ON MULTIVALUED EVOLUTION EQUATIONS IN HILBERT SPACES

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ABSTRACT

The Cauchy problem $du/dt + Au + B(t,u) \ni 0$, $u(0) = u_0$ is studied in a separable Hilbert space setting, when A is a multivalued maximal monotone operator, and B is a multivalued operator which is measurable with respect to the time variable and upper semi-continuous with respect to the space variable. Under some boundedness conditions on B, an existence theorem is proved, with the extra assumption, in the infinite dimensional case that A is the subdifferential of a proper lower semi-continuous inf-compact convex function. A theorem of dependence upon the initial condition is also given.

Given a maximal monotone operator A and a multivalued upper semi-continuous operator B of a Hilbert space H, we give sufficient conditions for the existence of solutions of the Cauchy problem:

$$du/dt + Au + Bu \ni f;\ u(0) = u_0$$

where f is in some $L^p(0, T; H)$.

We use standard results on the solutions of evolution equations associated with monotone operators in Hilbert spaces, particularly recent results of Ph. Benilan and H. Brezis (see [1] and [5]) and obtain results closely related to those of A. Lasota and Z. Opial [18], Ch. Castaing and M. Valadier [11], and M. Valadier [23]. These results are related, when A is the subdifferential of a l.s.c. convex function on H, to some equations of econometrics (see C. Henry, [13] and [14]). The first section gives preliminary results and definitions; sections two and three deal with the finite dimensional case when B is, respectively, single-valued and multi-valued; in section four we consider the case of a separable Hilbert space

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with some examples of application.

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I. Preliminary results and definitions

Let H be a real Hilbert space, in which $|\cdot|$ and $(\cdot,\cdot)$ are the norm and scalar product.

1) Recall that a maximal monotone operator A on H is a multi-valued application from H into H satisfying

$$\forall [x_i, y_i] \in A \ (i = 1,2) \qquad (y_1 - y_2, x_1 - x_2) \geqq 0$$

and

$$R(I + A) = H$$

We set $D(A) = \{x \in H;\ Ax \neq \varnothing\}$ (Domain of A). It is known that $\overline{D(A)}$ is convex.

(1.1) Definition. Given f in $L^1\ (0, T; H)$, u is a strong solution of $du/dt + Au \ni f$ whenever u is in $C([0, T], H)$, u is absolutely continuous on every compact subset of $(0, T)$ (hence almost everywhere differentiable) and such that a.e. on $(0, T)$:

$$u(t) \in D(A) \text{ and } du/dt(t) + Au(t) \ni f(t).$$

Recall the following two fundamental results:

(1.2) Theorem (*Benilan-Brezis, see* [1] *and* [5]). *Let H be finite dimensional, A maximal monotone, f in $L^1(0, T; H)$, u_0 in $\overline{D(A)}$. There exists a unique strong solution u to the equation $du/dt + Au \ni f$ with $u(0) = u_0$. Furthermore*

*α) At every Lebesgue point t of f, u has a derivative from the right d^+u/dt, $u(t)$ belongs to $D(A)$, and $d^+u(t)/dt = (f(t) - Au(t))^0$.**

β) The following inequalities hold: If f (resp. g) is in $L^1\ (0, T; H)$ and u(resp. v) is a corresponding strong solution, we have

$$\text{(i)} \qquad \bigvee_{[0,T]} u = \left|\frac{du}{dt}\right|_{L^1} \leqq C[(1 + T + |f|_{L^1})(1 + |u|_{L^\infty}) + |u(0)|^2]$$

where C is a constant depending only upon A ($\bigvee_{[0,T]}$ is the total variation on $[0, T]$).

* If C is a nonempty closed convex set in H, we denote by C° the projection of O on C. If A is maximal monotone, recall that Ax is closed convex.

(ii) $\forall\, 0 \leqq s \leqq t \leqq T$

$$|u(t) - v(t)| \leqq |u(s) - v(s)| + \int_s^t |f(\sigma) - g(\sigma)|\, d\sigma$$

(iii) *In particular,* $\forall\, [x,y] \in A$ $\forall\, 0 \leqq s \leqq t \leqq T$

$$|u(t) - x| \leqq |u(s) - x| + \int_s^t |f(\sigma) - y|\, d\sigma$$

Recall that if Φ is a l.s.c. proper convex function on H (i.e. with values in $(-\infty, +\infty]$, and $\Phi \not\equiv +\infty$) its subdifferential $\partial\Phi$ is maximal monotone (it is defined by: $z \in \partial\Phi(x) \Leftrightarrow \forall y \in H \;\; \Phi(y) - \Phi(x) \geqq (z, y - x)$).

(1.3) THEOREM (*Brezis* [5] *and* [6]): *Let H be a general real Hilbert space. Given the subdifferential A of a proper l.s.c. convex function Φ on H, f in $L^2(0,T;H)$ and u_0 in $\overline{D(A)}$, there exists a unique strong solution of $du/dt + Au \ni f$; $u(0) = u_0$. In addition*

α) (i) $\sqrt{t}\; du/dt \in L^2(0,T;H)$, $t \to \Phi(u(t))$ is absolutely continuous on every compact subset of $(0,T]$, and $|du/dt|^2 + d\Phi(u)/dt = (f, du/dt)$ a.e. on $(0,T)$.

(ii) If $u(0) \in D(\Phi)$, $\Phi \geqq 0$, then $du/dt \in L^2(0,T;H)$, $|du/dt|_{L^2}^{\frac{1}{2}} \leqq \sqrt{\Phi(u(0))} + [\int_0^T |f(t)|^2 dt]$ and $\Phi(u(t))$ is absolutely continuous on $[0,T]$.

β) If Φ is the indicator function I_c of a closed convex set C ($I_c = 0$ on C, $+\infty$ outside of C) and if $f \in L^p(0,T,H,)$ with $1 \leqq p \leqq +\infty$ then $du/dt \in L^p(0,T;H)$.
We shall denote $F_{u_0}(f) = u$, the unique solution of $du/dt + Au \in f$; $u(0) = u_0$ ($u_0 \in \overline{D(A)}$). By (1.2) (ii), F_{u_0} is continuous from $L^1(0,T;H)$ into $\mathscr{C}([0,T];H)$.

2) We recall the following definition (see [2]).

