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ON GROUPS THAT ARE RESIDUALLY OF FINITE RANK

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ABSTRACT

Let r be a fixed positive integer. A group G has (Prüfer) rank r if every finitely generated subgroup of G can be generated by r elements and r is the least such integer. In this paper we consider groups that are residually of rank r. Among other things we prove that a periodic group that is residually (of rank r and locally finite) is locally finite and obtain the structure of groups that are residually (of rank r and locally soluble). A number of examples are also given to illustrate the limitations of these theorems.

1. Introduction

Let r be a fixed positive integer. A group G has (Prüfer) rank r if every finitely generated subgroup of G can be generated by r elements and r is the least such integer. Throughout this paper we shall say that a group G has rank r if, in the above sense, it has rank at most r; no confusion should arise as a result.

The property of having finite rank (i.e. rank r for some r) is fairly decisive in a number of circumstances. For example, a locally nilpotent group of finite rank is hypercentral [10; Section 6.3] and a locally soluble group of finite rank is

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hyperabelian and has some term of its derived series periodic and locally nilpotent [10; Lemma 10.39]. Also there is the famous theorem of Šunkov [13] that a locally finite group all of whose abelian subgroups have finite rank is itself of finite rank and is almost locally soluble (i.e. has a locally soluble subgroup of finite index). Although rank restrictions on abelian subgroups are not discussed in the present article, the fact that a locally finite group of finite rank is almost locally soluble is fundamental to our considerations. Another result to be mentioned is that a locally (soluble-by-finite) group of finite rank is almost locally soluble [2], [4].

Our concern here is with groups that are residually of finite rank. It is easily deduced from Šunkov's theorem that a locally finite group with this property is (locally soluble)-by-(residually finite), and somewhat more can be said about such a group by considering its locally soluble radical. However, we shall be restricting attention to groups that are residually of rank r (which we write as res(rank r)) for some finite r, our two main results on groups with this property being as follows.

THEOREM 1: *Let G be a periodic group and suppose that G is residually (of rank r and locally finite) for some positive integer r. Then G is locally finite.*

THEOREM 2: *Let G be a group that is residually (of rank r and locally soluble), for some integer r. Then there are subgroups M, N of G with $M \triangleleft N \triangleleft G$ such that*

(i) *M is hyperabelian and locally nilpotent,*
(ii) *N/M is residually (linear of r-bounded degree) and*
(iii) *G/N is soluble of r-bounded derived length.*

By "r-bounded" we mean "bounded in terms of r only". This is a convenient point at which to state the following result from [12], to the first part of which Theorem 2 bears some resemblance.

THEOREM 3 (Segal [12]): *Let G be a finitely generated group that is* res*(finite soluble of rank r). Then G has a normal nilpotent subgroup Q such that G/Q is a subdirect product of finitely many linear groups over fields. If, moreover, every finite quotient of G is soluble then G is almost nilpotent-by-abelian.*

As indicated in [12], Theorem 3 allows one to deduce results about groups that are residually (finite of rank r) without appealing to the theory of analytic pro-p groups [3]. Theorem 2 is, in part, proved by adopting a similar approach. In the case where G is itself locally soluble and res(rank r) we obtain easily from Theorem 2 and Zassenhaus' theorem on soluble linear groups [10; Theorem 3.23] that G is locally nilpotent-by-(soluble of r-bounded derived length). That such a group G is hyperabelian is already known from work of Amberg and Sysak [1]. The following result shows that a little more can be said. We recall that a group G is called **radical** if it is the terminus of an ascending normal series with locally nilpotent factors.

THEOREM 4: *Let G be a group that is residually of rank r. The following are equivalent.*

(i) *G is locally soluble.*
(ii) *G is hyperabelian.*
(iii) *G is radical.*

It is easy to find a locally finite group that is res(rank r) but not almost locally soluble, for example the direct product of infinitely many copies of some finite non-abelian simple group has this property. In general a locally finite group that is res(rank r) need not be hyperfinite, even in the soluble case, and so there is no obvious analogue of the theorem of Amberg and Sysak mentioned above.

THEOREM 5: *Let p and q be distinct primes and let $G = P \wr Q$, where P is of order p and Q is an infinite elementary abelian q-group. Then G is* res*(rank q) (but has no non-trivial finite normal subgroups).*

Again with reference to Theorem 1, certainly there exist periodic residually finite groups that are not locally finite, for instance the example of Grigorčuk [5] has this property. The following well-known result is also relevant here.

THEOREM 6: *Let p be a fixed prime, F a countable free group. Then F has a descending normal series of subgroups N_i such that F/N_i is a finite p-group of rank 9 for all i and $\bigcap_{i=1}^{\infty} N_i = 1$.*

Theorem 6 suggests the following definition: for a class (or property) $\mathcal{P}$ of groups, let us say that a group G is res*($\mathcal{P}$) if it has a countable descending series of normal subgroups N_i intersecting in the identity such that each G/N_i is a $\mathcal{P}$-group. Thus a countable free group is res*(finite of rank 9). Clearly a locally finite group that is res*(rank r) is of rank r and hence almost locally soluble, and one avenue of investigation is that of attempting to determine for which other groups G does the presence of the property res*(rank r) imply that G is almost locally soluble. We prove the following.

THEOREM 7: *Let r be a fixed positive integer and let G be a group in which every finitely generated subgroup is soluble-by-finite and* res*(rank r). Then G is almost locally soluble.*

In particular, therefore, a locally (soluble-by-finite) group G that is res*(rank r) is almost locally soluble and we have a partial generalization of the theorem of Černikov referred to above (see also [4]). Such a group G need not, however, be of finite rank, even in the case where G is finitely generated and soluble.

