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DUNFORD–PETTIS OPERATORS ON L^1 AND THE RADON–NIKODYM PROPERTY

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ABSTRACT

Using the duality between Dunford–Pettis operators on L^1 and Pettis–Cauchy martingales, we prove that the Dunford–Pettis operators from L^1 into L^1 form a lattice. We show also that a Banach space X has the Radon–Nikodým property provided the Dunford–Pettis members of $\mathscr{L}(L^1, X)$ are representable. The lifting of dual valued Dunford–Pettis operators is investigated. Some remarks are included.

Introduction

$[0,1]$, m will be the Lebesgue space. For each $n \in \mathbf{N}$, we let Σ_n be the finite algebra of subsets of $[0,1]$ generated by the intervals $I_{n,k} = [(k-1)2^{-n}, k2^{-n}]$ where $k = 1, \cdots, 2^n$. We use the notation E_n for the conditional expactation with respect to Σ_n. Let X be a fixed Banach space. By X-valued martingale, we mean a sequence (ξ_n) in $L^1_X[0,1]$ so that $\xi_n = E_n[\xi_{n+1}]$. The martingale (ξ_n) is uniformly bounded provided $\sup_n \|\xi_n\|_\infty < \infty$.

It is well-known that the uniformly bounded X-valued martingales correspond to the operators $\mathscr{L}(L^1, X)$. This correspondence is obtained by taking

$$T(\varphi) = \lim_{n\to\infty} \int \xi_n(t)\varphi(t)dt \qquad \text{if } (\xi_n) \text{ is the martingale}$$

and

$$\xi_n(t) = 2^{-n} \sum_k h_{n,k}(t) T(h_{n,k}) \qquad \text{if } T : L^1 \to X \text{ is the operator}$$

(the Haarfunction $h_{n,k}$ is the normalized characteristic function of $I_{n,k}$).

For more details, we refer to [5].

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Dunford–Pettis operators and Pettis–Cauchy martingales

Let us recall that $T: Y \to X$ is a Dunford–Pettis (D–P) operators, provided T maps weakly convergent sequences on norm onvergent sequences. For $1 \leqq p \leqq \infty$, $i_p : L^p \to L^1$ will be the canonical injection. The next result is essentially known (cf. [7]), but we take it up for the sake of completeness.

PROPOSITION 1. *For an operator $T: L^1 \to X$, the following assertions are equivalent*:

(1) *T is a Dunford–Pettis operator,*

(2) *Ti_p is a compact operator for some $1 < p \leqq \infty$,*

(3) *Ti_∞ is a compact operator.*

PROOF. (1) ⇒ (2). Because the L^p-ball is weakly compact in L^1 for $p > 1$.

(2) ⇒ (3). Obvious.

(3) ⇒ (1). If T is not D–P, then there is a weakly nullsequence (φ_n) in L^1 and $\varepsilon > 0$ such that $\|T(\varphi_n)\| > \varepsilon$. Let $\delta = \frac{1}{4}\varepsilon \|T\|^{-1}$. By passing eventually to a subsequence, we may assume $\|E_n[\varphi_n]\|_1 < \delta$. Since (φ_n) is uniformly integrable, there is some constant $K > 0$ with $\|\varphi_n - \psi_n\|_1 < \delta$, taking $\psi_n(t) = \varphi_n(t)$ if $|\varphi_n(t)| \leqq K$ and $\psi_n(t) = 0$ otherwise. Define $\eta_n = \psi_n - E_n[\psi_n]$. It is clear that $\|\eta_n\|_\infty \leqq 2K$ and $\lim_{n\to\infty} \eta_n = 0$ $\sigma(L^\infty, L^1)$.

Now $\|\varphi_n - \eta_n\|_1 \leqq \|\varphi_n - \psi_n\|_1 + \|E_n[\psi_n]\|_1 \leqq 2\|\varphi_n - \psi_n\|_1 + \|E_n[\varphi_n]\|_1 < 3\delta$ and hence $\|T(\eta_n)\| > \varepsilon - 3\delta\|T\| = \varepsilon/4$. Since $\lim_{n\to\infty} i_\infty(\eta_n) = 0$ $\sigma(L^1, L^\infty)$, Ti_∞ is not compact.