DEFINITION. Let X and Y be two topological spaces. A multi-valued operator B from X into Y is said to be upper semi-continuous (u.s.c.) when

— $\forall x \in X$, Bx is a compact subset of Y;

— $\forall x \in X$, for every neighborhood V in Y of the subset Bx of Y, there is a neighborhood U of x in X, such that

$$y \in U \Rightarrow By \subset V.$$

The domain of B is $D(B) = \{x \in X; Bx \neq \varnothing\}$. Recall that if $R(B)$ is compact Hausdorff, B is u.s.c. if and only if B is closed as a subset of $X \times Y$ (closed graph property).

3) *A few notations.* Let I be an interval of the type $[0,T]$ ($T < +\infty$) or $[0, +\infty)$. It will often be referred to as the time set. As usual, $L^p_{loc}(I;H)$ (resp. $L^p_{lcc}(I)$) denotes the space of H-valued (resp. $\mathbb{R}$-valued) measurable functions on I

such that their restriction to every compact subset of I is of pth power integrable (if $p = +\infty$, then essentially bounded). We shall denote by $w - L^p(I; H)$, the space $L^p(I; H)$ with its weak topology for finite p and with its weak-* topology for $p = +\infty$ (i.e., for the duality with $L^1(I; H)$).

We shall say that $B(t,x)$ is a time dependent multi-valued operator on H whenever for (almost) all t in I, $B(t, \cdot)$ is a multi-valued operator of H. Let us end this section with the following definition:

Definition. Let A be maximal monotone on H, B be a time-dependent multi-valued operator on H, u_0 belong to $\overline{D(A)}$. A function u is a solution of the initial value problem.

(P): $$\frac{du}{dt} + Au + B(\cdot, u) \ni 0; \qquad u(0) = u_0$$

if and only if

— u is in $\mathscr{C}(I; H)$ and $u(0) = u_0$

— u is absolutely continuous on every compact subset of interior of I (hence almost everywhere differentiable)

— for almost all t in I, the following holds:

$$u(t) \in D(A); \ \frac{du}{dt}(t) + Au(t) + B(t, u(t)) \ni 0.$$

II. Case of B single-valued continuous

We assume in this section that H is *finite dimensional*. We shall prove the following:

(2.1) Theorem. *Let A be maximal monotone on the finite dimensional Hilbert space H. Let B be a measurable mapping from $I \times \overline{D(A)}$ into H, which for almost all t in I is continuous on $D(A)$ and such that there exist two functions γ and δ in $L^1_{loc}(I)$ with*

$$|B(t,x)| \leqq \gamma(t)|x| + \delta(t)$$

for all x in $\overline{D(A)}$ and almost all t in I. Then there exists at least one solution u for (P). Furthermore, for almost all t in I, u is right-differentiable and $d^+u/dt = -(B(t,u(t)) + Au(t))^0$. If B is continuous in both t and x, then for every t of interior of I, u is right-differentiable and $d^+u/dt = -(B(t,u(t)) + Au(t))^0$.

Proof. The proof is in three parts.

1) We first assume $I = [0, T]$ and $\gamma \equiv 0$. Put

$$G: L^1(0, T; H) \to \mathscr{C}([0, T]; H) \text{ with } G(u) = F_{u_0}(-B(\cdot, u(\cdot))).$$

G is defined on $E = [u \in L^1(0, T; H); u(t) \in \overline{D(A)}$ a.e. in $t\}$. We shall prove that G has a fixed point (which obviously will be a solution of (P)).

(2.2) Proposition. *G is continuous from E (with L^1 topology) into $\mathscr{C}([0, T]; H)$ and its range is included in a compact convex set of E.*

Proof of (2.2). Let $u_n \to u$ in E, $v_n = G(u_n)$, $v = G(u)$. From (1.2) (ii) we have $|v_n - v|_\infty \leqq |B(\cdot, u_n) - B(\cdot, u)|_{L^1}$.

Let u_ν be a subsequence of u_n such that u_ν converges to u almost everywhere on $(0, T)$.

$B(\cdot, u_\nu)$ converges to $B(\cdot, u)$ almost everywhere. Since B is dominated by δ which is in $L^1(0, T)$, we have, by Lebesgue's theorem, that $B(\cdot, u_\nu)$ converges to $B(\cdot, u)$ in $L^1(0, T; H)$ so that $|v_\nu - v|_\infty$ converges to 0. This implies that the full sequence v_n converges to v in $\mathscr{C}([0, T]; H)$.

We use the following lemma for the result concerning the range of G.

(2.3) Lemma. *F_{u_0} is a compact operator from $L^1(0, T; H)$ into $L^p(0, T; H)$ for $1 \leqq p < +\infty$ (i.e. the images by F_{u_0} of bounded sets are conditionally compact sets) (in the case dim. $H < \infty$).*

Proof of the Lemma. Let $S_M = \{f \in L^1(0, T; H); |f|_{L^1} \leqq M\}$. By (1.2) iii) with $s = 0$, we find a constant $C_1(M)$ such that for all f in S_M) $|F_{u_0}(f)|_\infty \leqq C_1(M)$. By (1.2) i) we find a constant $C_2(M)$ such that for all f in S_M $\bigvee_{[0,T]}(F_{u_0}(f)) \leqq C_2(M)$. Then $F_{u_0}(S_M) \subset E(M)$ where

$$E(M) = \{u \in \mathscr{C}(0, T; H): |u|_\infty \leqq C_1(M), \quad \bigvee_{[0,T]}(u) \leqq C_2(M), \ u(t) \in \overline{D(A)}$$

for all t in $[0, T]\}$. By Frechet-Kolmogorov's theorem (see [24], pp. 275–277), $\overline{E(M)}$ is compact in $L^p(0, T; H)$ $(1 \leqq p < +\infty)$ and is convex because $\overline{D(A)}$ is convex. ($\overline{E(M)}$ denotes the closure of $E(M)$ in $L^1(0, T; H)$.)

We now return to the proof of Theorem (2.1). Let us consider $\overline{E(M)}$ with $M = |\delta|_{L^1(0,T)}$. It is a compact convex subset of $L^1(0, T; H)$ and G maps $\overline{E(M)}$ into $E(M)$. By Schauder's fixed point theorem, G has a fixed point u in $E(M) \subset \mathscr{C}([0, T]; H)$.

Since $t \to B(t, u(t))$ is in $L^1(0, T; H)$, then by Theorem (1.2), $d^+u/dt(t) + (Au(t) + B(t, u(t)))^0 = 0$ almost everywhere in $(0, T)$. If B is continuous in both t and x, then $t \to B(t, u(t))$ is continuous and $d^+u/dt + (Au + B(t,u(t))^0 = 0$ for all t in $(0, T)$. This completes the proof in the case $I = [0, T]$, $\gamma \equiv 0$.