THEOREM 8: *The group $G = \mathbb{Z} \wr \mathbb{Z}$ is* res* *(rank 2) (but of infinite rank).*

It is easy to see that a group is residually of rank 1 if and only if it is abelian; however, if a group G is res*(rank 1) then it is a subcartesian product of countably many locally cyclic groups and therefore of cardinality at most $2^{\aleph_0}$. A res*(rank 1) group of precisely this cardinality is of course the group of p-adic integers. In contrast, the direct square of a nontrivial finite abelian group cannot be res*(rank 1). There does not seem to be a satisfactory theory in the case of (locally) nilpotent groups. An easy example of a nilpotent group that is not even residually of finite rank is the central product of infinitely many copies of a non-abelian group H of order p^3 (p a prime). If one replaces H here with a free nilpotent group of class two on two generators then it is easily shown that the resulting group again fails to be residually of finite rank. Since finiteness of rank exerts such a strong influence in the theory of locally nilpotent groups, particularly in the torsionfree case, where it is known that a torsionfree locally nilpotent group of rank r is nilpotent of bounded rank [10; Lemma 6.37], it is reasonable to ask whether there are comparable results for the property res*(rank r). That this is not so is indicated by our next result.

THEOREM 9:

(i) *For each positive integer n there exists a 2-generator, metabelian, torsionfree nilpotent group G that is* res* *(rank 2), nilpotent of class exactly n and of rank exactly $n+1$.*
(ii) *There exists a metabelian, locally nilpotent, torsionfree group G that is* res* *(rank 2) but neither nilpotent nor of finite rank.*

The group G of part (ii) of the above theorem is hypercentral. With some further effort we are able to establish the last of our main results concerning the existence of examples, namely the following, which shows even more clearly that the property res*(rank r) is far from being decisive in this area.

THEOREM 10: *There exists a locally nilpotent group C that is* res* *(rank 9) but not hypercentral (and not even hypercentral-by-soluble).*

In view of the existence of all these examples which, it is hoped, are of interest in their own right, it is nice to be able to conclude this introductory section with a pair of results of a "positive" nature.

THEOREM 11: *Suppose that the group G is* res*(*finite p and of rank r*), *where p is a fixed prime. Then there exist integers c, d that depend only on r such that*

(i) *if G is locally soluble then G is soluble and $G^{(d)}$ is a finite p-group of rank r, and*

(ii) *if G is locally nilpotent then G is nilpotent and $\gamma_c(G)$ is a finite p-group of rank r.*

2. Proofs

Firstly, it is easy to see that the residual properties with which we are concerned are all inherited by subgroups; this fact is used without further mention. Suppose that F is a finite semi-simple group of rank r and let N denote the socle of F. By the Feit–Thompson Theorem each (non-abelian) simple direct factor of N contains a 2-element and so N is the direct product of at most r such factors. Further, F/N embeds in $\operatorname{Aut} N$ and, by considering more closely the structure of N and using the fact that the outer automorphism group of a finite non-abelian simple group is soluble (the Schreier Conjecture), we deduce the following (see [4; Lemma 2.2] for the details of the argument sketched here).

LEMMA 1: *Let F be a finite semi-simple group of rank r and N the socle of F. Then there is a normal subgroup E of index at most $(r!)^r$ in F such that $N \leq E$ and E/N is soluble.*

This lemma has two interesting corollaries. For the first of these we need only note that every finite group is semi-simple modulo its soluble radical.

COROLLARY 1: *Let G be a finite group of rank r. Then the number of non-abelian composition factors of G is bounded in terms of r only.*

COROLLARY 2: *Let G be a group that is* res* (*finite and of rank r*). *Then G has a normal subgroup N of finite index that is* res* (*finite soluble and of rank r*).

For the proof of Corollary 2 it suffices to note that among all finite images of G of rank r we may choose one that has a maximal number of non-abelian composition factors.

Proof of Theorem 1: Suppose that G satisfies the hypotheses of the theorem; assuming the result false we may suppose further that G is finitely generated and infinite. Thus G is res(finite and of rank r). Let G/K be an arbitrary finite quotient of rank r and let S/K be its soluble radical, N/S the socle of G/S. By Lemma 1 there is a normal subgroup E of r-bounded index in G such that $N \leq E$ and E/N is soluble. Since there are only finitely many subgroups of a given finite index in G we deduce easily from consideration of appropriate intersections of subgroups that G has a normal subgroup H of finite index that contains subgroups L, M with $L \triangleleft M \triangleleft H$ such that H/M and L are res(finite soluble and of rank r) and M/L is residually (finite of rank r). Further, the finite quotients involved in the residual system for M/L are direct products of non-abelian simple groups.

Now a periodic linear group is locally finite [14; 9.1(i)] and, since H is finitely generated, we deduce from Theorem 3 that H/M is nilpotent-by-finite and hence finite, so that M is finitely generated. If M/L is finite then the same argument gives the contradiction that L (and hence M) is finite. Thus M/L is infinite and we may assume that $L = 1$ and hence that M is a subcartesian product of finite non-abelian simple groups S_i of rank r. Certainly no S_i has a section isomorphic to $C_p \wr C_n$ for any prime p and $n > r$, and so Lemma 4.1 of [15] tells us that there exists an integer m such that each S_i is linear of degree m. Then, by Lemma 4.2 of [15], either M is soluble-by-finite or M contains a free subgroup of rank 2. But M is periodic and once again we obtain the contradiction that M is finite, thus proving the theorem. ∎

Next we require the following easy extension of a well-known result.

