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AUTOMORPHISMS OF GROUPS AND OF SCHEMES OF FINITE TYPE

BY

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ABSTRACT

We show first that certain automorphism groups of algebraic varieties, and even schemes, are residually finite and virtually torsion free. (A group virtually has a property if some subgroup of finite index has it.) The rest of the paper is devoted to a study of the groups of automorphisms Aut(Γ) and outer automorphisms Out(Γ) of a finitely generated group Γ, by using the finite-dimensional representations of Γ. This is an old idea (cf. the discussion of Magnus in [11]). In particular the classes of semi-simple n-dimensional representations of Γ are parametrized by an algebraic variety $S_n(\Gamma)$ on which Out(Γ) acts. We can apply the above results to this action and sometimes conclude that Out(Γ) is residually finite and virtually torsion free. This is true, for example, when Γ is a free group, or a surface group. In the latter case Out(Γ) is a "mapping class group."

1. Automorphisms of schemes of finite type

Let k be a commutative ring. Let V be a scheme of finite presentation over k, and $\mathrm{Aut}_k(V)$ its group of k-scheme automorphisms.

(1.1) THEOREM. *Suppose that* $k = \mathbf{Z}$.

(a) $\mathrm{Aut}_k(V)$ *is residually finite.*

(b) *If V is flat over* $\mathbf{Z}$ *then* $\mathrm{Aut}_k(V)$ *is virtually torsion free. Hence there is a bound on the orders of the finite subgroups of* $\mathrm{Aut}_k(V)$.

Recall that a group Γ is *residually finite* if its subgroups of finite index have trivial intersection. Equivalently the natural map from Γ to its *profinite completion*

$$\hat{\Gamma} = \varprojlim_{\Gamma/N \text{ finite}} \Gamma/N$$

is injective.

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To say that V is *flat* over $\mathbf{Z}$ signifies that its local rings are flat over $\mathbf{Z}$, i.e. torsion free as $\mathbf{Z}$-modules.

(1.2) COROLLARY. *Let k be arbitrary and let Γ be a finitely generated subgroup of* $\mathrm{Aut}_k(V)$.

(a) Γ *is residually finite.*

(b) *If V is flat over* $\mathbf{Z}$ *then* Γ *is virtually torsion free.*

Before deducing (1.2) from (1.1) we give a mildly strengthened version of the affine case.

(1.3) COROLLARY. *Let k be a commutative ring, A a finitely generated commutative k-algebra, and Γ a finitely generated group of k-algebra automorphisms of A.*

(a) Γ *is residually finite.*

(b) *If A is* $\mathbf{Z}$*-torsion free then* Γ *is virtually torsion free.*

If $k = \mathbf{Z}$ then (a) *and* (b) *hold with all of* $\mathrm{Aut}(A)$ *in place of* Γ.

In fact, let X be a finite set of k-algebra generators of A and $S = S^{-1}$ a finite set of generators of Γ. For $s \in S$ and $x \in X$ we have $s(x) = f_{s,x}(X)$, where $f_{s,x}(X)$ is a polynomial in X with coefficients in k. Let k_0 be the subring of k generated by all coefficients of $f_{s,x}$ for all $s \in S$ and $x \in X$; let A_0 denote the k_0-algebra in A generated by X. Evidently Γ stabilizes A_0, and acts faithfully on A_0. Since A_0 is a finitely generated $\mathbf{Z}$-algebra (k_0 being so), which is $\mathbf{Z}$-torsion free if A is so, we are reduced to the case $k = \mathbf{Z}$, which is just the affine case of Theorem (1.1).

To similarly deduce (1.2) from (1.1) we require a more sophisticated reduction, due to Grothendieck [4]. First, since V is finitely presented over k, there is a subring k_0 of k finitely generated over $\mathbf{Z}$ and a k_0-scheme V_0 of finite type such that $V \cong k \otimes_{k_0} V_0$ as k_0-schemes ([4], proposition (8.9.1)). Let (k_λ) denote the family of finitely generated k_0-subalgebras of k, ordered by inclusion, and put $V_\lambda = k_\lambda \otimes_{k_0} V_0$. These form a projective system with $V \cong \varinjlim V_\lambda$. It follows from [4], théorème (8.8.2) and the finite presentation of the k_λ-schemes V_λ that

$$\mathrm{Hom}_k(V, V) = \varinjlim \mathrm{Hom}_{k_\lambda}(V_\lambda, V_\lambda).$$

Consequently also

$$\mathrm{Aut}_k(V) = \varinjlim \mathrm{Aut}_{k_\lambda}(V_\lambda). \tag{1}$$

(If $s \in \mathrm{Aut}_k(V)$ then s and s^{-1} come, for some λ, from some s_λ, $t_\lambda \in \mathrm{Hom}_{k_\lambda}(V_\lambda, V_\lambda)$, and then $s_\lambda t_\lambda$ and $t_\lambda s_\lambda$ become the identity of V_μ for some $\mu \geqq \lambda$.)

Now let Γ be a finitely generated subgroup of $\mathrm{Aut}_k(V)$. In view of (1) there is a λ and a finitely generated subgroup Γ_λ of $\mathrm{Aut}_k(V_\lambda)$ such that $\Gamma = k \otimes_{k_\lambda} \Gamma_\lambda$. Consider the commutative diagram

(2)
$$\begin{array}{ccccc} V & & \xrightarrow{\varphi} & & V_\lambda \\ \downarrow & \searrow & & \circlearrowleft & \downarrow \\ & & W & \hookrightarrow & \\ & & & \searrow & \\ \mathrm{Spec}(k) & & \longrightarrow & & \mathrm{Spec}(k_\lambda) \end{array}$$

where W is the schematic closure of the image of V in V_λ. The group Γ_λ acts, via the natural homomorphism $\Gamma_\lambda \to \Gamma$, on V, so that φ is Γ_λ-equivariant. It follows that W is Γ_λ-invariant. If $\gamma \in \Gamma_\lambda$ denote by γ_W the induced automorphism of W, and put $\Gamma_W = \{\gamma_W \mid \gamma \in \Gamma_\lambda\}$. Then we clearly have a commutative diagram of surjective homomorphisms

$$\begin{array}{ccc} \Gamma & \longleftarrow & \Gamma_\lambda \\ & {}_{\alpha}\searrow \quad \swarrow & \\ & & \end{array} \qquad \begin{array}{ccc} K \otimes_{k_\lambda} \gamma & \longleftarrow & \gamma \\ & \searrow \quad \swarrow & \\ & \gamma_W & \end{array}$$

We claim that α is injective (and hence bijective). For suppose that $\gamma \in \Gamma$ and $\gamma_W = 1_W$. Then $(1_{\mathrm{Spec}(k)}, 1_V, \gamma)$ defines an automorphism of the (peripheral) cartesian square in (2) so it follows that $1_V = k \otimes_{k_\lambda} \gamma$.

In conclusion, we have shown that $\Gamma \cong \Gamma_W \subset \mathrm{Aut}_{k_\lambda}(W)$. As a closed subscheme of V, W (like V) is of finite type over k_λ. Since k_λ is a finitely generated $\mathbf{Z}$-algebra, W is of finite type over $\mathbf{Z}$. Finally it follows from the construction of W that, for U open in W, $\mathcal{O}_w(U) \to \mathcal{O}_V(\varphi^{-1}(U))$ is injective. Consequently W is flat over $\mathbf{Z}$ if V is so. Thus Corollary (1.2) follows by applying Theorem (1.1) to $\Gamma_W \subset \mathrm{Aut}_{\mathbf{Z}}(W)$.

(1.4) Remarks. (1) The properties in (a) and (b) above are well known to hold for finitely generated *linear* groups $\Gamma \subset \mathrm{GL}_n(k)$. In fact this is a special case of (1.3) if we let such a Γ act on the symmetric algebra of the k-module k^n.