2) We now show the existence of local solutions in the general case, using the following result due to R. T. Rockafellar (see [20] for the proof).

(2.4) THEOREM. *If A_1 and A_2 are two maximal monotone operators on a general real Hilbert space H and if* $(\operatorname{int} D(A_1)) \cap D(A_2)$ *is not empty, then $A_1 + A_2$ is again maximal monotone.*

Let V be a bounded closed convex neighborhood of u_0 in H, let ψ_V be the indicator function of V; then since $(\operatorname{int} V) \cap D(A) \neq \varnothing$, $A + \partial\psi_V$ is maximal monotone. We use part 1 of the proof of Theorem (2.1) to get a solution u for the problem

$$\frac{du}{dt} + (A + \partial\psi_v)u + B(\cdot, u) \ni 0; \qquad u(0) = u_0$$

on any compact interval $[0, T_0]$ of I. Indeed for any y in $\overline{D(A)} \cap V$, we have $|B(t, y)|$ bounded by $\gamma(t) \cdot \sup\{|x|\,; x \in V\} + \delta(t)$ which is in $L^1(0, T_0)$.

Since u is continuous there is a T_1 with $0 < T_1 \leqq T_0$ such that for every t in $[0, T_1)$, $u(t)$ belongs to $\operatorname{int} V$; but then we have $\partial\psi v\, u(t) = \{0\}$ for $t \in [0, T_1)$. Hence u is solution of

$$\frac{du}{dt} + Au + B(\cdot, u) \in 0; \qquad u(0) = u_0 \text{ on } [0, T_1).$$

3) We now prove that a maximal solution of (P) is everywhere defined on I.

Let u be a maximal solution of (P), let $[0, T_1)$ be its domain; assume T_1 is finite. We shall show that $\lim_{t \uparrow T_1} u(t)$ exists; since this limit will be in $\overline{D(A)}$ it will be possible to extend u locally to the right of T_1 by using step two of the proof, thus getting a contradiction.

Put $\beta(t) = -B(t, u(t))$; u is solution on $[0, T_1)$ of $du/dt + Au \ni \beta$; $u(0) = u_0$

(2.5) $\qquad |\beta(t) \leqq \gamma(t)|u(t)| + \delta(t)$ a.e. with γ and δ in $L^1(0, T_1)$.

Using estimate (1.2) iii) we get for any $[x, y]$ in A:

$$\begin{aligned}
|u(t) - x| &\leqq |u_0 - x| + \int_0^t (|y| + |\beta(\sigma)|)d\sigma \\
&\leqq |u_0 - x| + \int_0^t (|y| + \delta(\sigma))d\sigma + \int_0^t \gamma(\sigma)|u(\sigma)|\,d\sigma \\
&\leqq |u_0 - x| + \int_0^t (|y| + \delta(\sigma) + |x|\gamma(\sigma))d\sigma + \int_0^t \gamma(\sigma)|u(\sigma) - x|\,d\sigma
\end{aligned}$$

Clearly $|u_0 - x| + \int_0^t(|y| + \delta(\sigma) + |x|\gamma(\sigma))d\sigma$ is bounded when t tends to T_1, so $|u(t) - x| \leqq k + \int_0^t\gamma(\sigma)|u(\sigma) - x|d\sigma$. This classically implies $|u(t) - x| \leqq k + \int_0^t\gamma(\sigma)|u(\sigma) - x|\,d\sigma \leqq k \exp\left(\int_0^t\gamma(\sigma)d\sigma\right)$. Hence $|u(t)|$ is bounded when $t \uparrow T_1$. By (2.5) β belongs to $L^1(0, T_1; H)$ so that by estimate (1.2) i):

$$\bigvee_{[0,t]}u \leqq C[(1 + T_1 + |\beta|_{L^1(0,T_1;H)})(1 + \sup_{[0,T_1)}|u|) + |u_0|^2] \ \forall t \in [0, T_1].$$

Thus u is of bounded variation as $t \uparrow T_1$ so that $\limsup_{s,t\uparrow T_1}|u(s) - u(t)| = 0$. Finally, this shows that $\lim_{t\uparrow T_1}u(t)$ exists, which completes the proof of (2.1).

Using theorem (1.3), we get the following regularity result:

(2.6) THEOREM. *Under the hypotheses of Theorem* (2.1), *and if A is the subdifferential of a proper l.s.c. convex function Φ, γ and δ are in $L^2_{loc}(I)$, then the derivative of the solution u of* (P) *satisfies*:

$$\sqrt{t}\,\frac{du}{dt} \in L^2_{loc}(I; H) \left(\textit{resp. } \frac{du}{dt} \in L^2_{loc}(I; H) \textit{ when } u_0 \in D(\Phi)\right).$$

Furthermore, if Φ is the indicator function Ψ_c of a closed convex set C of H, γ and δ are in $L^p_{loc}(I)$, $(1 \leqq p \leqq +\infty)$, then the derivative of u is in $L^p_{loc}(I; H)$.

REMARK. If in the hypotheses of Theorem (2.1), γ and δ are in $L^1(0, +\infty)$ and $\operatorname{Int} A^{-1}(0) \neq \emptyset$ then by the same proof as in part 3, one can prove that $u(t)$ has a limit as t tends to infinity.

III. Case of B multi-valued upper semi-continuous

We first give the following definition:

(3.1) DEFINITION. A multi-valued mapping B from $I \times \overline{D(A)}$ into H will be said to satisfy condition R_p whenever

(a) for almost all t of I, $B(t, \cdot)$ is multi-valued upper semi-continuous defined on $\overline{D(A)}$ with *non-empty* convex compact values in H.

(b) for all ξ in H, and all X in $\overline{D(A)}$, the function $b_{x,\xi}: t \to \sup\{(y, \xi); y \in B(t, x)\}$ is measurable on I.

(c) there exist two functions γ and δ in $L^p_{loc}(I)$ such that for almost all t in I, and for all x of $\overline{D(A)}$, the following holds: $\sup_{y\in B(t,x)}|y| \leqq \gamma(t)|x| + \delta(t)$.

We recall (see C. Castaing [10], corollary 6.1) that condition (b), when H is separable, is equivalent to:

(b′) For all x in $\overline{D(A)}$, the mapping $t \to B(t, x)$ is multi-valued measurable in the following sense: for every closed set F of H, the set $E_x = [t \in I; B(t, x) \cap F \neq \emptyset\}$ is measurable in I.