LEMMA 2: *Let G be a locally soluble group of rank r. Then there exists $d = d(r)$ such that $G^{(d)}$ is a direct product of finite p-groups for different primes p.*

Proof: By Lemma 10.39 of [10] there exists $n = n(r)$ such that $G^{(n)}$ is periodic and hypercentral. In particular, $G^{(n)}$ is a direct product of p-groups and it now suffices to prove the result for p-groups. Suppose that G is a p-group of rank r with divisible radical D, which of course has finite index in G since a p-group of finite rank is Černikov. Let $C = C_G(D)$; then G/C embeds in Aut D, which is linear of degree at most r over the ring of p-adic integers. By a result of Zassenhaus [10; Theorem 3.23], G/C has r-bounded derived length. Since C is centre-by-finite, C' is finite and the result follows. ∎

Our other requirement for the proof of Theorem 2 is the following.

LEMMA 3: *Let K be the cartesian product of finite p-groups $P_j, j \in J$, for possibly different primes p where each P_j has rank at most r. Then K has a normal subgroup M such that M is hyperabelian and locally nilpotent and K/M is residually (linear of r-bounded degree).*

Proof: Let A_j be a maximal normal abelian subgroup of P_j, for each j, and set $A = \prod_{j \in J} A_j$. Let e_j be the exponent of A_j and $R = \prod_{j \in J}(\mathbb{Z}/e_j\mathbb{Z})$. In a natural way, A is an r-generator module for the commutative ring R. The conjugation action of K on A commutes with the action of R and there is a homomorphism $K \longrightarrow \mathrm{Aut}_R A$ induced by conjugation. Since $C_K(A) = A$ we have an embedding of K/A into $\mathrm{Aut}_R A$ and it follows from [14; Theorem 13.5] that K/A has a normal, hyperabelian, locally nilpotent subgroup M/A such that K/M is res(linear of r-bounded degree). Let B_j be a minimal K-invariant subgroup of A_j for each j; then it is easy to see that $\prod_{j \in J} B_j$ is central in K and that repeating this argument shows that A is in the hypercentre of K and hence of M. The result follows. ∎

Proof of Theorem 2: Let $\{K_i\}_{i \in I}$ be a collection of normal subgroups of G intersecting in the identity such that each G/K_i is of rank r and locally soluble. For each i we define $N_i = K_i G^{(d)}$, where d is as in Lemma 2, and note that N_i/K_i is a direct product of p-groups of rank r. Writing N for the intersection of the N_i we observe that each $N/N \cap K_i \cong NK_i/K_i$ is a direct product of finite p-groups of rank r. It follows that N is a subcartesian product of finite p-groups of rank r. Since $G^{(d)} \leq N$ the result is now a consequence of Lemma 3. ∎

Before turning to our examples let us deal with Theorems 4, 7 and 11.

Proof of Theorem 4: The result of Amberg and Sysak [1] mentioned earlier indicates that we only need show that (iii) implies (i). Suppose G is a finitely generated radical group which is res(rank r) and let $\{N_i\}_{i \in I}$ be a collection of normal subgroups of G intersecting in the identity such that each G/N_i has rank r. Now G/N_i is a soluble minimax group by [10; Theorem 10.38] and the finite residual R_i/N_i of G/N_i is abelian by [10; Theorem 10.33]. Let $R = \bigcap_{i \in I} R_i$ and note that R is abelian since $R' \leq [R_i, R_i] \leq N_i$, for each $i \in I$. Furthermore, since each G/R_i is residually finite it follows that G/R is residually (finite soluble of rank r) and Theorem 3 now implies that G/R is soluble. The result follows. ∎

Proof of Theorem 7: Let G be as stated. Since the property of being almost locally soluble is a “countably recognisable” one [4; Lemma 3.5], we may assume that G is the ascending union of finitely generated subgroups $G_i, i = 1, 2, \ldots$. For each i, let R_i denote the soluble radical of G_i. It is routine to show that G_i/R_i is also res*(rank r) (this only requires that R_i be a maximal normal (derived length d_i)-subgroup for some d_i); thus G_i/R_i has rank at most r. For each $i, G_i/R_i$ is a semi-simple section of G_{i+1}/R_{i+1} and Proposition 2.1 of [4] now applies to show that there is an upper bound n, say, for the indices $|G_i : R_i|$. The subgroup $R = \langle R_i | i \in \mathbb{N} \rangle$ is locally soluble and is easily shown to have index at most n in G. The result follows. ∎

Our next result states somewhat more than we need but is perhaps of independent interest. We let $\mathbb{Z}_p$ denote the ring of p-adic integers.

PROPOSITION 4: *Let G be* res* *(finite p and of rank r), where p is a fixed prime. Then*

(i) *G is (finite p and of rank r)-by-(torsionfree abelian)-by-(a subgroup of $\mathrm{GL}(r, \mathbb{Z}_p)$) and*

(ii) *G has a normal torsionfree subgroup N such that N is centre-by-(a subgroup of $\mathrm{GL}(r, \mathbb{Z}_p)$) and G/N is a finite p-group of rank r.*

Proof: Let $\{N_i\}_{i \in \mathbb{N}}$ be a descending series of normal subgroups of G intersecting in the identity such that G/N_i is a finite p-group of rank r for each i. The group $\hat{G} = \varprojlim G/N_i$ is a pro-p completion of G and has rank at most r as a pro-p-group and so there exists an open characteristic uniform subgroup H of $\hat{G}$ [3; Corollary 4.3]. Now H is torsionfree [3; Theorem 4.8] and $\mathrm{Aut}\, H$ embeds in $\mathrm{GL}(r, \mathbb{Z}_p)$ [3; Corollary 4.18]. Hence $H/Z(H)$ embeds in $\mathrm{GL}(r, \mathbb{Z}_p)$. Viewing G as a subgroup of $\hat{G}$ in the natural way we may set $N = G \cap H$ and hence obtain part (ii) of the proposition.