(2) This suggests extending other results for linear groups to automorphism groups like $\mathrm{Aut}_k(V)$. For example, does a finitely generated subgroup Γ of $\mathrm{Aut}_k(V)$ satisfy the "Tits alternative": either (i) Γ is virtually solvable, or (ii) Γ contains a non-abelian free group? Further, one would like some control over the solvable subgroups of $\mathrm{Aut}_k(V)$. A basic example is the automorphism group of the polynomial algebra $k[x_1, \cdots, x_n]$, the so-called *integral Cremona group.*

PROOF OF (1.1). Put $\Gamma = \mathrm{Aut}(V)$. For every commutative ring F we have the F-valued points

$$V(F) = \mathrm{Mor}(\mathrm{Spec}(F), V)$$

of V, and Γ acts naturally on $V(F)$. If the ring F is finite then the set $V(F)$ is finite. (For instance, if $V = \mathrm{Spec}(\mathbf{Z}[x_1, \cdots, x_n])$ is affine then $V(F) \hookrightarrow F^n$; in general V is covered by finitely many such affine schemes, since V is of finite type over $\mathbf{Z}$.)

To prove (a) it suffices to show that if $s \in \Gamma$ acts trivially on $V(F)$ for all finite F then $s = 1$. Let x be a point of V, A_x its local ring, $m_x = \mathrm{rad}(A_x)$, and $k(x) = A_x/m_x$. If x is a closed point then $k(x)$ is a finite field. The trivial action of s on $V(k(x))$ implies that s fixes x and hence acts on A_x. Since s acts trivially on $V(A_x/m_x^r)$ it does likewise on A_x/m_x^r. Since $\bigcap_r m_x^r = 0$, s acts trivially on A_x. Thus, since s fixes all closed points and acts trivially on their local rings, $s = 1$.

To prove (b) we need a little preliminary discussion. Call a subset X of V *effective* if V admits an open covering by affine schemes $U_i = \mathrm{Spec}(A_i)$ such that, for each i, the natural map

$$A_i \to \prod_{x \in X \cap U_i} A_x$$

is injective. Our interest in this notion is that if $s \in \Gamma$ fixes each $x \in X$ and acts trivially on each A_x then $s = 1$. (If V is reduced this follows by noting that s acts trivially on the ring of rational functions on V, since it does so on enough of the local rings therein. In general, one deduces from this that s acts trivially on $V_{\mathrm{red}} \subset V$, hence fixes all points of V, and then one sees easily that s acts trivially on each $\mathrm{Spec}(A_i)$ as above.)

Let $V = U_1 \cup \cdots \cup U_n$ with $U_i = \mathrm{Spec}(A_i)$ open. Let $X = X_1 \cup \cdots \cup X_n$ where $X_i \subset U_i$ is a finite set of closed points such that, for each $\mathfrak{G} \in \mathrm{Ass}(A_i)$, $\mathfrak{G} \subset m_x$ for some $x \in X_i$. The latter is precisely the condition that guarantees the injectivity of $A_i \to \Pi_{x \in X_i} A_x$. Thus X is a finite effective set of closed points.

Suppose further that V is flat over $\mathbf{Z}$, so that each A_i is torsion free. Then for each $\mathfrak{G} \in \mathrm{Ass}(A_i)$, $A_i/\mathfrak{G}$ has characteristic zero. It follows (since A_i is finitely generated over $\mathbf{Z}$) that $A_i/\mathfrak{G}$ has finite residue class fields of all but finitely many possible characteristics. Hence we can choose another finite effective set X' of closed points as above so that

$$(*) \qquad \operatorname{char}(k(x)) \neq \operatorname{char}(k(x'))$$

for all $x \in X$, $x' \in X'$.

Let

$$A = \prod_{x \in X} A_x, \qquad J = \operatorname{rad}(A) = \prod_{x \in X} m_x,$$

and

$$\Gamma_x = \operatorname{Ker}(\Gamma \to \operatorname{Aut}(V(A/J^2))).$$

Then Γ_x has finite index in Γ and fixes each x in X, and so acts on A, inducing the trivial action on A/J^2.

It follows that Γ_x acts trivially on

$$\operatorname{gr}(A) = \bigoplus_{r \geqq 0} J^r/J^{r+1} = (A/J)[J/J^2],$$

and so acts unipotently on each A/J^r. By Lemma (1.5) below therefore, the image of Γ_X in $\operatorname{Aut}(A/J^r)$ is "X-torsion", i.e. each element has order divisible only by the primes $\operatorname{char}(k(x))$ $(x \in X)$. Since X is effective Γ_X acts faithfully on A. Since $\bigcap_r J^r = 0$ it follows that every finite subgroup of Γ_X is represented faithfully on some A/J^r, and so is X-torsion.

Now if we repeat the above considerations with X' in place of X we obtain Γ_X of finite index in Γ whose only torsion is X'-torsion. In view of $(*)$ above therefore, $\Gamma_X \cap \Gamma_{X'}$ is torsion free, and clearly of finite index in Γ. This proves Theorem (1.1).

(1.5) LEMMA. *Let n be an integer $\geqq 1$ and A a $(\mathbf{Z}/n\mathbf{Z})$-algebra.*

(a) *If $x \in A$ is nilpotent there is a positive integer N such that* $(1-x)^{n^N} = 1$.

(b) *If s is a unipotent automorphism of an A-module M then s has order dividing some power of n.*

Applying (a) to $x = 1 - s \in \operatorname{End}_A(M)$ gives (b).

To prove (a) we can, by the Chinese Remainder Theorem, reduce to the case where n is a prime power p^r. Choose $t > 0$ so that $x^{p^t} = 0$. Then $(1-x)^{p^t} = 1 - py$ for some $y \in A$. For any $z \in A$ and $i > 0$ we have $(1 - p^i z)^p = 1 - p^{i+1} z'$ for some z'. It follows that $(1-py)^{p^{r-1}} = 1$, so $(1-x)^{p^e} = 1$ with $e = t + r - 1$.

(1.6) PROBLEM. Let V be a scheme of finite type over $\mathbf{Z}$ and let $\Gamma = \mathrm{Aut}(V)$. For each finite ring F put

$$\Gamma_F = \mathrm{Ker}(\Gamma \to \mathrm{Aut}(V(F))).$$

These subgroups of finite index define a topology on Γ analogous to that defined by congruence subgroups in the case of linear groups. We shall accordingly call this the *congruence topology* on Γ. One can thus pose the "congruence subgroup problem" for Γ: Does every subgroup of finite index in Γ contain some Γ_F as above? The answer is almost certainly "no" in general, but there may be cases where one can give a reasonable description of the kernel

$$C = \mathrm{Ker}(\hat{\Gamma} \to \bar{\Gamma})$$

of the map from the profinite completion $\hat{\Gamma}$ to the congruence completion $\bar{\Gamma}$.

2. Automorphisms of groups of finite type

Let Γ be a group. We have the exact sequence

$$1 \to Z(\Gamma) \to \Gamma \xrightarrow{\mathrm{ad}} \mathrm{Aut}(\Gamma) \to \mathrm{Out}(\Gamma) \to 1$$

where $\mathrm{ad}(x): y \to xyx^{-1}$ for $x, y \in \Gamma$; its kernel is the center $Z(\Gamma)$; its image

$$\mathrm{ad}(\Gamma) = \mathrm{I\,Aut}(\Gamma)$$

is the group of inner automorphisms, and $\mathrm{Out}(\Gamma) = \mathrm{Aut}(\Gamma)/\mathrm{I\,Aut}(\Gamma)$, the group of outer automorphisms.

We recall some familiar results which we shall use.

(2.1) PROPOSITION (J. Smith [16]). *Let Γ be a profinite group (= a projective limit of finite groups) which is topologically finitely generated. Then its group* $\mathrm{Aut}^c(\Gamma)$ *of continuous automorphisms is a profinite group, and hence so also is* $\mathrm{Out}^c(\Gamma) = \mathrm{Aut}^c(\Gamma)/\mathrm{I\,Aut}(\Gamma)$.

Let q be an integer $\geqq 1$. Since Γ is finitely generated it has only finitely many open subgroups of index $\leqq q$; their intersection Γ_q is therefore open, and $\Gamma = \varprojlim_q \Gamma/\Gamma_q$. The functoriality of the Γ_q's shows that the groups $\mathrm{Aut}(\Gamma/\Gamma_q)$ likewise form a projective system, and the natural map

$$\mathrm{Aut}^c(\Gamma) \to \varprojlim_q \mathrm{Aut}(\Gamma/\Gamma_q)$$

is easily seen to be an isomorphism. Since $\mathrm{ad}: \Gamma \to \mathrm{Aut}^c(\Gamma)$ is continuous, its (compact) image is closed, so $\mathrm{Out}^c(\Gamma) = \mathrm{coker}(\mathrm{ad})$ is also profinite.

