is $(m-1)$-dimensional, it has interior (by Theorem VI3 of [H–W]). Since P contains the first factor, the projection of each element of $\mathscr{K}$ to P is a proper subset of P. The rest of the selection process now proceeds exactly as in the case $k \geqq 2$.

Finally, let $S = \bigcup_{\alpha\in\mathbf{R}} S_\alpha$ and $E = \bigcup_{\alpha\in\mathbf{R}} E_\alpha$. By (2), $E \subset \mathbf{R}^m - S$. By (7), S intersects each element of $\mathscr{K}$ in exactly one point. By (5), we obtain statement (a) of Proposition 4.2. By (6), E is dense, and by (4), E has cardinality $2^{\aleph_0}$. By (3), no $z \in \text{seq}(S, E)$ has $r(z) \in \mathscr{R}$. This completes the proof of all statements in Proposition 4.2 except the remark about the dimension of X.

In order to force the construction to yield a set X of dimension $\leqq 1$, let $\{A_i\}$ be a countable basis of round open balls in $\mathbf{R}^m$. For each i, let $\{B_i^j\}$ be a countable basis for the topology of ∂A_i, consisting of the intersections of round balls in $\mathbf{R}^m$ with ∂A_i, so that the boundaries of the B_i^j are round $(m-2)$-spheres. Let $Y_{m-1} = \bigcup \partial A_i$, and let $Y_{m-2} = \bigcup \partial B_i^j$. In the construction of X, choose S_0 and E_0 to lie in the complement of Y_{m-1}, and in all steps where the sets X_{m-1} and X_{m-2} of Lemma 4.3 are used, use instead the sets $X_{m-1} \cup Y_{m-1}$

and $X_{m-2} \cup Y_{m-2}$. The construction will then yield a space X for which X is disjoint from Y_{m-2}, and any such X must have dimension $\leqq 1$. This completes the proof of Proposition 4.2.

Combining our results will now yield the examples. If $f: \mathbf{R}^k \to \mathbf{R}^{m-k}$ is a function, let $S_f \subseteq \mathbf{R}^k \times \mathbf{R}^{m-k}$ denote its graph.

THEOREM 4.5. *Let k and m be integers with $1 \leqq k < m$, and let $\mathscr{R} = \{r_\nu\}$ be a set of cardinality less than $2^{\aleph_0}$, with each r_ν a point in the interior of the standard simplex of dimension $n_\nu \geqq 2$. Then there exist a function $f: \mathbf{R}^k \to \mathbf{R}^{m-k}$ and a set $E \subseteq \mathbf{R}^m$ disjoint from the graph S_f of f with the following properties.*

(a) *S_f is connected.*

(b) *E has cardinality $2^{\aleph_0}$ and E is dense in $\mathbf{R}^m$.*

(c) *If $X = S_f \cup E$, then for no nondegenerate sequence $z = (z_0, z_1, z_2, \ldots, z_{n+1})$ in X with $z_0, z_{n+1} \in E$ is the ratio sequence $r(z)$ an element of $\mathscr{R}$.*

Moreover, f may be chosen so that $\dim(X) \leqq 1$.

PROOF. The set S constructed in Proposition 4.2 is the graph of a function from $\mathbf{R}^k$ to $\mathbf{R}^{m-k}$, because of condition (b) there. Because of condition (a) there, Proposition 3.4 implies that S is connected.

Taking $\mathscr{R} = \{\text{barycenter}(\sigma_n) \mid n \geqq 2\}$, we have as an immediate consequence the case of interest:

COROLLARY 4.6. *For each $m \geqq 2$, there exist a connected dense subset $S \subseteq \mathbf{R}^m$ and a dense subset $E \subseteq \mathbf{R}^m$ disjoint from S and of cardinality $2^{\aleph_0}$ such that for no pair of points e and e' in E is there a sequentially equidistant sequence of* 4 *or more distinct points of $S \cup E$ starting at e and ending at e'.*

The next theorem shows that many of our examples — in particular, all the examples in Corollary 4.6 — will have dimension 1.

THEOREM 4.7. *Let X be a dense subset of $\mathbf{R}^m$, containing a dense subset E. Let r be a point in the interior of the standard* 2*-dimensional simplex, or in the interior of the standard* 3*-simplex. If* $\dim(X) \geqq 2$, *then there is a sequence of distinct points of X, with endpoints in E, whose ratio sequence is r.*

The proof will use the following lemma.

LEMMA 4.8. *Let X be a dense subset of $\mathbf{R}^m$, containing a dense subset E. Assume that the dimension of X is at least* 2. *Then there is a point $z \in X$ with the following property. Given any two positive numbers s and t, there exists a* *positive number ε with the property that if e is any point of E with $d(e, z) < \varepsilon$, then there are infinite sets $\{x_i\} \subseteq X$ and $\{e_i\} \subseteq E$ of distinct points with the property that for each i, $d(z, x_i) = sd(e, z)$ and $d(x_i, e_i) = td(e, z)$.*

PROOF. By definition of inductive dimension, there exists a point z in X with the following property. Writing Σ_δ for the $(m-1)$-sphere of radius δ centered at z, there is an $\varepsilon_1 > 0$ such that for each positive $\delta < \varepsilon_1$, there exists a point $y \in X \cap \Sigma_\delta$ such that all sufficiently small neighborhoods of y in $X \cap \Sigma_\delta$ have nonempty boundary.