Now let $K = C_{\hat{G}}(H)$ so that $\hat{G}/K$ embeds in $\mathrm{GL}(r, \mathbb{Z}_p)$. We have $K/Z(H) = K/K \cap H \cong KH/H \leq \hat{G}/H$, and so $K/Z(H)$ is a finite p-group. Evidently $Z(H) \leq Z(K)$ and so K is centre-by-(finite p) and K' is therefore finite and hence a p-group of rank r. It follows easily that K/K' is also res*(finite p and of rank r), thus the torsion subgroup of K/K' is a finite p-group. Part (i) now follows easily on considering $G \cap K$. ∎

Proof of Theorem 11: Since a locally soluble linear group of degree r has r-bounded derived length [10; Theorem 3.23], part (i) of the theorem follows immediately from part (i) of Proposition 4.

Now suppose that G is locally nilpotent. By [14; Theorem 8.2(iii)] there exists $c = c(r)$ such that $\gamma_c(L)$ is periodic for every locally nilpotent linear group L of degree r. If N is as in part (ii) of Proposition 4, therefore, we have $\gamma_c(N)$ periodic modulo $Z(N)$ and, since N is torsionfree, it follows from [11; 5.2.19] that $\gamma_{c+1}(N) = 1$. Clearly the torsion subgroup T of G is a finite p-group, and G/T is torsionfree and a finite extension of NT/T, hence $\gamma_{c+1}(G) \leq T$ [10; Lemma 6.33]. It follows easily that G is nilpotent, and Theorem 11 is proved. ∎

Proof of Theorem 5: Let G be as stated. Then G is residually finite [6; Theorem 3.2] and it suffices to prove that every finite image of G is res(rank q). Clearly an arbitrary finite image H of G is of the form $A \rtimes B$ where A and B are abelian of exponent p and q respectively. If $x \in H \setminus A$ then there is a cyclic image H/N such that $x \notin N$, so suppose that $x \in A$. By Maschke's Theorem, A is the direct product of minimal normal subgroups of H and so there is a maximal (proper) H-invariant subgroup M of A with $x \notin M$. Let $C = C_B(A/M)$; then A/M is a faithful irreducible module for B/C, hence B/C has order at most q (see, for example, [9; Proposition II.1]) and A/M has rank at most $q-1$. Finally, $x \notin MC$ and H/MC has rank at most q, and the result follows. ∎

Proof of Theorem 6: Let p be a prime and let F be the group generated by the integral matrices

$$a = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}.$$

For each i, let H_i be the group of all 2×2 matrices x over $\mathbb{Z}/p^i\mathbb{Z}$ such that $x \equiv 1 \bmod p$ and let N_i be the kernel of the map $F \longrightarrow H_i$ given by $y \mapsto y \bmod p^i$. Then $\bigcap_{i=1}^{\infty} N_i = 1, F$ is freely generated by a and b and each H_i has rank at most 9 (see [10; Vol. 2, p. 179] for further details). Since every countable free group embeds in F, Theorem 6 follows. ∎

Proof of Theorem 8: Let $G = \mathbb{Z} \wr \mathbb{Z}$. The base group of G can be identified in the usual way with a free $\mathbb{Z}C_\infty$ module of rank 1 and hence with the Laurent polynomial ring $\Lambda = \mathbb{Z}[x, x^{-1}]$. Further, normal subgroups of G contained in the base group may be identified with (right) ideals of Λ and it therefore suffices to establish the following.

CLAIM: There exist ideals $I_1, I_2, \ldots$ of Λ such that $I_1 \geq I_2 \geq I_3 \geq \cdots$, $\bigcap_{k=1}^{\infty} I_k = 0$ and Λ/I_k is cyclic as an additive abelian group, for each k.

We require the following lemma where, for elements a, b of Λ, $(a,b)\Lambda$ denotes the ideal of Λ generated by a and b and of course $\mathbb{Z}[x]$ is regarded as a subring of Λ.

LEMMA 5: *Let l be an integer, p a prime and α a positive integer such that $p \nmid \alpha$. If $w \in \mathbb{Z}[x]$ and $w \in (p^l, x-\alpha)\Lambda$ then $w = zp^l + \lambda(x-\alpha)$ for some $z \in \mathbb{Z}, \lambda \in \Lambda$ and this representation is unique.*

Proof: Let $\mathbb{Z}[1/\alpha] = \{a/\alpha^t \mid a \in \mathbb{Z}, t \geq 0\}$, a subring of $\mathbb{Q}$, and define θ : $\Lambda \longrightarrow \mathbb{Z}[1/\alpha]$ via $\lambda(x, x^{-1}) \longmapsto \lambda(\alpha, 1/\alpha)$. Thus $\ker\theta = (x-\alpha)\Lambda$. Writing $w = \lambda_1 p^l + \lambda_2(x-\alpha)$, where $\lambda_1, \lambda_2 \in \Lambda$, we have $\theta(w) = \lambda_1(\alpha, 1/\alpha)p^l$. Since

$w \in \mathbb{Z}[x], \theta(w)$ is an integer and hence $\lambda_1(\alpha, 1/\alpha)$ is also an integer, as $p \nmid \alpha$. Thus $w \equiv \lambda_1(\alpha, 1/\alpha)p^l \bmod \ker\theta$, and we have $w = zp^l + \lambda(x-\alpha)$ as required. If $w = z'p^l + \lambda'(x-\alpha)$, where $z' \in \mathbb{Z}$ and $\lambda' \in \Lambda$, then $(\lambda - \lambda')(x-\alpha) = (z'-z)p^l$ and a simple degree argument gives $\lambda = \lambda'$ and $z = z'$, which shows that the expression for w is unique. ∎