If $\xi \in L^1_X$, we introduce its Pettis norm $|||\xi|||$ by

$$|||\xi||| = \sup\left\{ \int |\langle \xi(t), x^*\rangle| \, dt;\ x^* \in X^*, \|x^*\| \leqq 1 \right\}.$$

We say that an X-valued martingale (ξ_n) is Pettis–Cauchy, provided (ξ_n) is a Cauchy-sequence for the Pettis norm.

For our next purpose, we need the following property.

PROPOSITION 2. *The martingale (ξ_n) is Pettis–Cauchy if and only if $\lim_{n\to\infty} \|\int \xi_n \varphi_n\| = 0$, whenever (φ_n) is an L^∞-bounded weakly null sequence in L^1.*

PROOF. Suppose (ξ_n) not Pettis–Cauchy. Then there is $\delta > 0$ and an increasing sequence (n_k) so that $|||\xi_{n_k} - \xi_{n_{k-1}}||| > \delta$ for each k. Fix k. Let $x_k^* \in X^*$ satisfy $\|x_k^*\| = 1$ and $\int |\langle \xi_{n_k} - \xi_{n_{k-1}}, x_k^*\rangle| > \delta$. It is possible to find a Σ_{n_k}-measurable function ε_k such that $\|\varepsilon_k\|_\infty = 1$ and

$$|\langle \xi_{n_k} - \xi_{n_{k-1}}, x_k^*\rangle| = \langle \xi_{n_k} - \xi_{n_{k-1}}, x_k^*\rangle \,.\, \varepsilon_k.$$

Consequently $\|\int (\xi_{n_k} - \xi_{n_{k-1}}) . \varepsilon_k \| > \delta$. If we define $\varphi_k = \varepsilon_k - E_{n_{k-1}}[\varepsilon_k]$, then $\|\varphi_k\|_\infty \leqq 2$, $\lim_{k\to\infty} \varphi_k = 0$ weakly and $\|\int \xi_{n_k}\varphi_k\| > \delta$.

Let now conversely (φ_n) be a weakly null sequence in L^1 and $\delta > 0$ such that $\|\varphi_n\|_\infty \leqq 1$ and $\overline{\lim}_n \|\int \xi_n\varphi_n\| > \delta$. It is straightforward to find an increasing sequence (n_k) so that

$$\left\| \int \xi_{n_{k-1}}\varphi_{n_k} \right\| < \frac{\delta}{2} \quad \text{and} \quad \left\| \int \xi_{n_k}\varphi_{n_k} \right\| > \delta.$$

Thus for each k we have

$$\int |\langle \xi_{n_k} - \xi_{n_{k-1}}, x_k^* \rangle| \geqq \left\| \int (\xi_{n_k} - \xi_{n_{k-1}})\varphi_{n_k} \right\| > \frac{\delta}{2}, \qquad \text{for some } x_k^* \in X^*, \quad \|x_k^*\| = 1.$$

Consequently $|||\xi_{n_k} - \xi_{n_{k-1}}||| > \delta/2$ for all k and (ξ_n) is not Pettis–Cauchy. This completes the proof.

THEOREM 1. *A uniformly bounded X-valued martingale (ξ_n) is Pettis–Cauchy iff the corresponding operator $T: L^1 \to X$ is D–P.*

PROOF. (1) If T is not D–P, then by Proposition 1 we get an L^∞-bounded weakly null sequence (φ_n) in L^1 for which $\overline{\lim}_{n\to\infty} \|T(\varphi_n)\| > 0$. We may assume without restriction that φ_n is Σ_n-measurable. Then $T(\varphi_n) = \int \xi_n . \varphi_n$ and Proposition 2 asserts that (ξ_n) is not Pettis–Cauchy.

(2) Suppose now (φ_n) an L^∞-bounded weakly null sequence in L^1 and $\lim_{n\to\infty} \|\int \xi_n . \varphi_n\| > 0$. Clearly $\int \xi_n . \varphi_n = T(E_n[\varphi_n])$ and $(E_n[\varphi_n])$ is still a weakly null sequence. Hence T is not D–P, completing the proof.