We now prove the following:

(3.2) THEOREM. *Let H be a finite dimensional Hilbert space. If B satisfies condition R_1, then problem (P) $(du/dt + Au + B(\cdot, u) \in 0;\ u(0) = u_0 \in \overline{D(A)})$ has at least one solution u on I. More precisely:*

i) *there exists a measurable section $\beta: I \to H$ such that $\beta(t) \in B(t, u(t))$ almost everywhere on I,*

ii) *u is the strong solution of $du/dt + Au + \beta \ni 0$; $u(0) = u_0$.*

REMARK. Using ii) above and Theorem (1.3), one obtains regularity results similar to those of (2.6).

The proof of Theorem (3.2) is, like that of Theorem (2.1), in three steps. We leave it to the reader to complete the last two steps. Here is a proof of the first step, i.e., $I = [0, T]$ and $\gamma \equiv 0$.

In order to use a fixed-point method in a functional framework, we introduce the following multi-valued operator:

(3.3) DEFINITION. $\mathbb{B}_p$ is defined by its graph in the following manner $\mathbb{B}_p = \{[u, v] \in (L^p(0, T; H))^2$; almost everywhere on $(0, T)$: $u(t) \in \overline{D(A)}$ and $v(t) \in B(t, u(t))\}$

(3.4) PROPOSITION. *$\mathbb{B}_p$ is demi-closed in $L^p(0, T; H)$ (i.e., its graph is closed in $L^p \times w - L^p$) for $1 \leqq p \leqq +\infty$ when H is separable.*

PROOF OF (3.4). By condition $R_p c)$, $\mathbb{B}_p$ takes its values in the set $X^p_\delta = \{f \in L^p(0, T; H); |f(t)| \leqq \delta(t)$ almost everywhere$\}$.

It is clear that X^p_δ is bounded closed convex in $L^p(0, T; H)$ so that for $p \neq 1$ it is compact in $w - L^p(0, T; H)$. For $p = 1$, if H is finite dimensional, applying the Dunford-Pettis criterion of weak conditional compactness in $L^1(0, T)$ (see [12], p. 292) we find that X^1_δ is still weakly compact.† Since $L^p(0, T; H)$ is separable (for $p \neq +\infty$), the weak topology on the weakly compact set X^p_δ of $L^p(0, T; H)$ is metrizable (see [12], p. 434). For $p = +\infty$, it is clear that the weak-* topology on the weak-* compact set X_δ of $L^\infty(0, T; H)$ is metrizable since $L^1(0, T; H)$ is separable. Thus, it is enough to show the demi-closedness of $\mathbb{B}_p$ on sequences. The result for $p = +\infty$ is a consequence of the result for p finite that we now show.

† This is still true when H is not finite-dimensional; see C. Castaing, Theorem 3 of *Proximité et mesurabilité, un théorème de compacité faible*, Colloque sur l'optimisation, Bruxelles 1969.

Let $u_n \xrightarrow{L^p} u$, $v_n \in \mathbb{B}_p u_n$ and $v_n \xrightarrow{w-L^p} v$. We can assume without loss of generality that u_n converges almost everywhere on $(0,T)$ to u. Since v_n converges weakly to v in L^p, for any integer m, we can find g_m, a finite convex combination of the v_n's with $n \geqq m$ and such that $|g_m - v|_{L^p} \leqq 1/m$ (use the weakly convergent sequence $(v_{n+m})_{n \in N}$). The sequence g_n so defined converges strongly to v in $L^p(0,T;H)$, so that there exists a subsequence g_{n_k} which converges almost everywhere on $(0,T)$ to v. Thus on a set E, whose complement in $(0,T)$ is a null set, we have for all t in E:

$$u_n(t) \underset{n\to\infty}{\longrightarrow} u(t), \quad g_{n_k}(t) \underset{k\to\infty}{\longrightarrow} v(t), \quad u_n(t) \in \overline{D(A)}, \quad v_n(t) \in B(t,u_n(t)),$$

$x \to B(t,x)$ is upper semi-continuous.

Fixing t in E, we shall show that $v(t)$ belongs to $B(t,u(t))$; this will complete the proof of (3.4). Since $B(t,\cdot)$ is u.s.c., for every neighborhood V of $B(t,u(t))$ there is a neighborhood U of $u(t)$ such that for all x in U, $B(t,x) \subset V$. Since $u_n(t)$ converges to $u(t)$ there exists an N such that $n \geqq N$ implies $v_n(t) \in V$; thus $g_n(t)$ belongs to the convex hull Conv V. Hence $\lim g_{n_k}(t)$ belongs to $\overline{\text{Conv}\, V}$ for every neighborhood V of $B(t,u(t))$. The latter being convex compact is the intersection of its closed convex neighborhoods so that $v(t) \in B(t,u(t))$.

The question of whether $\mathbb{B}_p$ is non-empty is answered by the following

(3.5) Proposition. *$\mathbb{B}_p$ is an upper semi-continuous multi-valued operator with convex compact values from $L^p(0\ T;H)$ into $w - L^p(0,T;H)$ for $1 \leqq p \leqq +\infty$ when H is separable. Furthermore, $\mathbb{B}_p u$ is nonempty whenever u is in $L^p(0,T;H)$ and $u(t)$ in $\overline{D(A)}$ a.e.*

Proof of (3.5). It is clear that: a) $\mathbb{B}_p$ is convex-valued since $B(t,x)$ is so for almost all t. b) $\mathbb{B}_p$ is weakly-conditionally-compact-valued since it takes values in X^p_δ. By (3.4) it is, in fact, weakly closed-valued so that it is weakly-compact-valued.

Since the graph of $\mathbb{B}_p$ is closed in $L^p \times w - L^p$ by (3.4) we conclude by (1.4) that $\mathbb{B}_p$ is upper semi-continuous from $L^p(0,T;H)$ into $w - L^p(0,T;H)$.

Let u be a measurable step function on $[0,T]$ with values $x_1,\cdots,x_n$ distinct in $\overline{D(A)}$. Consider the multi-valued mapping Γ: $t \to \Gamma(t) = B(t,u(t))$ defined (almost everywhere) on $(0,T)$. We show that Γ is multi-valued measurable. Let E be a closed set of H, by condition R_p b'), the set $E_i = \{t \in (0,T);\, B(t,x_i) \cap E \neq \varnothing\}$ is measurable. This is also true of $E_i \cap u^{-1}(x_i)$ and of

$$\bigcup_{i=1}^{n} E_i \cap u^{-1}(x_i) = \{t \in (0,T)\colon B(t,u(t)) \cap E \neq \varnothing\},$$

so that Γ is measurable. By theorem 5.1 of [10] due to C. Castaing, $\Gamma(t)$ has a measurable section, which being dominated by δ in $L^p(0,T;H)$ is in $\mathbb{B}_p u$.