We now return to the proof of the theorem. We use the obvious fact that every non-zero ideal of Λ contains a non-zero element of $\mathbb{Z}[x]$. Let $w_1, w_2, \ldots$ be a list of all the non-zero elements of $\mathbb{Z}[x]$ and let p be a prime. We now define inductively ideals I_k of Λ as follows. Let $I_1 = (p, x-1)\Lambda$ and suppose that for some k we have $I_k = (p^l, x-\alpha)\Lambda$ for some $l > 0$ and $\alpha \in \mathbb{Z}$ such that $p \nmid \alpha$ and $0 < \alpha < p^l$. Let w be the first element in our list that belongs to I_k. As a polynomial in x, w has only finitely many zeros and so there exists a positive integer m and integers $b_1, \ldots, b_m$ satisfying $0 \leq b_i < p$ for all i such that $w(\alpha') \neq 0$, where $\alpha' = \alpha + b_1p^l + b_2p^{l+1} + \cdots + b_mp^{l+m-1}$. Set $I^* = (p^{l+m}, x-\alpha')\Lambda$ and observe that $I^* \subseteq I_k, p \nmid \alpha'$ and $0 < \alpha' < p^{l+m}$. If $w \notin I^*$ then we set $I_{k+1} = I^*$. Supposing that $w \in I^*$, we may write $w = zp^{l+m} + \lambda(x-\alpha')$ for some $z \in \mathbb{Z}, \lambda \in \Lambda$, by Lemma 5. Since $w(\alpha') \neq 0$ we have $z \neq 0$ and hence $zp^{l+m} = sp^q$, where $p \nmid s$ and of course $q \geq l+m$. Now set $I_{k+1} = (p^{q+1}, x-\alpha')\Lambda$. If $w \in I_{k+1}$ then, again by Lemma 5, we have $w = z'p^{q+1} + \lambda'(x-\alpha')$ for some $z' \in \mathbb{Z}, \lambda' \in \Lambda$. From the uniqueness of the expression for w (in terms of p^{l+m} and $x - \alpha'$) we deduce that $zp^{l+m} = z'p^{q+1}$, contradicting the definition of q. Thus $w \notin I_{k+1}$. We have now defined I_k for all k, and it is clear that the intersection of all the I_k is trivial. Further, with the above notation, Λ/I_k has exponent at most p^l as an additive abelian group and is an image of the locally cyclic additive group $\Lambda/(x-\alpha) \cong \mathbb{Z}[1/\alpha]$ so that Λ/I_k is cyclic and the claim is established. ∎

In fact a little more may be proved: Writing $G = \Lambda \rtimes \langle g \rangle$, we see that the centralizer C_k of each I_k in $\langle g \rangle$ has finite index in $\langle g \rangle$ and of course C_kI_k is normal in G and the C_k's form a descending chain. Further, the intersection of all the C_kI_k is trivial and we have the following.

COROLLARY 3: *$\mathbb{Z} \wr \mathbb{Z}$ is* res* *(finite and of rank 2).*

Proof of Theorem 9: (i) Let n be a fixed positive integer and let $A_n = \langle a_1 \rangle \times \cdots \times \langle a_n \rangle$ be a free abelian group of rank n. Define an automorphism x of A_n via $[a_i, x] = a_{i+1}, 1 \leq i \leq n$, where a_{n+1} is interpreted as 1, and set $G = A_n \rtimes \langle x \rangle$. Clearly G is nilpotent of class precisely n. For any prime p there exists $k = k(p)$ such that x^{p^k} centralizes A_n/A_n^p; then $\langle x^{p^k}, A_n \rangle / A_n^p$ has rank exactly $n+1$ and therefore so does G. Now let π be an infinite set of primes and let $p \in \pi, K_p =$

$\langle a_1^p a_2^{-1} \rangle^G$. It is easy to see that $K_p = \langle a_1^p a_2^{-1}, a_2^p a_3^{-1}, \dots, a_{n-1}^p a_n^{-1}, a_n^p \rangle$ and hence that A_n/K_p is cyclic of order p^n. For each subset σ of π let $K_\sigma = \bigcap_{p\in\sigma} K_p$ (where $K_{\{p\}} = K_p$). We claim that $K_\pi = 1$. If $n = 1$ then $K_p = \langle a_1^p \rangle^G = A_n^p$ for each $p \in \pi$, and the result is clear in this case. By induction on n we may assume that $K_\pi \leq \langle a_n \rangle$ and, since $K_p \cap \langle a_n \rangle = \langle a_n^p \rangle$ for each p, we have $K_\pi \leq \bigcap_{p\in\pi} \langle a_n^p \rangle = 1$, as claimed. Since A_n/K_σ is cyclic for every finite subset σ of π the groups G/K_σ are all of rank 2 and the result now follows easily.

(ii) For each $n \geq 2$ let A_n be as in (i) and let $A = \mathop{\mathrm{Dr}}\limits_{n=1}^{\infty} A_n$. Let x be the automorphism of A that acts on each A_n as above and set $G = A \rtimes \langle x \rangle$, which is clearly non-nilpotent and of infinite rank but locally nilpotent. Now let $\pi_1, \pi_2, \dots$ be a sequence of pairwise disjoint, infinite sets of primes. Again as in (i), each A_n has a descending series of $\langle x \rangle$-invariant subgroups K_σ intersecting in the identity such that each A_n/K_σ is a cyclic π_n-group. This easily gives us a descending series of $\langle x \rangle$-invariant subgroups N_i of A intersecting in the identity, such that each A/N_i is locally cyclic, that is of rank 1. Again the result follows. ∎

As was the case for Theorem 8, there is an improved version of Theorem 9. With the notation as in (i), let C_σ denote the centralizer in $\langle x \rangle$ of A_n/K_σ for each finite subset σ of π. Corresponding to the descending chain of K_σ's there is a descending chain of $C_\sigma K_\sigma$'s. Each of these is normal in G and their intersection is trivial and so G is res*(finite and of rank 2). By choosing our subgroups N_i a little more carefully in part (ii) we can show that the group G is also res*(finite and of rank 2); all we need is for each A/N_i to be finite cyclic (with the N_i still forming a descending chain), but it is easy to arrange for this employing what might be termed a diagonal argument. It is convenient to state our conclusion in the following manner.