D–P operators from L^1 into L^1

We say that an operator $T: L^1 \to L^1$ is positive if $T(\varphi) \geqq 0$ whenever $\varphi \in L^1$ and $\varphi \geqq 0$. It is well-known that this gives a lattice ordering of $\mathcal{L}(L^1, L^1)$. More precisely, we have that

$$T^+(\varphi) = \sup\{T(\psi); 0 \leqq \psi \leqq \varphi\} \quad \text{and} \quad |T|(\varphi) = \sup\{T(\psi); |\psi| \leqq \varphi\}$$

for $\varphi \in L^1$, $\varphi \geqq 0$.

Denote by Σ the Borel-σ-algebra of $[0,1]$ and consider for each n the product σ-algebra $\mathfrak{S}_n = \Sigma_n \otimes \Sigma$ on $[0,1] \times [0,1]$.

Assume $T: L^1 \to L^1$ an operator and (ξ_n, Σ_n) the corresponding martingale. It is clear that if $A \in \mathfrak{S}_m$ for some m and we define η_n by $\eta_n(t)(u) = \xi_n(t)(u) . \chi_A(t,u)$, then $(\eta_n, \Sigma_n)_{n \geqq m}$ is still a uniformly bounded L^1-valued martingale. This allows us to introduce the operator $T_A(\varphi) = \lim_{n\to\infty} \int \eta_n(t)\varphi(t)dt$ $(\varphi \in L^1)$.

PROPOSITION 3. *Let* $T \in \mathcal{L}(L^1, L^1)$ *and* $A \in \bigcup_n \mathfrak{S}_n$.

(1) $T_A \leqq T^+$.

(2) *If* T *is D–P, then* T_A *is also D–P.*

PROOF. If $A \in \mathfrak{S}_m$, then $A = \bigcup_k (I_{m,k} \times A_k)$, where $A_k \in \Sigma$ for each $k = 1, \cdots, 2^m$. Consequently A has also the form $A = \bigcup_i (S_i \times P_i)$, where $(S_i) \subset \Sigma_m$ and (P_i) is a measurable partition of $[0,1]$.

(1) Fix $n \geqq m$ and $\varphi \geqq 0$, φ Σ_n-measurable. Then

$$T_A(\varphi) = \int \eta_n(t)\varphi(t)dt = \sum_i \left(\int \xi_n(t)\varphi(t)\chi_{S_i}(t)dt \right) \chi_{P_i}$$

$$= \sum_i T(\varphi \cdot \chi_{S_i}) \cdot \chi_{P_i} \leqq \sum_i T^+(\varphi) \cdot \chi_{P_i} = T^+(\varphi),$$

as required.

(2) By Theorem 1 and Proposition 2, it is enough to show that $\lim_{n\to\infty} \| \int \eta_n \varphi_n \| = 0$, if (φ_n) is L^∞-bounded and weakly null in L^1. Now

$$\left\| \int \eta_n \varphi_n \right\| = \int \left| \int \eta_n(t,u)\varphi_n(t)dt \right| du$$

$$\leqq \sum_k \int \left| \int_{I_{m,k}} \xi_n(t)(u)\varphi_n(t)dt \right| du = 2^{-m} \sum_k \left\| \int \xi_n h_{m,k}\varphi_n \right\|$$

and $(h_{m,k}\varphi_n)$ is weakly null for all $k = 1, \cdots, 2^m$. Since (ξ_n) is Pettis–Cauchy, Proposition 2 completes the proof.

It is clear that we may associate to each L^1-valued martingale (ξ_n, Σ_n) the real martingale $(f_n, \mathfrak{S}_n)$, taking $f_n(t,u) = \xi_n(t)(u)$. This gives the identification of $\mathcal{L}(L^1, L^1)$ and the real martingales $(f_n, \mathfrak{S}_n)$ for which $\sup_n \| \int |f_n(t,u)| du \|_\infty < \infty$ holds.