Now let u be in $L^p(0,T;H)$ with values in $\overline{D(A)}$ (for almost all t). Let u_n be a sequence of step functions with values in $\overline{D(A)}$ converging to u in $L^p(0,T;H)$ (for p finite). Let v_n be in $\mathbb{B}_p u_n$ (we have just shown that such v_n's exist); since v_n is in X^p_δ which is weakly compact, (v_n) has weak cluster points in X^p_δ as n tends to infinity; by the demi-closedness of $\mathbb{B}_p$, any such weak cluster point is in $\mathbb{B}_p u$ which, therefore, is not empty. The previous result for p finite obviously implies that (3.5) holds also for $p = +\infty$.

We now turn to some properties of the operator F_{u_0} when H is finite dimensional.

(3.6) Proposition. *Let A be maximal monotone on H finite dimensional, u_0 in $\overline{D(A)}$, and p in $[1,+\infty)$, then F_{u_0} is continuous from X^p_δ (with the $w-L^p(0,T;H)$ topology) to $L^q(0,T;H)$ for all q in $[1,+\infty)$*

The following proof of (3.6) stems from an idea of P. Benilan. It is enough to show the result when $p = 1$. We can always assume that $\delta \geqq 1$ on $(0,T)$.

Let $f_n \xrightarrow{w-L^1} f$, $u_n = F_{u_0} f_n$, $u = F_{u_0} f$. Fix $r > 1$.

Put $g_n = f_n\, \delta^{(1/r)-1}$ and $g = f\,\delta^{(1/r)-1}$; g_n and g belong to $L^r(0,T;H)$ and are bounded above by $\delta^{1/r}$ which belongs to $L^r(0,T)$.

Put $v_n = F_{u_0} g_n$, $v = F_{u_0} g$. By Theorem 1.2.iii) the u_n's are uniformely bounded on $[0,T]$, therefore their convergence to u in any $L^q(0,T;H)$ will be implied by their convergence almost everywhere. We shall show that, in fact, $u_n(t)$ converges to $u(t)$ for all t in $[0,T]$. Indeed, we have

$$|u_n(t) - u(t)| \leqq |u_n(t) - v_n(t)| + |v_n(t) - v(t)| + |v(t) - u(t)|.$$

By (1.2), ii), one gets

$$|u_n(t) - v_n(t)| \leqq \int_0^t |f_n - g_n|\,d\sigma \leqq \int_0^t \delta(1 - \delta^{(1/r)-1})\,d\sigma \leqq \int_0^T \delta(1 - \delta^{(1/r)-1})\,d\sigma$$

and also

$$|u(t) - v(t)| \leqq \int_0^T \delta(1 - \delta^{(1/r)-1})\,d\sigma.$$

Given a positive ε, one can find $r > 1$ such that

$$\int_0^T \delta(1 - \delta^{(1/r)-1})\,d\sigma < \varepsilon \text{ (by Lebesgue's theorem).}$$

With such an r, we have $|u_n(t) - u(t)| \leqq 2\varepsilon + |v_n(t) - v(t)|$ and we now show that $v_n(t)$ converges to $v(t)$ for all t of $[0, T]$.

Since v_n and v are locally absolutely continuous, $g_n - dv_n/dt \in A\, v_n(t)$ for almost all t in $[0, T]$. The same holds between g and v. Applying the monotonicity of A, one obtains

$$\frac{1}{2}|v_n(t) - v(t)|^2 \leqq \int_0^t (g_n(\sigma) - g(\sigma), v_n(\sigma) - v(\sigma))d\sigma. \tag{3.7}$$

Put $w_n = v_n - v$. By Lemma (2.3), $\{w_n\}$ is conditionally compact in any $L^s(0, T; H)$, $1 \leqq s < +\infty$, in particular for $1/s + 1/r = 1$. Hence there is a subsequence w_{n_k} converging to a w in $L^s(0, T; H)$ and with w in $L^\infty(0, T; H)$ (since the w_n's are uniformly bounded in $L^\infty(0, T; H)$). We then get

$$\frac{1}{2}|v_n(t) - v(t)|^2 \leqq \int_0^t (g_n(\sigma) - g(\sigma), w_n - w)d\sigma + \int_0^t (f_n(\sigma) - f(\sigma), \delta(\sigma)^{(1/r)-1} w(\sigma))d\sigma$$

$$\leqq \|g_n - g\|_{L^r} \|w_n - w\|_{L^s} + \int_0^t (f_n(\sigma) - f(\sigma), \delta(\sigma)^{(1/r)-1} w(\sigma))d\sigma$$

Therefore, for all t in $[0, T]$,

$$\lim_{k \to +\infty} |v_{n_k}(t) - v(t)|^2 = 0, \text{ since } |w_{n_k} - w|_{L^s} \to 0,$$

and since

$$f_{n_k} \to f \text{ in } w - L^1(0, T; H) \text{ and } \delta^{(1/r)-1}.\, w \in L^\infty(0, T; H).$$

Since $\lim_{k\to+\infty} |w_{n_k}(t)| = 0$ for all t in $[0, T]$, we find that $w = 0$ in $L^s(0, T; H)$. Thus w_n converges to 0 in $L^s(0, T; H)$ i.e., v_n converges to v in $L^s(0, T; H)$. Then using (3.7), we find that $v_n(t)$ converges to $v(t)$ for all t of $[0, T]$. This shows that $u_n(t)$ converges to $u(t)$ for all t in $[0, T]$.

Remark. Using a demi-closedness property for $(F_{u_0})^{-1}$ and Lemma (2.3), one can actually show that for $p > 1$, F_{u_0} is continuous from the whole of $w - L^p(0, T; H)$ to $L^q(0, T; H)$ for all q in $[1, +\infty)$.

One can give a more precise continuity result in the following case.

(3.8) Proposition. *Under the asumptions of* (3.6), *and if* $D(A)$ *is closed and* A^0 *is bounded on every compact subset of* $D(A)$†, *then* F_{u_0} *is continuous from*

X_δ^p (with the $w - L^p(0, T; H)$ topology) to $\mathscr{C}([0, T]; H)$.