COROLLARY 4: *In each of parts* (i) *and* (ii) *of Theorem* 9, *the group* G *is* res* *(finite and of rank* 2*).*

Finally, we turn to the proof of Theorem 10. First we require some basic properties of certain matrix groups. Let p be a prime and n a positive integer. We let $G = G(p, n)$ denote the kernel of the natural map

$$\mathrm{GL}(2, \mathbb{Z}/p^n\mathbb{Z}) \longrightarrow \mathrm{GL}(2, \mathbb{Z}/p\mathbb{Z}).$$

Then G consists of all matrices of the form

$$(*) \qquad \begin{pmatrix} 1+a & b \\ c & 1+d \end{pmatrix},$$

where $a, b, c, d \in p(\mathbb{Z}/p^n\mathbb{Z})$. (Clearly each such matrix is invertible since its determinant is congruent to 1 modulo the radical $p(\mathbb{Z}/p^n\mathbb{Z})$ of $\mathbb{Z}/p^n\mathbb{Z}$.) Therefore G has order $(p^{n-1})^4$ and, in particular, G is a finite p-group.

Let D denote the subgroup of all scalar matrices in G. For each i with $1 \le i \le n$, let H_i denote the subgroup of G that consists of all matrices of the form (*) where $a, b, c, d \in p^i(\mathbb{Z}/p^n\mathbb{Z})$, and let $G_i = DH_i$. Then

LEMMA 6:

(i) *For all i, j with $1 \le i, j \le n$, we have $[G_i, G_j] \le G_{i+j}$, where $G_m = 1$ for all $m \ge n$.*

(ii) *Let $A \equiv \begin{pmatrix} 1+p^j & 0 \\ 0 & 1 \end{pmatrix} \bmod G_{j+1}$ and $B \equiv \begin{pmatrix} 1 & 0 \\ p^j & 1 \end{pmatrix} \bmod G_{j+1}$. If $x \in G$ and if $[x, A], [x, B] \in G_{i+j+1}$ for some $i \ge 0$, where $i + j < n$, then $x \in G_{i+1}$.*

(iii) *For all $i \ge 0$ and $j \ge 1$ with $i + j < n$, the centralizer in G of G_j/G_{i+j+1} is precisely G_{i+1}. In particular (setting $j = 0$) the subgroups $G_n, \ldots, G_1$ are (all of) the terms of the upper central series of G.*

(iv) *G has rank at most 9.*

Proof: (i) Note that $[G_i, G_j] = [H_i, H_j]$ since D is central in G. Let $M_2(\mathbb{Z}/p^n\mathbb{Z})$ denote the full 2×2 matrix ring over $\mathbb{Z}/p^n\mathbb{Z}$ and, for each i with $1 \le i \le n$, let $M_2(i)$ denote the kernel of the natural ring homomorphism $M_2(\mathbb{Z}/p^n\mathbb{Z}) \longrightarrow M_2(\mathbb{Z}/p^i\mathbb{Z})$. Note that $H_i = 1 + M_2(i)$ in the obvious sense and that $M_2(i)M_2(j) \subseteq M_2(i+j)$. Let $y \in H_i$ and $x \in H_j$. Then $y = 1 - Y$ and $x = 1 - X$ for some $Y \in M_2(i)$ and $X \in M_2(j)$. Now $y^{-1} = 1 + Y + Y^2 + \cdots + Y^l$ and $x^{-1} = 1 + X + X^2 + \cdots + X^s$ where $Y^{l+1} = 0 = X^{s+1}$. Thus

$$\begin{aligned}[y, x] &= (1 + Y + Y^2 + \cdots + Y^l)(1 + X + X^2 + \cdots + X^s)(1 - Y)(1 - X) \\ &\equiv (1 + Y + Y^2 + \cdots + Y^l + X + X^2 + \cdots + X^s)(1 - Y - X) \\ &\qquad \bmod M_2(i+j) \\ &\equiv 1 \bmod M_2(i+j)\end{aligned}$$

and it follows that $[y, x] \in H_{i+j} \le G_{i+j}$, thus proving (i).

(ii) For $i = 0$ the result is trivial. Assume that $i > 0$ and that the result holds for $i - 1$. Since $G_{i+j+1} \le G_{i+j}$ we then have $x \in G_i$ by induction. Suppose for a contradiction that $x \notin G_{i+1}$ and that $[x, A], [x, B] \in G_{i+j+1}$ where $i + j < n$. Now $[dx, g] = [x, g]$ for all $d \in D$ and $g \in G$ and so it suffices to assume that

$$x = \begin{pmatrix} 1 & b \\ c & 1+d \end{pmatrix}$$

where $p^i \mid b, c, d$ and derive a contradiction. We calculate $[x, A]$ and on examining the off-diagonal entries deduce that $p^{i+1} \mid c, b$. On computing $[x, B]$ and examining its $(2,1)$-entry we deduce that $p^{i+1} \mid d$. Thus $x \in H_{i+1}$, and we obtain the desired contradiction.