(2.2) Corollary (Baumslag). *Let* Γ *be a finitely generated group. If* Γ *is residually finite so also is* $\mathrm{Aut}(\Gamma)$.

Indeed since $\Gamma \to \hat{\Gamma}$ is injective so also is $\mathrm{Aut}(\Gamma) \to \mathrm{Aut}^c(\hat{\Gamma})$.

Unfortunately the same reasoning does not show that $\mathrm{Out}(\Gamma)$ is residually finite, since $\mathrm{Out}(\Gamma) \to \mathrm{Out}^c(\hat{\Gamma})$ need not be injective. Sufficient conditions for this are given in the next proposition.

Write $x \sim y$ if x and y are conjugate in Γ. Call Γ *conjugacy separable* if $x \sim y$ whenever x and y become conjugate in all finite quotients of Γ.

(2.3) Proposition (E. Grossman [3]). *Let* Γ *be a finitely generated group satisfying*

(a) Γ *is conjugacy separable, and*

(b) *if* $\alpha \in \mathrm{Aut}\ (\Gamma)$ *and* $\alpha(x) \sim x$ *for all* $x \in \Gamma$ *then* $\alpha \in \mathrm{I\ Aut}(\Gamma)$.

Then $\mathrm{Out}(\Gamma)$ *is residually finite.*

It suffices to show that $\mathrm{Out}(\Gamma) \to \mathrm{Out}^c(\hat{\Gamma})$ is injective. Let $\alpha \in \mathrm{Aut}(\Gamma)$ and suppose that $\hat{\alpha} \in \mathrm{I\ Aut}(\hat{\Gamma})$. If $x \in \Gamma$ then $\alpha(x) \sim x$ by (a), and so $\alpha \in \mathrm{I\ Aut}(\Gamma)$ by (b).

(2.4) Examples. (1) Let Γ be a *free group*. Then (a) and (b) are well known, even in more precise forms. For (a) see [9], ch. I, prop. (4.8). For (b) see [3], lemma 1.

(2) Let Γ be a *surface group* of genus g, i.e. fundamental group of a compact orientable surface S of genus g. Then (a) has been proved by Stebe [17], and (b) by Grossman [3], using rather involved word and cancellation arguments. This case is of particular interest because the mapping class group $\pi_0(\mathrm{Diff}(S))$ is naturally isomorphic to $\mathrm{Out}(\Gamma)$ (cf. [1], theorem 1.4).

We present below another method for proving such results, using the representation theory of Γ. (See Theorem (4.3).)

(3) $\mathrm{Aut}(\Gamma)$ and $\mathrm{Out}(\Gamma)$ are not very functorial in Γ. Here is a useful exception. Suppose that Γ is perfect, i.e. $H_1(\Gamma, \mathbf{Z}) = 0$, and let C be the Schur multiplier $H_2(\Gamma, \mathbf{Z})$. Then there is a universal central extension (cf. [12])

$$(\varepsilon) \qquad 1 \to C \to \bar{\Gamma} \to \Gamma \to 1,$$

with $\bar{\Gamma}$ also perfect, such that for any central extension

$$(\varepsilon') \qquad 1 \to C' \to G \to P \to 1,$$

every homomorphism $\rho : \Gamma \to P$ lifts to a unique homomorphism $\bar{\rho} : \bar{\Gamma} \to G$. This is classically applied to lift projective representations $\rho : \Gamma \to \mathrm{PGL}_n$ to ordinary representations $\bar{\rho}:\bar{\Gamma}\to \mathrm{GL}_n$. When applied to $(\varepsilon')=(\varepsilon)$ one obtains an isomorphism

$$\mathrm{Aut}(\Gamma) \underset{\alpha\leftrightarrow\bar{\alpha}}{\simeq} \mathrm{Aut}(\bar{\Gamma}, C)$$

where the right hand group is the stabilizer of C in $\mathrm{Aut}(\bar{\Gamma})$. This leads to a commutative diagram

$$\begin{array}{ccccccc} \bar{\Gamma} & \xrightarrow{\text{ad}} & \mathrm{Aut}(\bar{\Gamma}, C) & \longrightarrow & \mathrm{Out}(\Gamma, C) & \longrightarrow & 1 \\ \downarrow & & \wr\Vert & & \wr\Vert & & \\ \Gamma & \longrightarrow & \mathrm{Aut}(\Gamma) & \longrightarrow & \mathrm{Out}(\Gamma) & \longrightarrow & 1 \end{array}$$

If C is characteristic in $\bar{\Gamma}$, e.g. if Γ is centerless, so that C is the center of $\bar{\Gamma}$, then $\mathrm{Aut}(\bar{\Gamma}, C)=\mathrm{Aut}(\bar{\Gamma})$ and $\mathrm{Out}(\bar{\Gamma}, C)=\mathrm{Out}(\bar{\Gamma})$. For automorphism questions therefore, one can work with Γ or $\bar{\Gamma}$, whichever is more convenient.

3. Automorphisms and representations; groups of type (TI)

(3.1) Proposition. *Let Γ be a finitely generated group and let $x, y\in\Gamma$. The following conditions are equivalent:*

(a) *x and y become conjugate in every finite quotient of Γ.*

(b) *x and y become conjugate in $\hat{\Gamma}$.*

(c) *There is an algebraically closed field F of characteristic zero such that for every representation $\rho:\Gamma\to\mathrm{GL}_n(F)$ we have $\chi_\rho(x)=\chi_\rho(y)$. (By definition, $\chi_\rho(z)=\mathrm{Tr}(\rho(z))$ for $z\in\Gamma$.)*

(d) *For every commutative ring k and every representation $\rho:\Gamma\to\mathrm{GL}_n(k)$ we have $\chi_\rho(x)=\chi_\rho(y)$.*

(a) $\Rightarrow$ (b): For each finite quotient Γ/N of Γ put $S(N)=\{z\in\Gamma/N \mid zx_Nz^{-1}=y_N\}$ (where $x_N=x \bmod N$, etc.). Then $S=\varprojlim_N S(N)$, being a projective limit of non-empty finite sets, is a non-empty set in $\hat{\Gamma}$. If $z\in S$ then z conjugates the image of x in $\hat{\Gamma}$ into that of y.

(b) $\Rightarrow$ (a): Trivial.

(c) $\Rightarrow$ (a): This follows because, over a field F as in (c), the characters of finite dimensional representations separate the conjugacy classes in any finite group.

(d) $\Rightarrow$ (c): Trivial.

(a) $\Rightarrow$ (d): Suppose $\rho:\Gamma\to\mathrm{GL}_n(k)$ and $a=\chi_\rho(x)-\chi_\rho(y)$ is not zero. Since Γ is finitely generated $\rho(\Gamma)\subset\mathrm{GL}_n(A)$ for some finitely generated ring $A\subset k$. Such an A is residually finite, so we can find an ideal J of finite index in A such that $a \notin J$. Then the composite $\Gamma \xrightarrow{\rho} \mathrm{GL}_n(A) \to \mathrm{GL}_n(A/J)$ is a representation σ such that $\chi_\sigma(x) \neq \chi_\sigma(y)$, and so x and y are not conjugate in the finite quotient $\sigma(\Gamma)$ of Γ; this contradicts (a).

Without assuming Γ finitely generated the implications (a) $\Leftrightarrow$ (b) $\Leftarrow$ (c) $\Leftarrow$ (d) remain valid.

(3.2) DEFINITION. Let Γ be a group, and let $x, y \in \Gamma$. We call x and y *trace equivalent* in Γ, and write $x \underset{T}{\sim} y$, if they satisfy condition (d), and hence all of the conditions, of (3.1). We put

$$\mathrm{T\ Aut}(\Gamma) = \{\alpha \in \mathrm{Aut}(\Gamma) \mid \alpha(x) \underset{T}{\sim} x \quad \forall x \in \Gamma\}.$$

This is a normal subgroup of Aut (Γ) containing I Aut (Γ). We say that Γ is of *type* (*TI*) if T Aut (Γ) = I Aut (Γ).