Let $\varepsilon = \varepsilon_1/s$, and suppose that e is any point of E with $d(e, z) < \varepsilon$. Denote $d(e, z)$ by d_1, and let $d_2 = sd_1$ and $d_3 = td_1$. Since $sd_1 < \varepsilon_1$, there exists $y \in \Sigma_{d_2}$ such that arbitrarily small neighborhoods of y in $X \cap \Sigma_{d_2}$ have nonempty boundary. Conceivably, y may equal e.

Consider the ray starting at z and passing through y. Let p be the point on this ray, at distance d_3 past y. For points p' on this ray at distance less than d_3 past y but sufficiently close to p, the intersection of the sphere of radius d_3 centered at p' with the set $X \cap \Sigma_{d_2}$ is the boundary of a small neighborhood of y in $X \cap \Sigma_{d_2}$. Choosing an $e_1 \in E$ extremely close to such a p' (and distinct from e) the intersection of the sphere of radius d_3 centered at e_1 with $X \cap \Sigma_{d_2}$ is the boundary of a small neighborhood of y in $X \cap \Sigma_{d_2}$, hence is nonempty. Choosing x_1 in this intersection, we have $d(z, x_1) = d_2$ and $d(x_1, e_1) = d_3$, giving the desired ratio sequence. Infinitely many more pairs may be obtained by varying the choice of e_1.

PROOF OF THEOREM 4.7. Assume first that $r = (r_1, r_2, r_3)$ (in barycentric coordinates) is a point in the interior of the standard 2-simplex. Applying Lemma 4.8 using $s = r_2/r_1$ and $t = r_3/r_1$ yields a sequence (e, z, x_1, e_1) with ratio sequence r. Now suppose $r = (r_1, r_2, r_3, r_4)$ is in the interior of the standard 3-simplex. Apply Lemma 4.8 twice, obtaining two sequences (e, z, x_1, e_1) and (e, z, x_2, e_2), with x_1, x_2, e_1, and e_2 distinct, such that $d(x_1, z) = r_2 d(e, z)$, $d(x_1, e_1) = r_1 d(e, x)$, $d(z, x_2) = r_3 d(e, z)$, and $d(x_2, e_2) = r_4 d(e, z)$. The sequence (e_1, x_1, z, x_2, e_2) has ratio sequence r. This completes the proof of Theorem 4.7.

REMARK 4.9. The question of whether any example as in Theorem 4.5 can have dimension greater than 1 remains open. Certainly X cannot contain any open subset of $\mathbf{R}^m$, so X has dimension at most $m-1$. In particular, when $m = 2$, X must be 1-dimensional. The construction used to prove Proposition 4.2 must always yield a set X of dimension $\leqq 2$ (when $\mathcal{R}$ is nonempty). To see this, first note that since E is dense in $\mathbf{R}^m$, the number 0 is a limit point of the set $\mathscr{D}$ defined in the proof of Lemma 4.3. Let $x \in X$ and $\varepsilon > 0$. Choose $\delta \in \mathscr{D}$ with $0 < \delta < \varepsilon$. Let Σ_δ be the round $(m-1)$-sphere in $\mathbf{R}^m$ centered at x and having radius δ. Suppose that $y \in X \cap \Sigma_\delta$; let $\varepsilon_1 > 0$. For sufficiently small $\delta_1 \in \mathscr{D}$, the sphere $\Gamma_{\delta_1} = \Gamma(y, x; \delta_1)$ is the boundary of an m-ball containing y in its interior and contained in the ball of radius ε_1 centered at y. Now the point y lies in $S_\alpha \cup E_\alpha$ for some $\alpha \in \mathbf{R}$. The number δ occurs by applying Lemma 4.3 to some (S_β, E_β) and δ_1 occurs similarly for some (S_γ, E_γ). Let $\mu = \max\{\alpha, \beta, \gamma\}$. Then $E \cap \Sigma_\delta \subseteq E_\mu$ and $S \cap (\Sigma_\delta \cap \Gamma_{\delta_1}) \subseteq S_\mu$, since all remaining selections in the construction prohibit the choice of more points of E in Σ_δ or more points of S in $\Sigma_\delta \cap \Gamma_{\delta_1}$. Thus

$$X \cap (\Sigma_\delta \cap \Gamma_{\delta_1}) \subseteq (E \cap \Sigma_\delta) \cup (S \cap (\Sigma_\delta \cap \Gamma_{\delta_1})) \subseteq E_\mu \cup S_\mu,$$

so $X \cap (\Sigma_\delta \cap \Gamma_{\delta_1})$ has cardinality less than $2^{\aleph_0}$. This shows that there are arbitrarily small neighborhoods of y in $X \cap \Sigma_\delta$ with boundaries of dimension $\leqq 0$. Since $\delta < \varepsilon$, x has arbitrarily small neighborhoods in X whose boundaries have dimension $\leqq 1$. Therefore X has dimension $\leqq 2$.

Remark 4.10. We do not know whether the construction in Proposition 4.2 (when $\mathscr{R}$ is nonempty) can ever yield a set X of dimension 2; in fact, we do not know whether any set $X = S \cup E$ as in the statement of Proposition 4.2 can have dimension greater than 1.

Remark 4.11. With minor modifications all arguments and constructions in sections 3 and 4 apply when $\mathbf{R}^m$ is replaced by any open convex subset of $\mathbf{R}^m$. In particular, there are examples of *bounded* sets S and E with the property in Corollary 4.6.

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