Finally, (iii) follows easily from (ii), while (iv) is an immediate consequence of Lemma 7.44 of [10]. ∎

We now define a group C as follows. Let $p_1 < p_2 < \cdots$ be an infinite sequence of odd primes and for each $\lambda \geq 1$, let $G(\lambda) = G(p_\lambda, 2^\lambda)$ be as above. Further, for all $\lambda \geq 1$ and all k such that $1 \leq k \leq \lambda$, let $G_k(\lambda)$ denote the subgroup $G_k(p_\lambda, 2^\lambda)$, again as above. Now let H be the cartesian product of all the $G(\lambda)$ and, for each $j \geq 1$, let C_j be the cartesian product of the $G_{2^{\lambda-j}}(\lambda)$, a subgroup of H. By Lemma 6(i) $G_{2^{\lambda-j}}(\lambda)$ is nilpotent of class at most 2^j (and has derived length at most j) and so $C = \bigcup_{j=1}^{\infty} C_j$ is locally nilpotent. It is also res*(rank 9) by Lemma 6(iv).

We now show that C is not hypercentral. For each λ and for each k satisfying $0 \leq k < \lambda$, let

$$A_{\lambda,k} = \begin{pmatrix} 1 + p_\lambda^{\alpha(\lambda,k)} & 0 \\ 0 & 1 \end{pmatrix}, \quad B_{\lambda,k} = \begin{pmatrix} 1 & 0 \\ p_\lambda^{\alpha(\lambda,k)} & 1 \end{pmatrix},$$

where $\alpha(\lambda, k) = 2^{\lambda-(k+1)}$. Then of course $A_{\lambda,k}, B_{\lambda,k} \in G_{\alpha(\lambda,k)}(\lambda) \setminus G_{\alpha(\lambda,k)+1}(\lambda)$ for all k.

An arbitrary element g of the cartesian product of the $G(\lambda)$ may as usual be written as (g_λ). Let $x_1 = (x_{1,\lambda})$, where

$$x_{1,\lambda} = \begin{pmatrix} 1 & p_\lambda^{2^{\lambda-1}} \\ 0 & 1 \end{pmatrix} \quad \text{for each } \lambda;$$

thus $x_{1,\lambda} \in G_{\alpha(\lambda,0)}(\lambda) \setminus G_{\alpha(\lambda,0)+1}(\lambda)$. Let n be a positive integer and let λ be greater than n. By repeated application of Lemma 6(ii), the element $x_{n,\lambda} = [x_{1,\lambda}, y_{1,\lambda}, \ldots, y_{n,\lambda}]$ belongs to $G_{\beta(\lambda,n)}(\lambda) \setminus G_{\beta(\lambda,n)+1}(\lambda)$, where

(i) $\beta(\lambda, n) = 2^{\lambda-1} + 2^{\lambda-2} + \cdots + 2^{\lambda-(n+1)}$ and

(ii) $y_{i,\lambda} = A_{\lambda,i}$ or $B_{\lambda,i}$, the choice being the obvious one suggested by Lemma 6(ii).

(Note that $\beta(\lambda, n) \leq 1 + 2 + 2^2 + \cdots + 2^{\lambda-1} = 2^\lambda - 1$ and so $x_{n,\lambda}$ is indeed non-trivial.) For each $i \geq 1$, let $y_i = (y_{i,\lambda})$, where $y_{i,\lambda}$ is as above for $\lambda > i$ and $y_{i,\lambda} = 1$ for $\lambda \leq i$. Then $x_n (= [x_1, y_1, \ldots, y_n])$ is non-trivial for each positive integer n. If x_1 belongs to the cth term of the upper central series of C for some (possibly transfinite) ordinal c, then the above sequence of elements x_n gives rise to an infinite descending chain of ordinals starting at α. By this contradiction, C is not hypercentral.

The proof that $C^{(n)}$ is not hypercentral for any positive integer n is essentially similar. However, one needs to establish that $C^{(n)}$ contains elements that are suitable substitutes for the elements $x_1, y_1, y_2, \ldots$. Indeed, $C^{(n)}_{n+1}$ can be shown to contain an element z_1 such that

$$z_{1,\lambda} \equiv \begin{pmatrix} 1 & p_\lambda^{2^{\lambda-1}} \\ 0 & 1 \end{pmatrix} \bmod G_{2^{\lambda-1}+1}(\lambda) \quad \text{for all sufficiently large } \lambda,$$

and substitutes for $y_{i,\lambda}$ are obtained similarly. We omit the details. ∎

3. Concluding remarks

In this final section we present a few additional results and comments. Firstly we note that in the hypotheses of Theorem 1 the condition that the relevant rank r images be locally finite may be weakened, as any class of groups for which periodic implies locally finite may be substituted here. Trivial though this observation is, it allows us to state (for example) a result in which the hypotheses more closely resemble those of Theorem 2.

3.1: *Let G be a periodic group that is residually $\mathfrak{X}_r$, where $\mathfrak{X}_r$ denotes the class of locally (soluble-by-finite) groups of rank r. Then G is locally finite.*

With the same notation, we also note the following.

3.2: *A periodic group that is* res$^*\mathfrak{X}_r$ *is locally finite and hence of rank r and almost locally soluble.*

We recall that a locally finite group G of finite rank is not only almost locally soluble but has a normal subgroup H of finite index such that H' is locally nilpotent ($\mathbf{L}\mathfrak{N}$). This result is due to Kargapolov [7] who shows further that H modulo its Hirsch–Plotkin radical has finite Sylow p-subgroups for all primes p. It follows that a locally finite group that is res(rank r) is res $\mathbf{L}\mathfrak{N}$-by-abelian-by-res (finite); more particularly we have:

3.3: *Let G be a locally finite group that is* res(*rank r*)*. Then there is a normal subgroup H of G such that G/H is* res(*of rank r and finite*) *and H' is locally nilpotent.*

It ought to be possible to say a little more about a locally finite group G that is res(rank r), particularly when the locally soluble radical of G is trivial (so that G is res(finite of rank r), by Šunkov's theorem). Our final remark on locally finite groups is that such a group is hyperfinite if it has finite rank; this again contrasts with the result of Theorem 5.