Call $\alpha \in \mathrm{Aut}(\Gamma)$ *residually inner* if $\hat{\alpha} \in \mathrm{I\,Aut}(\hat{\Gamma})$. When Γ is finitely generated this is equivalent to saying that α induces an inner automorphism of every finite characteristic quotient group of Γ. If RI Aut (Γ) denotes the group of residually inner automorphisms then evidently

$$\mathrm{I\ Aut}(\Gamma) \subseteq \mathrm{RI\ Aut}(\Gamma) \subseteq \mathrm{T\ Aut}(\Gamma) \tag{1}$$

and

$$\frac{\mathrm{RI\ Aut}(\Gamma)}{\mathrm{I\ Aut}(\Gamma)} = \mathrm{Ker}\,(\mathrm{Out}\,(\Gamma) \to \mathrm{Out}^c(\hat{\Gamma})). \tag{2}$$

(3.3) COROLLARY. *Let Γ be a finitely generated group and let F be an algebraically closed field of characteristic zero. The following conditions are equivalent*:

(a) Γ *is of type* (TI).

(b) *In order that $\alpha \in \mathrm{Aut}(\Gamma)$ be inner it suffices that $\rho \circ \alpha \cong \rho$ for every irreducible representation $\rho : \Gamma \to \mathrm{GL}_n(F)$.*

(c) *In order that $\alpha \in \mathrm{Aut}(\Gamma)$ be inner it suffices that $\chi_\rho \circ \alpha = \chi_\rho$ for every representation $\rho : \Gamma \to \mathrm{GL}_n(F)$.*

The equivalence of (b) and (c) follows by considering Jordan–Holder series and noting that an irreducible representation is determined up to isomorphism by its character.

The equivalence of (a) and (c) is immediate from Proposition (3.1) and Definition (3.2).

Groups of type (TI) have the following agreeable properties.

(3.4) COROLLARY. *Let* Γ *be a finitely generated group of type* (TI).

(1) $\mathrm{Out}(\Gamma) \to \mathrm{Out}^c(\hat{\Gamma})$ *is injective and (therefore)* $\mathrm{Out}(\Gamma)$ *is residually finite. If* Γ *is residually finite then*

$$N_{\hat{\Gamma}}(\Gamma) = \Gamma \cdot Z_{\hat{\Gamma}}(\Gamma). \tag{3}$$

(Here N and Z stand for normalizer and centralizer, respectively.)

(2) *Let k be a commutative ring containing* $\mathbf{Z}$ *and let* $(k\Gamma)^\times$ *denote the group of units in the group algebra* $k\Gamma$. *Then* $\mathrm{Out}(\Gamma) \to \mathrm{Out}_{k\text{-alg}}(k\Gamma)$ *is injective; equivalently*

$$N_{(k\Gamma)^\times}(\Gamma) = \Gamma \cdot Z(k\Gamma)^\times. \tag{4}$$

In (1) the injectivity assertion follows from the discussion after Definition (3.2), (1) and (2), and the normalizer formula (3) is just a translation of this injectivity. The residual finiteness claim follows from (2.1).

To prove (2), suppose that $\alpha \in \mathrm{Aut}(\Gamma)$ becomes inner in $k\Gamma$, i.e. that $\alpha(x) = uxu^{-1}$ for all $x \in \Gamma$ and some $u \in (k\Gamma)^\times$. Since $\mathbf{Z} \subset k$ we have $\mathbf{Q} \otimes_{\mathbf{Z}} k \neq 0$, and so k admits a homomorphism to a field F of characteristic zero, which we may enlarge to be algebraically closed. Making the base change $k \to F$ we see that α again becomes inner in $F\Gamma$. It follows now by (c) of Corollary (3.3) that α is inner.

We can now reformulate E. Grossman's results [3].

(3.5) PROPOSITION. *Let* Γ *be a finitely generated group satisfying* (a) *and* (b) *of Proposition* (2.3). *Then* Γ *is of type* (TI). *In particular this is so when* Γ *is a free group or a surface group (Examples* (2.4) (1) *and* (2)).

Let $\alpha \in \mathrm{T\,Aut}(\Gamma)$. Then for all $x \in \Gamma$, $\alpha(x) \sim x$ in $\hat{\Gamma}$, so $\alpha(x) \sim x$ in Γ, by (a), so α is inner by (b).

In order to verify property (TI) we are led to consider stabilizers of characters. Explicitly, let Γ be a group. For each irreducible representation $\rho : \Gamma \to \mathrm{GL}_n(\mathbf{C})$ put

$$\begin{aligned} \mathrm{Aut}(\Gamma)_{(\rho)} &= \{\alpha \in \mathrm{Aut}(\Gamma) \mid \rho \circ \alpha \cong \rho\} \\ &= \{\alpha \in \mathrm{Aut}(\Gamma) \mid \chi_\rho \circ \alpha = \chi_\rho\}. \end{aligned} \tag{5}$$

It follows from (3.3) that, when Γ is finitely generated,

$$\mathrm{T\,Aut}(\Gamma) = \bigcap_\rho \mathrm{Aut}(\Gamma)_{(\rho)} \tag{6}$$

where ρ varies over all complex irreducible representations of Γ. Therefore to show that Γ is of type (TI) it would suffice, for example, to show that $\mathrm{Aut}(\Gamma)_{(\rho)} = \mathrm{I\,Aut}(\Gamma)$ for some ρ as above. (See (3.6) below.)

Let $\mathrm{Irr}_n(\Gamma)$ denote the set of isomorphism classes (ρ) of irreducible representations $\rho : \Gamma \to \mathrm{GL}_n(\mathbf{C})$. Then

(7)
$$\begin{gathered}\mathrm{Out}(\Gamma) \text{ acts on } \mathrm{Irr}_n(\Gamma) \text{ with stabilizers}\\ \mathrm{Out}(\Gamma)_{(\rho)} = \mathrm{Aut}(\Gamma)_{(\rho)}/\mathrm{I\,Aut}(\Gamma)\end{gathered}$$

and, when Γ is finitely generated,

(8)
$$\begin{gathered}\Gamma \text{ is of type (TI) iff } \mathrm{Out}(\Gamma) \text{ acts faithfully}\\ \text{on } \coprod_{n \geqq 1} \mathrm{Irr}_n(\Gamma) \text{ (disjoint union).}\end{gathered}$$

For $n = 1$ we have the group of linear characters

$$X(\Gamma) = \mathrm{Irr}_1(\Gamma) = \mathrm{Hom}(\Gamma, \mathbf{C}^\times)$$

which acts on $\mathrm{Hom}(\Gamma, \mathrm{GL}_n(\mathbf{C}))$ by tensor product: for $\rho \in \mathrm{Hom}(\Gamma, \mathrm{GL}_n(\mathbf{C}))$ and $\chi \in X(\Gamma)$ we have $\rho \otimes \chi = \rho \cdot \chi$ defined by

$$(\rho \otimes \chi)(x) = \rho(x)\chi(x).$$

Further $\mathrm{Aut}(\Gamma)$ acts on $X(\Gamma)$ and on $\mathrm{Hom}(\Gamma, \mathrm{GL}_n(\mathbf{C}))$, so we can form the semi-direct product,

(9)
$$\begin{gathered}\mathrm{Aut}(\Gamma) \ltimes X(\Gamma)\\ (\alpha, \chi) \cdot (\alpha', \chi') = (\alpha \circ \alpha', (\chi \circ \alpha') \cdot \chi')\end{gathered}$$

which acts on $\mathrm{Hom}(\Gamma, \mathrm{GL}_n(\mathbf{C}))$ by

(10)
$$\rho \cdot (\alpha, \chi) = (\rho \circ \alpha) \cdot \chi \qquad (= (\rho \circ \alpha) \otimes \chi).$$

We can thus form the stabilizer

(11)
$$(\mathrm{Aut}(\Gamma) \ltimes X(\Gamma))_{(\rho)} = \{(\alpha, \chi) \mid (\rho \circ \alpha) \otimes \chi \cong \rho\},$$

which contains $\mathrm{Aut}(\Gamma)_{(\rho)}$ (as those (α, χ) for which $\chi = 1$).