† This is true, in particular, of the case of the subdifferential of the indicator function of a closed convex set of H.

PROOF. As before it is enough to show this for $p = 1$. Let f_n converge to f in $w - L^1(0, T; H)$. Put $u_n = F_{u_0} f_n$ and $u = F_{u_0} f$. By (1.2), iii), where $x = u_n(s)$, $y = A^0 u_n(s)$ (which exists since $D(A)$ is closed) one obtains

$$|(u_n(t) - u_n(s)| \leqq \int_s^t |f_n(\sigma) - A^0 u_n(s)| d\sigma \text{ for } 0 \leqq s \leqq t \leqq T.$$

Since the set $\{u_n(t); t \in [0, T], n \in N\}$ is bounded in $\overline{D(A)}$, it is conditionally compact. Thus there exists an M such that for all $t \in [0, T]$, for all $n \in N$, $|A^0 u_n(t)| \leqq M$. Therefore $|u_n(t) - u_n(s)| \leqq \int_s^t \delta(\sigma) d\sigma + M(t - s)$ for $0 \leqq s \leqq t \leqq T$. By Ascoli's theorem, the family $\{u_n\}$ is conditionally compact in $\mathscr{C}([0, T]; H)$. This, together with Proposition 3.6 implies that the sequence u_n converges to u in $\mathscr{C}([0, T]; H)$.

PROOF OF THEOREM (3.2). (Recall that $\gamma \equiv 0$ and $I = [0, T]$.) To solve $du/dt + Au + B(\cdot, u) \ni 0$, $u(0) = u_0$, we interpret the problem as follows: There exists β in $\mathbb{B}_1(u)$ such that $u = F_{u_0}(-\beta)$. The classical Kakutani, Ky-Fan, Tychonof fixed point theorem for multivalued u.s.c. mappings does not apply to the equation $u \in F_{u_0}(-\mathbb{B}_1 u)$ (it is not convex valued), but as was noticed by F. Browder, it does apply to the equation $\beta \in \mathbb{B}_1(F_{u_0}(-\beta))$.

Using (3.5) and (3.6), we find that $\beta \to \mathbb{B}_1(F_{u_0}(-\beta))$ is u.s.c. from X_δ into itself (X_δ with the $w - L^1$ topology), and nonempty convex compact valued. Thus, it has a fixed point β which, together with $u = F_{u_0}(-\beta)$, satisfies the conclusions of Theorem (3.2).

IV. Infinite dimensional case

In this section H will be a separable real Hilbert space and ϕ a proper convex l.s.c. function on H; $A = \partial\phi$.

We shall prove the following

(4.1) THEOREM. *Let H be a separable real Hilbert space, ϕ be a proper l.s.c. convex on H such that for all real M the set $C(M) = \{x \in H; |x| \leqq M, \phi(x) \leqq M\}$ is (convex) compact in H*†. *Also let B be a time dependent multivalued operator on H satisfying condition R_2 ($p = 2$) (cf (3.1)).*

Then the problem (P) has a solution u on I. More precisely:

† This is clearly equivalent to: for all $M \in \mathbb{R}$ $\{X \in H; \phi(x) + |x|^2 \leqq M\}$ is convex compact in H.

i) There exists a measurable section $\beta: I \to H$ such that $\beta(t) \in B(t, u(t))$ almost everywhere on I.

ii) u is the strong solution of $du/dt + \partial\phi(u) + \beta \ni 0$, $u(0) = u_0$. ($u_0 \in \overline{D(\phi)}$).

Proof. We first show that we can take ϕ bounded below. Indeed since ϕ is l.s.c. proper convex on H, it is bounded below by some affine functional $(a, x) + b$. If we replace ϕ by $\phi - (a, \cdot)$ and B by $B + a$ (the subdifferential of $\phi - (a, \cdot)$ is $\partial\phi - a$), condition R_2 is still satisfied and all we have to show is that the sets $\{|x| \leqq M, \phi(x) - (a, x) \leqq M\}$ are still compact in H. But they are closed and included in $C(M(1 + |a|))$. From here on, we assume that ϕ is bounded below on H. We use the same method as in Theorem (3.2), with $\gamma = 0$ and $I = [0, T]$. Thus all we have to show, in view of (3.5), is the following:

(4.2) Proposition. *The operator F_{u_0} is continuous from X_δ^2 (with $w - L^2$ topology) to $\mathscr{C}([0, T]; H)$.*

Proof.† We first take u_0 in $D(\phi)$.

Let $f_n \rightharpoonup f$ (in X_δ^2) and put $u_n = F_{u_0} f_n$, $u = F_{u_0} f$. By Theorem (1.3), α), ii) $|du_n/dt|_{L^2}$ is bounded uniformly in n; therefore $\{u_n\}$ is equicontinuous on $[0, T]$ and uniformly bounded. Moreover, from (1.3), α), i) we also get that $\phi(u_n(t))$ is absolutely continuous on $[0, T]$ and $(d/dt)\phi(u_n(t)) \leqq (f_n, (du_n/dt))$. Therefore, $\forall 0 \leqq t \leqq T$ $\phi(u_n(t)) \leqq \phi(u_0) + |f_n|_{L^2} |du_n/dt|_{L^2}$. Therefore, the set $\{u_n(t); t \in [0, T], n \in N\}$ is included in some $C(M)$, which is compact. By Ascoli's theorem, the family $\{u_n\}$ is conditionally compact in $\mathscr{C}([0, T]; H)$.

Let u_{n_k} converge uniformly to a cluster point v.
We have

$$\frac{1}{2}|u_{n_k}(t) - u(t)|^2 \leqq \int_0^t (f_{n_k}(\sigma) - f(\sigma), u_{n_k}(\sigma) - u(\sigma)) d\sigma.$$

Letting k go to infinity we get $u_{n_k}(t) \to u(t)$ for all t in $[0, T]$. Therefore v equals u and the whole u_n converges to u in $\mathscr{C}([0, T]; H)$.

Take now u_0 in $\overline{D(\phi)}$. Let $u_{0,m}$ be in $D(\phi)$ and converge to u_0. It is enough to show that $v_{n,m} = F_{u_{0,m}} f_n$ (resp. $v_m = F_{u_{0,m}} f$) converge uniformly in n to $u_n = F_{u_0} f_n$ (resp. $u = F_{u_0} f$), when m tends to infinity. By the monotony of $\partial\phi$ we have $|v_{n,m}(t) - u_n(t)| \leqq |v_{n,m}(0) - u_n(0)|$ for all t in $[0, T]$ (and the same for v_m and u); since $v_{n,m}(0) = v_m(0) = u_{0,m}$ and $u(0) = u_n(0) = u_0$, the uniform convergence holds.