Next we record the following result on locally soluble groups.

3.4: *Let G be a locally soluble group that is* res*(*rank r*). *Then some r-bounded term of the derived series of G is a subcartesian product of finite p-groups of rank r, for different primes p.*

It is the fact that the primes are distinct that distinguishes 3.4 from Theorem 2—we recall from the proof of Theorem 2 that the subgroup N was easily shown to be a subcartesian product of p-groups of rank r, while Lemma 3 gave us the indicated structure of N. To establish 3.4 we use Theorem 11. We already know that G has a normal subgroup U containing $G^{(d)}$ for some $d = d(r)$ such that U is res*(finite nilpotent of rank r). For each prime p, let $R_{p,i}/N_i$ denote the p'-radical of U/N_i for each i, where the N_i form the obvious residual system for U. Then set $R_p = \bigcap_{i=1}^{\infty} R_{p,i}$; by Theorem 11, $U^{(e)}$ is a finite p-group of rank r modulo R_p, for some $e = e(r)$. It now follows easily that $U^{(e)}$ is a subcartesian product of such finite p-groups (one for each prime p that occurs), and 3.4 is proved.

In view of Theorem 8, our next result may be of some interest.

3.5: *Let G be a finitely generated soluble group that is* res*(*rank r*). *If G has no sections isomorphic to $\mathbb{Z} \wr \mathbb{Z}$ then G has finite rank.*

We have been unable to find an example to show that "sections" cannot be replaced by "subgroups" in the above. The proof of 3.5 is easy, though it depends on a deep result. With G as stated, let A be a maximal normal abelian subgroup such that G/A has smaller derived length than G; then G/A is res*(rank r) and hence of finite rank, by induction. The torsion subgroup T of A is res*(rank r) and hence of rank r, and if $a \in A \setminus T$ then $\langle a \rangle^{\langle g \rangle}$ has finite rank for all $g \in G$, since there are no $\mathbb{Z} \wr \mathbb{Z}$ sections. In the terminology of [8], A is thus a constrained G-module and hence G is minimax.

Our final remark concerns Theorem 7 and whether one can replace "locally (soluble-by-finite)" by "locally $\mathfrak{X}$" for some more general class $\mathfrak{X}$. In order to apply our current methods of proof, one would probably require a class $\mathfrak{X}$ for which finiteness of rank implies almost locally soluble. Certainly the extensive class defined by Černikov in [2] will not do in this context as his class contains all free groups (and Theorem 6 provides a suitable counterexample).

References

[1] B. Amberg and Y. P. Sysak, *Locally soluble products of two minimax subgroups*, in *Groups–Korea '94* (A. C. Kim and D. Johnson, eds.), de Gruyter, Berlin, 1995, pp. 9–14.

[2] N. S. Černikov, *A theorem on groups of finite special rank*, Ukrainskiĭ Matematicheskiĭ Zhurnal **42** (1990), 962–970; English transl.: Ukrainian Mathematical Journal **42** (1990), 855–861.

[3] J. D. Dixon, M. F. P. du Sautoy, A. Mann and D. Segal, *Analytic Pro-p Groups*, London Mathematical Society Lecture Note Series, Volume 157, Cambridge University Press, 1991.

[4] M. R. Dixon, M. J. Evans and H. Smith, *Locally (soluble-by-finite) groups of finite rank*, Journal of Algebra **182** (1996), 756–769.

[5] R. I. Grigorčuk, *On the Burnside problem for periodic groups*, Funkcionalnyi Analiz i ego Priloženiya **14** (1980), 53–54; English transl.: Functional Analysis and its Applications **14** (1980), 41–43.

[6] K. W. Gruenberg, *Residual properties of infinite soluble groups*, Proceedings of the London Mathematical Society (3) **7** (1957), 29–62.

[7] M. I. Kargapolov, *Some problems in the theory of soluble and nilpotent groups*, Doklady Adakemii Nauk SSSR **127** (1959), 1164–1166 (Russian).

[8] P. H. Kropholler, *On finitely generated soluble groups with no large wreath product sections*, Proceedings of the London Mathematical Society (3) **49** (1984), 155–169.

[9] C. R. Leedham-Green and W. Plesken, *Some remarks on Sylow subgroups of general linear groups*, Mathematische Zeitschrift **191** (1986), 529–535.

[10] D. J. S. Robinson, *Finiteness Conditions and Generalized Soluble Groups*, Vols. 1 and 2, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 62 and 63, Springer-Verlag, Berlin, Heidelberg, New York, 1972.

[11] D. J. S. Robinson, *A Course in the Theory of Groups*, Graduate Texts in Mathematics, Vol. 80, Springer-Verlag, Berlin, Heidelberg, New York, 1996.

[12] D. Segal, *A footnote on residually finite groups*, Israel Journal of Mathematics **94** (1996), 1–5.

[13] V. P. Šunkov, *On locally finite groups of finite rank*, Algebra i Logika **10** (1971), 199–225; English transl.: Algebra and Logic **10** (1971) 127–142.

[14] B. A. F. Wehrfritz, *Infinite Linear Groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete Band 76, Springer-Verlag, New York, Heidelberg, Berlin, 1973.

[15] J. S. Wilson, *Two generator conditions for residually finite groups*, The Bulletin of the London Mathematical Society **23** (1991), 239–248.