(3.6) Proposition. *Let Γ be a group and $\rho : \Gamma \to \mathrm{GL}_n(\mathbf{C})$ an irreducible representation. There is a homomorphism π_ρ making the following diagram commute*

(12)
$$\begin{array}{ccc} \Gamma & \xrightarrow{\quad\rho\quad} & \mathrm{GL}_n(\mathbf{C}) \\ {\scriptstyle \mathrm{ad}}\downarrow & & \downarrow{\scriptstyle \mathrm{ad}} \\ \mathrm{Aut}(\Gamma)_{(\rho)} & & \\ \cap & & \\ (\mathrm{Aut}(\Gamma) \ltimes X(\Gamma))_{(\rho)} & \xrightarrow[\quad\pi_\rho\quad]{} & \mathrm{PGL}_n(\mathbf{C}). \end{array}$$

we have

(13)
$$\pi_\rho(\mathrm{Aut}(\Gamma)_{(\rho)}) \subset \mathrm{ad}(N_{\mathrm{GL}_n(\mathbf{C})}(\rho\Gamma))$$

and

(14)
$$\pi_\rho((\mathrm{Aut}(\Gamma) \ltimes X(\Gamma))_{(\rho)}) \subset N_{\mathrm{PGL}_n(\mathbf{C})}(\mathrm{ad}\,\rho\Gamma).$$

If ρ is injective then π_ρ maps $\mathrm{Aut}(\Gamma)_{(\rho)}$ *isomorphically onto* $\mathrm{ad}(N_{\mathrm{GL}_n(\mathbf{C})}(\rho\Gamma))$. *If* $\mathrm{ad}\circ\rho$ *is injective, i.e. if ρ is injective and Γ is centerless, then π_ρ maps* $(\mathrm{Aut}(\Gamma) \ltimes X(\Gamma))_{(\rho)}$ *isomorphically onto* $N_{\mathrm{PGL}_n(\mathbf{C})}(\mathrm{ad}\,\rho\Gamma)$.

(3.7) COROLLARY. *Assume that Γ is finitely generated. Let $\rho : \Gamma \to \mathrm{GL}_n(\mathbf{C})$ be a faithful irreducible representation. If*

$$N_{\mathrm{GL}_n(\mathbf{C})}(\rho\Gamma) = (\rho\Gamma)\cdot \mathbf{C}^\times,$$

for example if

$$N_{\mathrm{PGL}_n(\mathbf{C})}(\mathrm{ad}\,\rho\Gamma) = \mathrm{ad}\,\rho\Gamma$$

then Γ is of type (TI).

Indeed it follows then from (3.6) that $\mathrm{Aut}(\Gamma)_{(\rho)} = \mathrm{ad}(\Gamma)$.

One should be able to use this Corollary to give direct proofs of E. Grossman's result (Proposition (3.5)) that free groups and surface groups are of type (TI).

PROOF OF (3.6). Let (α, χ) satisfy $(\rho\circ\alpha)\cdot\chi \cong \rho$. Then there is a $\sigma \in \mathrm{GL}_n(\mathbf{C})$ such that

(15)
$$\sigma\rho(x)\sigma^{-1} = \rho(\alpha x)\cdot\chi(x) \qquad (x \in \Gamma).$$

Since ρ is irreducible,

(16)
$$\pi_\rho(\alpha, \chi) = \mathrm{ad}(\sigma) \in \mathrm{PGL}_n(\mathbf{C})$$

is well defined. A straightforward calculation shows that π_ρ is a homomorphism.

If $\alpha = \mathrm{ad}(x)$ for some $x \in \Gamma$ and $\chi = 1$ then $\rho(\alpha(y)) = \rho(x)\rho(y)\rho(x)^{-1}$, so we can choose $\sigma = \rho(x)$ above, whence $\pi_\rho(\mathrm{ad}(x), 1) = \mathrm{ad}(\rho(x))$, which expresses the commutativity of (12). The inclusions (13) and (14) are immediately seen from (15) and (16).

Suppose that ρ is injective. Then if $\sigma \in N_{\mathrm{GL}_n(\mathbf{C})}(\rho\Gamma)$ there is a unique $\alpha \in \mathrm{Aut}(\Gamma)$ satisfying (15) with $\chi = 1$, so $\pi_\rho : \mathrm{Aut}(\Gamma)_{(\rho)} \to \mathrm{ad}(N_{\mathrm{GL}_n(\mathbf{C})}(\rho\Gamma))$ is an isomorphism.

Suppose further that Γ is centerless. If (15) holds with σ a scalar then $\chi(x) = \rho(\alpha x)^{-1}\rho(x)$ is a scalar in $\rho(\Gamma)$ so $\chi = 1$, and so also $\alpha = 1$. This shows that π_ρ is injective. Suppose that $\sigma \in \mathrm{GL}_n(\mathbf{C})$ and that $\mathrm{ad}(\sigma)$ normalizes $\mathrm{ad}\,\rho(\Gamma)$. Since $\mathrm{ad} \circ \rho$ is injective we can define $\alpha \in \mathrm{Aut}(\Gamma)$ by $\mathrm{ad}(\sigma)\,\mathrm{ad}\,\rho(x)\,\mathrm{ad}(\sigma)^{-1} = \mathrm{ad}\,\rho(\alpha x)$ for $x \in \Gamma$. We can then define χ by $\sigma\rho(x)\sigma^{-1} = \rho(\alpha x) \cdot \chi(x)$. It is clear then that $\chi \in X(\Gamma)$, $(\alpha, \chi) \in (\mathrm{Aut}(\Gamma) \ltimes X(\Gamma))_{(\rho)}$, and $\pi_\rho(\alpha, \chi) = \mathrm{ad}(\sigma)$. This proves the Proposition.

We conclude this section by relating trace preserving automorphisms to "finite normal" automorphisms.

(3.8) Definition. An automorphism α of a group Γ is called *finite normal* if $\alpha(N) = N$ for all normal subgroups N of finite index in Γ. When Γ is profinite and α is continuous we require this only for open normal subgroups N; it then follows for all closed normal subgroups as well, the latter being intersections of open ones.

(3.9) Proposition. *Let Γ be a group, $\alpha \in \mathrm{Aut}(\Gamma)$, and $\hat{\alpha} \in \mathrm{Aut}^c(\hat{\Gamma})$ its profinite completion. Then α is finite normal iff $\hat{\alpha}$ is finite normal. If $\alpha \in \mathrm{T\,Aut}(\Gamma)$ (α is trace preserving) then α is finite normal.*

The first assertion follows from the bijective correspondence $N \mapsto \hat{N}$ between normal subgroups of finite index in Γ, and open normal subgroups in $\hat{\Gamma}$. If $\alpha \in \mathrm{T\,Aut}(\Gamma)$ then, by Proposition (3.1), $\hat{\alpha}$ preserves $\hat{\Gamma}$-conjugacy classes that meet Γ, and so $\hat{\alpha}$ is finite normal, again because of the above correspondence.

(3.10) Theorem (Jarden and Ritter [7], theorem B). *Let Γ be a group presented with d generators and r relations with $d \geqq r + 2$. Then every finite normal continuous automorphism of $\hat{\Gamma}$ is inner.*

(3.11) Corollary. *Let Γ be as in* (3.9). *If Γ is of type* (TI) *then every finite normal automorphism of Γ is inner.*

Let $\alpha \in \mathrm{Aut}(\Gamma)$ be finite normal. Theorem (3.10) and Proposition (3.9) imply then that $\hat{\alpha}$ is inner. Since Γ is of type (TI), $\mathrm{Out}(\Gamma)$ embeds in $\mathrm{Out}(\hat{\Gamma})$ ((3.4)(1)). Hence α is inner.

Corollary (3.11) applies in particular when Γ is a free group F_d in $d \geqq 2$ generators (this is theorem 1 of [8]), or a surface group Γ_q of genus $q \geqq 2$. These groups are of type (TI) by Proposition (3.5). For Γ_q we have $d - r = 2q - 1 \geqq 2$ when $q \geqq 2$. (In case $q = 1$, $\Gamma_1 \cong Z_2$ and the finite normal automorphisms are $\pm$ Id.) In the surface group case we can use the Nielsen isomorphism of $\mathrm{Out}(\Gamma_q)$ with the mapping class group [1] to reformulate Corollary (3.11) geometrically as follows.

(3.12) COROLLARY. *Let S be a compact orientable surface of genus $q \geqq 2$, and let $\alpha : S \to S$ be a homeomorphism. If α can be lifted to every finite normal covering of S then α is isotopic to the identity.*

4. Varieties of representations

Let Γ be a group generated by a finite set S. Let G be an affine algebraic group (over $\mathbf{C}$). Then

$$R(\Gamma, G) = \mathrm{Hom}(\Gamma, G)$$

can be identified with a closed subvariety of G^S. Let $A(\Gamma, G)$ denote the corresponding affine algebra.