† One can also use a compactness result; see J. L. Loins [19], pp. 141–143.

As in Theorem (3.2), the previous result implies the existence of local solutions in the general case of I, $\gamma \not\equiv 0$. To show that a maximal solution is defined on the whole of I, one uses the same technique as in (3.2) to show that u is bounded on $[0, T_0]$; then we use estimate ii) of (1.3) on $[\alpha, T_0]$ with $\alpha > 0$ (since $\phi(u(\alpha))$ is finite) to show that du/dt is in $L^2(\alpha, T_0; H)$, which implies the existence of a limit for $u(t)$ when $t \uparrow T_0$.

For applications, the following variation of Theorem (4.1) is of interest.

(4.3) Definition. A multivalued mapping B from $I \times \overline{D(A)}$ into H satisfies condition (R') when

a) for almost all t in I, the mapping $x \to B(t, x)$ is multivalued u.s.c. from $\overline{D(A)}$ (with the strong topology) to $w - H$ with convex weakly compact values ($w - H$ is H with its weak topology).

b) for all x in $\overline{D(A)}$, the mapping $t \to B(t, x)$ is multivalued measurable from I to $w - H$.

c) there exist two functions γ and δ in $L^\infty_{loc}(I)$, such that for almost all t in I, all x in $\overline{D(A)}$,

$$\sup_{y \in B(t,x)} |y| \leqq \gamma(t)|x| + \delta(t).$$

(4.4) Proposition. *Under the same assumptions on H and ϕ as in Theorem* (4.1), *and if B satisfies condition (R') (Definition* (4.3)), *the conclusions of Theorem* (4.1) *still hold.*†

Proof. All we have to show is that under condition (R') and when $I = [0, T]$ and $\gamma \equiv 0$, the operator $\mathbb{B} = \{[u, v] \in (L^2(0, T; H))^2$; almost everywhere on $[0, T]$ $u(t) \in \overline{D(A)}$ and $v(t) \in B(t, u(t))\}$ is upper semi-continuous multivalued with convex compact images from $L^2(0, T; H)$ into $w - L^2(0, T; H)$. The proof of Proposition (3.4) still holds verbatim, as well as Proposition (3.5) except for the fact that $\mathbb{B}u$ is nonempty when $u(t)$ belongs to $\overline{D(A)}$ almost everywhere.

Consider a measurable step function u on $[0, T]$ with values $x_1, \cdots, x_n$, distinct in $\overline{D(A)}$ and put $\Gamma(t) = B(t, u(t))$, defined (almost everywhere) on $(0, T)$. In fact Γ takes values in the ball of radius $|\delta|_{L^\infty(0,T)}$ of H, and the weak topology of this ball is metrisable (since H is separable). As in the proof of (3.5), it is easily seen that Γ is measurable from $(0, T)$ into that ball (with the weak topology). Hence,

† This result still holds when in condition $R'c$), one only assumes that γ and δ are in $L^2_{loc}(I)$. One can notice that this modified condition R' is weaker than R_2.

by theorem 5.1 of [10], Γ has measurable sections, which obviously are in $L^2(0,T;H)$. The proof ends as in Proposition (3.5).

(4.5) REMARK. *The multivalued mapping ψ which maps u_0 (in $\overline{D(\phi)}$) into the set of solution of problem (P) with initial condition u_0 is upper semi-continuous from $\overline{D(\phi)}$ into $w-L^2(0,T;H)$.*

PROOF. One can always restrict oneself to bounded subsets of $\overline{D(\phi)}$. It is obvious that ψ maps bounded sets of $\overline{D(\phi)}$ into bounded, hence weakly compact, sets of $L^2(0,T;H)$.

It is now enough to show that ψ has the closed graph property. If $u_{0,n} \xrightarrow{H} u_0$ and u_n, solution of $du_n/dt + \partial\phi u_n \ni \beta_n, u_n(0) = u_{0,n}$ with $\beta_n \in \mathbb{B}u_n$, converges to u in $w-L^2$, then the β_n's are bounded in $L^2(0,T;H)$. There exists a converging subsequence $\beta_{n_k} \rightharpoonup \beta$ in $w-L^2$. Since

$$
\begin{aligned}
| F_{u_0}\beta - u_n|_\infty &\leqq | F_{u_0}\beta - F_{u_0}\beta_n|_\infty + | F_{u_0}\beta_n - F_{u_{0,n}}\beta_n|_\infty \\
&\leqq | F_{u_0}\beta - F_{u_0}\beta_n|_\infty + | u_0 - u_{0,n}|
\end{aligned}
$$

by (4.2), we find that u_{n_k} converges to $F_{u_0}\beta$ in $\mathscr{C}([0,T];H)$; hence $F_{u_0}\beta = u$. On the other hand, since $\beta_{n_k} \rightharpoonup \beta$ in $w-L^2$, $u_{n_k} \to u$ in L^2. By the closed graph property of $\mathbb{B}$, we see that β belongs to $\mathbb{B}u$, so that $u = F_{u_0}\beta$ and $\beta \in \mathbb{B}u$.†

When B is single valued, one can transfer the compactness condition which was so far taken on ϕ, onto B itself, as in the following partial result noticed by P. Benilan.

(4.6) THEOREM. *Let H be a separable real Hilbert space, ϕ a proper l.s.c. convex function on H; let B be a single valued time dependent operator on $I \times \overline{D(\phi)}$ which is measurable in t on I, and continuous in x from $\overline{D(\phi)}$ (with the weak topology) into H (with the strong topology). Suppose there exist γ and δ in $L^2_{loc}(I)$ such that a.e. in t, for all x in $\overline{D(\phi)}$,*

$$| B(t,x)| \leqq \gamma(t)| x| + \delta(t).$$

Then the problem (P)

$$\frac{du}{dt} + \partial\phi u + B(\cdot,u) \ni 0;\ u(0) = u_0\ \ (u_0 \in \overline{D(\phi)})$$

has a solution.

† One can actually show that the mapping ψ is u.s.c. from $\overline{D(\phi)}$ into the space $C([0,T],H)$ with the topology of uniform convergence on every compact subset of $(0,T)$.