The group $\mathrm{Aut}(G) \times \mathrm{Aut}(\Gamma)$ acts on $R(\Gamma, G)$, hence also on $A(\Gamma, G)$, in the obvious way. In particular G acts by conjugation, via $G \xrightarrow{\mathrm{ad}} \mathrm{Aut}(G)$. Let

$$C(\Gamma, G) = A(\Gamma, G)^G,$$

the ring of G-invariants. Assume now that G is reductive. Then $C(\Gamma, G)$ is a finitely generated $\mathbf{C}$-algebra by a classical theorem of Hilbert–Weyl–Mumford (see [13], theorem 1). Hence

$$S(\Gamma, G) = \mathrm{Spec}(C(\Gamma, G))$$

is an "approximate" affine quotient of $R(\Gamma, G)$ by G. The action of $\mathrm{Aut}(\Gamma)$ on $R(\Gamma, G)$ induces an action of $\mathrm{Out}(\Gamma)$ on $S(\Gamma, G)$, or, equivalently, on $C(\Gamma, G)$. We may apply the results of section 1 to these actions.

(4.1) EXAMPLE. Let $G = \mathrm{PGL}_2$. In $R(\Gamma, G)$ consider the subset

$$R_0(\Gamma, G) = \left\{ \rho \in R(\Gamma, G \;\middle|\; \begin{array}{l} \rho \text{ is injective} \\ \rho(\Gamma) \text{ is a discrete subgroup of } \mathrm{PSL}_2(\mathbf{R}) \\ \mathrm{PSL}_2(\mathbf{R})/\rho(\Gamma) \text{ is compact.} \end{array} \right\}$$

We assume that $R_0(\Gamma, G) \neq \emptyset$, so that Γ is a Fuchsian group which admits a presentation of the form

$$\Gamma = \left\langle a_1, b_1, \cdots, a_g, b_g, c_1, \cdots, c_m \;\middle|\; \left(\prod_i [a_i, b_i]\right)\left(\prod_j c_j\right) = c_1^{e_1} = \cdots = c_m^{e_m} = 1 \right\rangle$$

where the e_j are positive integers, g is the genus, and $(g; e_1, \cdots, e_m)$ is called the *signature* of Γ. One has $m = 0$ if and only if Γ is torsion free, in which case Γ is called a *surface group* of genus g. Our hypothesis makes Γ centerless, so we can identify Γ with a subgroup of $\mathrm{Aut}(\Gamma)$.

The natural projection $R(\Gamma, G) \to S(\Gamma, G)$ sends $R_0(\Gamma, G)$ onto the set $S_0(\Gamma, G)$ of G-conjugacy classes of $R_0(\Gamma, G)$. Since $R_0(\Gamma, G)$ is (clearly) $\mathrm{Aut}(\Gamma)$-invariant, $S_0(\Gamma, G)$ is $\mathrm{Out}(\Gamma)$-invariant.

(4.2) THEOREM (Macbeath and Singerman, [10], theorem (9.15)). *Under the assumptions of* (4.1) *the kernel* Γ_1/Γ *of the action of* $\mathrm{Out}(\Gamma)$ *on* $S_0(\Gamma, G)$ *is finite, and even trivial except in the following cases:*

Signature of Γ	*Signature of* Γ_1	$[\Gamma_1 : \Gamma]$
$(2; -)$	$(0; 2, 2, 2, 2, 2, 2)$	2
$(1; e, e)$	$(0; 2, 2, 2, 2, e)$	2
$(1; e)$	$(0; 2, 2, 2, 2e)$	2
$(0; e, e, e, e)$ $(e \geqq 3)$	$(0; 2, 2, 2, e)$	4
$(0; e, e, f, f)$ $(\max(e, f) \geqq 3)$	$(0; 2, 2, e, f)$	2
$(0; e, e, e)$ $(e \geqq 4)$	$(0; 3, 3, e)$	3
$(0; e; e, e)$ $(e \geqq 4)$	$(0; 2, 3, 2e)$	6
$(0; e, e, f)$ $(e \geqq 3, e + f \geqq 7)$	$(0; 2, e, 2f)$	2

(4.3) THEOREM. *Let* Γ *be a surface group of genus* g. *Then the "mapping class group"* $\mathrm{Out}(\Gamma)$ *is residually finite and virtually torsion free.*

For $g = 1$, $\Gamma \cong \mathbf{Z}^2$, $\mathrm{Out}(\Gamma) \cong \mathrm{GL}_2(\mathbf{Z})$, and the result is clear. Suppose that $g \geqq 2$. Then we conclude from (4.2) that $\mathrm{Aut}(\Gamma)/\Gamma_1$ acts faithfully on the affine variety $S(\Gamma, G)$. Moreover $\mathrm{Aut}(\Gamma)$ is finitely generated, in fact finitely presented (cf. [1], theorem (5.1)), so it follows from Corollary (1.2) that $\mathrm{Aut}(\Gamma)/\Gamma_1$ is residually finite and virtually torsion free. Since $\Gamma_1 = \Gamma$ for $g \geqq 3$ the proof is complete in this case. For $g = 2$ we can appeal to Edna Grossman [3] (cf. Example (2.4) (2) above) for the residual finiteness of $\mathrm{Out}(\Gamma)$. Since Γ_1/Γ has order 2 we conclude from the following lemma that $\mathrm{Out}(\Gamma)$ is virtually torsion free.

(4.4) LEMMA. *Let Γ be a residually finite group, and N a finite normal subgroup. If Γ/N is virtually torsion free so also is Γ.*

In fact we can choose a normal subgroup H of finite index in Γ such that $N \cap H = \{1\}$. Then H embeds in Γ/N so H, and hence also Γ, is virtually torsion free.

REMARKS. (1) It is known that, when Γ is a surface group of genus $g \geqq 2$, every $\rho \in R_0(\Gamma, G)$ lifts to a homomorphism $\tilde{\rho}: \Gamma \to \mathrm{SL}_2(R)$, uniquely up to multiplication by one of the 2^{2g} characters in $\mathrm{Hom}(\Gamma, \{\pm I\})$. This is proved, for example, in S.J. Patterson, *On the cohomology of Fuchsian groups*, Glasg. Math. J. **16** (1975), 123–140.

(2) The fact that $\mathrm{Out}(\Gamma)$ as above is virtually torsion free can also be proved by Teichmüller theory, as L. Bers pointed out to us. By a theorem of Nielsen, every element of finite order in $\mathrm{Out}(\Gamma)$ has a fixed point in $S_0(\Gamma, G)$. Thus it suffices to produce a Γ_1 of finite index in $\mathrm{Out}(\Gamma)$ that acts freely on $S_0(\Gamma, G)$. We take the subgroup Γ_1 that acts trivially on $H_1(\Gamma, Z/3, Z)$. If $s \in \Gamma_1$ fixes $(\rho) \in S_0(\Gamma, G)$ then s defines an automorphism of the corresponding Riemann surface Σ_ρ, and s acts trivially on the elements of order 3 in its Jacobian. By a well-known lemma of Serre (Sém. H. Cartan **13** (1960/61), Appendix of the exposé of Grothendieck, pp. 17-18 to 17-20) such an s must be the identity.

Now let $G = \mathrm{GL}_n$. Then we shall use the abbreviations

$$\begin{aligned} R_n(\Gamma) &= R(\Gamma, \mathrm{GL}_n), \\ A_n(\Gamma) &= A(\Gamma, \mathrm{GL}_n), \\ C_n(\Gamma) &= C(\Gamma, \mathrm{GL}_n) = A_n(\Gamma)^{\mathrm{GL}_n}, \\ S_n(\Gamma) &= S(\Gamma, \mathrm{GL}_n) = \mathrm{Spec}(C_n(\Gamma)). \end{aligned} \tag{1}$$

In this case we have the following basic results.

For each $t \in \Gamma$ define the "character" $\chi(t) \in C_n(\Gamma)$ by

$$\chi(t): \rho \mapsto \chi_\rho(t) = \mathrm{Tr}(\rho(t)) \tag{2}$$

for $\rho \in R_n(\Gamma)$.

(4.5) THEOREM (Procesi, Sibirskii, Resmyslov).

(1) ([15], theorem 3.4) *$C_n(\Gamma)$ is generated, as a* **C**-*algebra by the characters $\chi(t)$, $t \in \Gamma$. In fact, if $\Gamma = \langle S \rangle$ it suffices to restrict t to elements of length $2^n - 1$ in the generators S.*

(2) ([14], theorem 4.1) $S_n(\Gamma)$ *parametrizes the isomorphism classes* (ρ) *of semi-simple representations* $\rho \in R_n(\Gamma)$.