PROOF. As usual it is enough to show the existence of a solution on $[0,T]$ for $\gamma \equiv 0$, and when ϕ is bounded below. We use a fixed point theorem for $G: u \to F_{u_0}(-B(\cdot, u(\cdot))$. First assume $u_0 \in D(\phi)$. Put $M = T|\delta|_{L^2} + \sqrt{\phi(u_0)}$ and $K_1 = \{u \in \mathscr{C}([0,T]; w-H)$; u is absolutely continuous on every compact subset of $(0,T)$; $u(0) = u_0$, $u(t) \in \overline{D(\phi)}$ a.e. and $|du/dt|_{L^2(0,T;H)} \leqq M\}$. K_1 is convex, and by Ascoli's theorem, conditionally compact (metrisable since all its elements have their range in a bounded set of $w-H$, which is therefore metrisable) in $\mathscr{C}([0,T]; w-H)$. Thus $K = \bar{K}_1$ is convex compact in $\mathscr{C}([0,T]; w-H)$.

G is continuous on K as follows:

Let u_n converge to u in K; the set $\{u_n(t); n \in N, t \in [0,T]\}$ is bounded hence weakly conditionally compact in H so that $B(t, \cdot)$ is uniformly continuous on it. Therefore $B(t, u_n(t))$ converges for all t to $B(t, u(t))$; thus $B(t, u_n(t))$ actually converges to $B(t, u(t))$ in $L^2(0,T;H)$; this in turn implies the uniform convergence of $G(u_n)$ to $G(u)$ in $\mathscr{C}([0,T];H)$. By Schauder's fixed point theorem, there is a solution of $u = G(u)$.

If u_0 is in $\overline{D(\phi)}$ we take $u_{n,0}$ in $D(\phi)$ converging to u_0, u_n a solution of $du_n/dt + \partial\phi(u_n) + B(t, u_n) \in 0 = u_{n\,0}$. By an estimate of [5] (p. III. 20) we have for all $\alpha \in (0,T)$, for all n,

$$\left|\frac{du_n}{dt}\right|_{L^2(\alpha,T;H)} \leqq |f_n|_{L^2(0,T;H)} + \frac{1}{\sqrt{2\alpha}}\int_0^\alpha |f_n(t)|\,dt + \frac{1}{\sqrt{2\alpha}}\operatorname{dist}(u_{n,0}, K_0)$$

where $f_n = -B(\cdot, u_n)$ and $K_0 = \phi^{-1}(\min \phi)$.

We see that the family $\{u_n\}$ is relatively compact in every $\mathscr{C}([\alpha,T];H)$ for $\alpha \in (0,T)$ (by Ascoli's theorem) therefore, taking a sequence α_n tending to 0, and by a diagonal sequence method, we get a subsequence u_{n_k} which converges for all $t \in (0,T]$. This subsequence obviously converges at $t = 0$ too, so that it converges for all t to a function u. Since B is continuous and dominated in L^2, $B(\cdot, u_{n_k})$ converges to $B(\cdot, u)$ in $L^2(0,T;H)$. If $v = F_{u_0}B(\cdot, u)$, we then have $|u_{n_k} - v|_{L^\infty} \leqq |u_{nk,0} - u_0| + |B(\cdot, u_{n_k}) - B(\cdot, u)|_{L^1}$ which shows that u_n converges to v uniformly on $[0,T]$; this shows that $u = v$, and u is as required.

The previous results can be applied to some multivalued partial differential equations. Here are two examples.

We take $H = L^2(\Omega)$ where Ω is an open bounded subset of $\mathbb{R}^n$, with a smooth boundary Γ. Let j be a positive proper l.s.c. convex function on $\mathbb{R}$ such that $\partial_j(0) \neq \emptyset$. We set

$$\phi(u) = \begin{cases} \frac{1}{2}\int_\Omega |\operatorname{grad} u|^2 dx + \int_\Omega j(u)dx & \text{if } u \in H_0^1(\Omega),\ j(u) \in L^1(\Omega) \\ +\infty \text{ otherwise} \end{cases}$$

Then, (see [4])

$$\partial\phi(u) = -\Delta u + \partial j(u) \text{ with}$$

$$D(\partial\phi) = \{u \in H^2(\Omega) \cap H_0^1(\Omega) \colon \text{there exists } g \in L^2(\Omega) \text{ with } g(x) \in \partial ju(x) \text{ a.e. on } \Omega\}$$

$$\overline{D(\partial\phi)} = \overline{D(\phi)} = \{u \in L^2(\Omega);\ u(x) \in \overline{D(j)} \text{ a.e. on } \Omega\}.$$

It easy to see that, for all M, the set

$$\{u \in L^2(\Omega);\ \phi(u) + |u|_{L^2}^2 \leqq M\} \text{ is bounded}$$

in $H^1(\Omega)$ and thus compact in $L^2(\Omega)$.

1) Let $f(t, y)$ be a bounded continuous function on $[0, T] \times \overline{D(j)}$. Let u_0 be in $L^2(\Omega)$, with $u_0(x) \in \overline{D(j)}$ a.e. on Ω. Then, there exists a function u in $\mathscr{C}([0, T]; L^2(\Omega))$ with $\sqrt{t}\, du/dt \in L^2\ (0, T; L^2(\Omega))$ satisfying

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) - \Delta_x u(t,x) + \partial j(u(t,x)) \ni f(t, u(t,x)) \text{ for almost all } (t,x) \text{ in } (0,T) \times \Omega \\ u(0,x) = u_0(x) \text{ on } \Omega \\ u(t,x)_{|\Gamma} = 0 \text{ a.e. on } (0,T). \end{cases}$$

2) Let $c(t, y)$ (resp. $d(t, y)$) be continuous in t and l.s.c. (resp. u.s.c.) in y on $[0, T] \times \overline{D(j)}$ with c and d bounded and

$$c(t, y) \leqq d(t, y).$$

Let u_0 be in $L^2(\Omega)$ with values in $\overline{D(j)}$. There exists u in $\mathscr{C}([0, T]; L^2(\Omega))$ and $h \in L^\infty((0, T); L^2(\Omega))$ with

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) - \Delta_x u(t,x) + \partial j(u(t,x)) \ni h(t,x) \\ c(t, u(t,x)) \leqq h(t,x) \leqq d(t, u(t,x)) \text{ a.e. on } (0,T) \times \Omega \\ u(0,x) = u_0(x) \text{ on } \Omega \\ u(t,x)_{|\Gamma} = 0 \text{ a.e. on } (0,T). \end{cases}$$

REMARK. One can easily transpose the above examples to get a Neumann boundary condition instead of the Dirichlet one.

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