(3) ([14], prop. 5.9) *The set* $\mathrm{Irr}_n(\Gamma)$ *of classes of irreducible representations* $\rho \in R_n(\Gamma)$ *is open in* $S_n(\Gamma)$.

(4.6) Proposition. *The kernel of the action of* $\mathrm{Out}(\Gamma)$ *on* $S_n(\Gamma)$ *is* $\mathrm{T}_n\ \mathrm{Aut}(\Gamma)/\mathrm{I}\ \mathrm{Aut}(\Gamma)$, *where*

$$\mathrm{T}_n\ \mathrm{Aut}(\Gamma) = \{\alpha \in \mathrm{Aut}(\Gamma) \mid \chi_\rho \circ \alpha = \chi_\rho \quad \text{for all } \rho \in R_n(\Gamma)\}.$$

We have $\mathrm{T}_n\ \mathrm{Aut}(\Gamma) \supset \mathrm{T}_{n+1}\ \mathrm{Aut}(\Gamma)$, *and* $\mathrm{T}\ \mathrm{Aut}(\Gamma) = \bigcap_{n \geq 1} \mathrm{T}_n\ \mathrm{Aut}(\Gamma)$. *Every finitely generated subgroup of* $\mathrm{Aut}(\Gamma)/\mathrm{T}_n\ \mathrm{Aut}(\Gamma)$ *is residually finite and virtually torsion free. Every finitely generated subgroup of* $\mathrm{Aut}(\Gamma)/\mathrm{T}\ \mathrm{Aut}(\Gamma)$ *is residually finite.*

The first assertions are obvious. The last assertion follows from the preceding ones. The main assertion, about subgroups of $\mathrm{Aut}(\Gamma)/\mathrm{T}_n\ \mathrm{Aut}(\Gamma)$, follows from the corresponding property of $\mathrm{Aut}(S_n(\Gamma))$, contained in Corollary (1.2).

(4.7) Remarks. We can similarly study $\mathrm{Aut}(\Gamma)$ itself via its action on $R_n(\Gamma)$. When this action is faithful then, again from (1.2), we conclude that finitely generated subgroups of $\mathrm{Aut}(\Gamma)$ are residually finite and virtually torsion free. In order for $\mathrm{Aut}(\Gamma)$ to act faithfully on $R_n(\Gamma)$ it suffices that Γ have a faithful representation $\rho \in R_n(\Gamma)$, or, more generally, that, given $x \neq 1$ in Γ, there is a $\rho \in R_n(\Gamma)$ such that $\rho(x) \neq 1$.

5. Schemes of representations

In order to obtain the results of (4.6) and (4.7) for the full automorphism groups, in place of their finitely generated subgroups, one would like to invoke Theorem (1.1) in place of its Corollary (1.2). In order to justify this we must realize the varieties used as schemes of finite type over **Z**.

Let the group Γ have a presentation $\langle S \mid W \rangle$ where S is a finite generating set, W is a subset of the free group based on S, and $w(S) = 1$ $(w \in W)$ are defining relations among the generators.

Let G be an affine group scheme of finite type over **Z** (e.g. GL_n). Then $R(\Gamma, G) =: \mathrm{Hom}(\Gamma, G)$ can be identified with the **Z**-subscheme of G^S consisting of all $\rho : S \to G$ such that

$$w(\rho(S)) = 1 \qquad \text{in } G \quad \text{for all } w \in W. \tag{1}$$

Note that, indeed, the equations (1) are defined over **Z**. We have

$$R(\Gamma, G) = \mathrm{Spec}(A(\Gamma, G))$$

where $A(\Gamma, G)$ is the quotient of the S-fold tensor product of the affine algebra A_G of G by the ideal generated by elements arising from the equations (1).

For any commutative ring k we put

$$A(\Gamma, G)_k = k \otimes_{\mathbf{Z}} A(\Gamma, G),$$

$$R(\Gamma, G)_k = \mathrm{Spec}(A(\Gamma, G)_k).$$

When $k = \mathbf{C}$ we thus recover the objects discussed in the preceding section.

The formation of $R(\Gamma, G)$ is functorial, contravariantly in Γ and covariantly in G (and the reverse for $A(\Gamma, G)$). In particular $\mathrm{Aut}(\Gamma) \times \mathrm{Aut}(G)$ acts on $R(\Gamma, G)$ and on $A(\Gamma, G)$. More precisely, if k is a commutative ring and $\mathrm{Aut}(G)(k)$ denotes the automorphisms defined over k of G, then $\mathrm{Aut}(\Gamma) \times \mathrm{Aut}(G)(k)$ acts on $R(\Gamma, G)_k$ and on $A(\Gamma, G)_k$.

To apply Theorem (1.1) to the action of $\mathrm{Aut}(\Gamma)$ on $R(\Gamma, G)$ we are troubled by the fact that the latter is not necessarily flat over $\mathbf{Z}$. To correct this we introduce

$$\bar{A}(\Gamma, G) = A(\Gamma, G)/T$$

where T is the ideal of ($\mathbf{Z}$-)torsion elements in $A(\Gamma, G)$. Then the subscheme $\bar{R}(\Gamma, G) = \mathrm{Spec}(\bar{A}(\Gamma, G))$ of $R(\Gamma, G)$ is flat over $\mathbf{Z}$, evidently invariant under $\mathrm{Aut}(\Gamma) \times \mathrm{Aut}(G)$, and $\bar{R}(\Gamma, G)_k = R(\Gamma, G)_k$ for any $\mathbf{Q}$-algebra k. From Theorem (1.1) we have:

(5.1) Proposition. *If* $\mathrm{Aut}(\Gamma)$ *acts faithfully on* $\bar{R}(\Gamma, G)$ *then* $\mathrm{Aut}(\Gamma)$ *is residually finite and virtually torsion free. This action is faithful provided that, for each* $x \neq 1$ *in* Γ, *there is a homomorphism* $\rho : \Gamma \to G(\mathbf{C})$ *such that* $\rho(x) \neq 1$.

Only the last assertion needs verification. Let $\alpha \in \mathrm{Aut}(\Gamma)$, $\alpha \neq 1$. Then there is a $y \in \Gamma$ such that $x = y^{-1}\alpha(y) \neq 1$. Choose $\rho : \Gamma \to G(\mathbf{C})$ so that $\rho(x) \neq 1$. Then $\rho(y) \neq \rho(\alpha(y))$, so $\rho \in R(\Gamma, G)(\mathbf{C}) = \bar{R}(\Gamma, G)(\mathbf{C})$ is not fixed by α, whence α acts non-trivially on $\bar{R}(\Gamma, G)$.

(5.2) Corollary. *Let* Γ *be a finitely generated subgroup of* $\mathrm{GL}_n(\mathbf{C})$. *Then* $\mathrm{Aut}(\Gamma)$ *is residually finite and virtually torsion free.*

Remark. One can also deduce such properties of $\mathrm{Aut}(\Gamma)$ as follows. Lubotzky shows in [8] that if Γ is virtually a residually p-group (for some prime p) then the same is true of $\mathrm{Aut}(\Gamma)$. Now a finitely generated linear group (over $\mathbf{C}$) is easily seen to be virtually a residually p-group for at least two (in fact, all but finitely many) primes p, and any group with this property is easily seen to be virtually torsion free.

To similarly study Out (Γ) via its action on a quotient of $R(\Gamma, G)$, we specialize now to the case where $G = \mathrm{GL}_n$ or SL_n. We let GL_n act on G (in both cases) by conjugation. This action factors through PGL_n. Since $\mathrm{PGL}_n(\mathbf{Z})$ is Zariski dense in $\mathrm{PGL}_n(\mathbf{C})$, it suffices to use $\mathrm{GL}_n(\mathbf{Z})$ to determine fixed points under any algebraic action of GL_n that factors through PGL_n in characteristic zero. Put

$$C(\Gamma, G) = A(\Gamma, G)^{\mathrm{GL}_n(\mathbf{Z})}.$$

It follows from the above remarks that for any commutative ring k which is flat over $\mathbf{Z}$ (i.e. torsion free) the map

$$C(\Gamma, G)_k = k \otimes_{\mathbf{Z}} C(\Gamma, G) \to A(\Gamma, G)_k^{\mathrm{GL}_n(k)} \tag{2}$$

is an isomorphism, which we shall view as an identification. We also have

$$\bar{C}(\Gamma, G) =: \bar{A}(\Gamma, G)^{\mathrm{GL}_n(\mathbf{Z})} = \bar{A}(\Gamma, G) \cap C(\Gamma, G)_{\mathbf{C}}.$$

For each $t \in \Gamma$ we have its "character"

$$\chi^G(t) \in C(\Gamma, G)$$

defined by

$$\chi^G(t): \rho \mapsto \chi_\rho(t) = \mathrm{Tr}(\rho(t)) \qquad \text{for } \rho \in R(\Gamma, G).$$

Let $\bar{\chi}^G(t)$ denote its image in $\bar{C}(\Gamma, G)$. Write

$\mathrm{Ch}(\Gamma, G) =$ the subring of $C(\Gamma, G)$ generated by all $\chi^G(t) \quad (t \in \Gamma)$,

$\overline{\mathrm{Ch}}(\Gamma, G) =$ the subring of $\bar{C}(\Gamma, G)$ generated by all $\bar{\chi}^G(t) \quad (t \in \Gamma)$.

A group homomorphism $\varphi: \Gamma \to \Gamma'$ induces a homomorphism $\varphi^*: \mathrm{Ch}(\Gamma, G) \to \mathrm{Ch}(\Gamma', G)$ carrying $\chi^G(t)$ to $\chi^G(\varphi t)$, and similarly for $\overline{\mathrm{Ch}}$. Consequently φ^* is surjective whenever φ is. Further this shows that $\mathrm{Ch}(\Gamma, G)$ and $\overline{\mathrm{Ch}}(\Gamma, G)$ are invariant under $\mathrm{Aut}(\Gamma)$.

It follows from Theorem (4.5) (1) that the commutative diagram

$$\begin{array}{ccc} \mathrm{Ch}(\Gamma, G) & \subset & C(\Gamma, G) \\ \downarrow & & \downarrow \\ \overline{\mathrm{Ch}}(\Gamma, G) & \subset & \bar{C}(\Gamma, G) \end{array}$$

becomes, after applying $\mathbf{C} \otimes_{\mathbf{Z}} -$, a square of isomorphisms of finitely generated $\mathbf{C}$-algebras.

The group Out(Γ) acts on both $\overline{\mathrm{Ch}}(\Gamma, G)$ and $\bar{C}(\Gamma, G)$ with the same kernel, $\mathrm{T}_G \operatorname{Aut}(\Gamma)/\mathrm{I} \operatorname{Aut}(\Gamma)$, where (in view of (4.5) (2))

$$(3) \qquad \mathrm{T}_G \operatorname{Aut}(\Gamma) = \{\alpha \in \operatorname{Aut}(\Gamma) \mid \chi_\rho \circ \alpha = \chi_\rho \quad \text{for all } \rho : \Gamma \to G(\mathbf{C})\}.$$

Thus we can deduce from Theorem (1.1):

(5.3) PROPOSITION. *Suppose that* $\mathrm{Ch}(\Gamma, G)$ *is a finitely generated* $\mathbf{Z}$*-algebra. Then* $\operatorname{Aut}(\Gamma)/\mathrm{T}_G \operatorname{Aut}(\Gamma)$ *is residually finite and virtually torsion free.*

Using functoriality in Γ, the finite generation of $\overline{\mathrm{Ch}}(\Gamma, G)$ will follow in general once it is shown when Γ is any free group.

(5.4) EXAMPLE. Let $G = \mathrm{SL}_2$. The ring $\overline{\mathrm{Ch}}(\Gamma, \mathrm{SL}_2)$ can be viewed as the $\mathbf{Z}$-algebra generated by the functions $\bar{\chi}(t) : \mathrm{Hom}(\Gamma, \mathrm{SL}_2(\mathbf{C})) \to \mathbf{C}$ $(t \in \Gamma)$, where $\bar{\chi}(t) : \rho \to \chi_\rho(t) = \mathrm{Tr}(\rho(t))$. This ring has been long studied, by Vogt [18] in 1889, and by Fricke [2] for Fuchsian groups Γ. Magnus [11] calls $\overline{\mathrm{Ch}}(\Gamma, \mathrm{SL}_2)$ the ring of "Fricke characters" of Γ, and proposes it as a natural tool for the study of Out(Γ), lamenting however ([11], p. 97) that "we do not know enough about automorphisms of rings."

Let Γ be a free group with basis $\{s_1, \cdots, s_d\}$. Then we have the following results.

(1) (Fricke [2] and Horowitz [5]). The ring $\overline{\mathrm{Ch}}(\Gamma, \mathrm{SL}_2)$ is finitely generated, in fact by the $2^d - 1$ elements $\bar{\chi}(t)$ where t is of the form $t = s_{i_1} \cdots s_{i_p}$, $1 \leqq i_1 < \cdots < i_p \leqq d$.

(2) (Horowitz [6]; cf. also Magnus [11]). The action of Out(Γ) on $\overline{\mathrm{Ch}}(\Gamma, \mathrm{SL}_2)$ is faithful for $d \geqq 3$. For $d \leqq 2$ the kernel of this action is generated by the class of the automorphism $\varepsilon : s_i \to s_i^{-1}$ for all i.

There are further results on the structure of $\overline{\mathrm{Ch}}(\Gamma, \mathrm{SL}_2)$ in Horowitz [6] and Whittemore [19], and on its full ring of fractions in Magnus [11].

Recall that there is a natural surjection $\mathrm{Out}(\Gamma) \to \mathrm{Aut}(\Gamma^{ab}) \cong \mathrm{GL}_d(\mathbf{Z})$, and that this is an isomorphism for $d \leqq 2$. Thus we conclude from the results above:

(5.5) COROLLARY. (a) *For any finitely generated group* Γ, $\overline{\mathrm{Ch}}(\Gamma, \mathrm{SL}_2)$ *is a finitely generated* $\mathbf{Z}$*-algebra.*

(b) *If* Γ *is a free group then* Out(Γ) *is residually finite and virtually torsion free.*

Of course (b) could also be deduced from (4.5) and (1.2) using the fact that Aut(Γ) is finitely generated.

When Γ is free on n generators there is an exact sequence

$$\begin{array}{c} 1 \to K \to \operatorname{Out}(\Gamma) \to \operatorname{Aut}(\Gamma^{ab}) \to 1 \\ \qquad\qquad\qquad\qquad\quad \wr\| \\ \qquad\qquad\qquad\qquad\quad \mathrm{GL}_n(\mathbf{Z}) \end{array}$$

and Baumslag–Taylor have shown that K is torsion free (cf. [9], ch. I, corollaries 4.12 and 4.13). This easily implies that $\operatorname{Out}(\Gamma)$ is virtually torsion free, since $\mathrm{GL}_n(\mathbf{Z})$ is so. Incidentally, the Baumslag–Taylor result is raised as an open problem in [11].

(5.6) REMARK. Let k be a commutative ring and R a finitely generated, not necessarily commutative k-algebra. One can study representations $\rho: R \to M_n(F)$ into $n \times n$ matrices over a commutative k-algebra F. These again form the F-valued points of an affine scheme $R_n(R)$ of finite type over k (cf. [14]). The group GL_n acts by conjugation giving an affine quotient $S_n(R)$ which, over a field F, parametrizes classes of semi-simple representations of R on F^n (loc. cit.). The group $\operatorname{Aut}(R)$ of k-algebra automorphisms of R acts on $R_n(R)$, and induces an action of $\operatorname{Out}(R) = \operatorname{Aut}(R)/\operatorname{ad}(R^\times)$ on $S_n(R)$. Even when R is a group algebra $\mathbf{Z}\Gamma$ these automorphism groups are much larger than the groups $\operatorname{Aut}(\Gamma)$ and $\operatorname{Out}(\Gamma)$. Still one can use the above methods to study them and draw conclusions, such as the following: If the finite-dimensional representations ρ as above separate points in R then every finitely generated subgroup of $\operatorname{Aut}(R)$ is residually finite. Indeed the hypothesis implies that $\operatorname{Aut}(R)$ acts faithfully on $\coprod_n R_n(R)$, so one can apply (1.2). There is no such natural condition for $\operatorname{Out}(R)$ to act faithfully on $\coprod_n S_n(R)$.

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