PROPOSITION 4. *Assume* $(f_n, \mathfrak{S}_n)$ L^1*-bounded. Then give* $\varepsilon > 0$ *there is some* $m \in \mathbf{N}$ *and some* $A \in \mathfrak{S}_m$ *such that* $\| f_n^+ - f_n \chi_A \|_1 < \varepsilon$ *for all* $n \geqq m$.

PROOF. We may consider m so that $\|f_m^+\|_1 > \sup_n \|f_n^+\|_1 - \varepsilon$. The set $A = [f_m \geqq 0]$ belongs to $\mathfrak{S}_m$. For $n \geqq m$, we have $\int_A f_n = \int_A f_m = \|f_m^+\|_1$. Consequently

$$\|f_n^+ - f_n\chi_A\|_1 = \int_{A\cap[f_n<0]} |f_n| + \int_{A^c} f_n^+$$

$$= -\int_A f_n + \int_{A\cap[f_n\geqq 0]} f_n + \int_{A^c} f_n^+ = -\int_A f_n + \int f_n^+ = \|f_n^+\|_1 - \|f_m^+\|_1 < \varepsilon,$$

which had to be obtained.

THEOREM 2. *Let $T \in \mathscr{L}(L^1, L^1)$ and (ξ_n) the corresponding martingale.*

(1) *The operator T^+ is determined by $T^+(\varphi) = \lim_{n\to\infty} \int \xi_n(t)^+ \varphi(t)dt$.*

(2) *If T is D–P, then also T^+ is D–P.*

PROOF. Choose $\varepsilon > 0$ and let m and A be as in Proposition 4. Consider the martingale $(\eta_n, \Sigma_n)_{n \geqq m}$ as above. If $n \geqq m$ and $\varphi \in L^\infty$, we find

$$\left\| \int \xi_n(t)^+ \varphi(t)dt - \int \eta_n(t)\varphi(t)dt \right\|_1 = \int \left| \int [f_n^+(t,u) - f_n(t,u)\chi_A(t,u)]\varphi(t)dt \right| du$$

$$\leqq \| f_n^+ - f_n \chi_A \|_1 \| \varphi \|_\infty \leqq \varepsilon \| \varphi \|_\infty .$$

This proves that $S(\varphi) = \lim_{n\to\infty} \int \xi_n(t)^+ \varphi(t)dt$ exists for $\varphi \in L^\infty$. But since

$$\| S(\varphi) \|_1 \leqq \overline{\lim_n} \int \| \xi_n(t)^+ \|_1 |\varphi(t)| dt \leqq \overline{\lim_n} \int \| \xi_n(t) \|_1 |\varphi(t)| dt \leqq \| T \| \| \varphi \|_1,$$

the operator S extends to L^1. In fact, the above computation shows that for each $\varepsilon > 0$ there exists some $A \in \bigcup_n \mathfrak{S}_n$ such that $\| T_A i_\infty - S i_\infty \| < \varepsilon$. Applying Proposition 3 and Proposition 1, we see that if T is D–P then $S i_\infty$ is compact and hence S is D–P.

It remains to prove that $S = T^+$. Obviously $S \geqq T^+$. Conversely, fix $\varphi \in L^\infty$, $\varphi \geqq 0$. Because $T_A(\varphi) \leqq T^+(\varphi)$ for all $A \in \bigcup_n \mathfrak{S}_n$ (Proposition 3), also $S(\varphi) \leqq T^+(\varphi)$ holds. A density argument completes the proof.

An immediate consequence is

COROLLARY 3. *The ideal of the D–P operators in $\mathscr{L}(L^1, L^1)$ is a sublattice.*

A characterization of the Radon–Nikodym property

We refer again the [5] for the following theorem:

THEOREM 4. *Let X be a Banach space, $T \in \mathscr{L}(L^1, X)$ and (ξ_n) the corresponding martingale. Then the following conditions are equivalent:*

(1) *T is representable, i.e. there exists some $\xi \in L_X^\infty$ such that $T(\varphi) = \int \xi(t)\varphi(t)dt$ for $\varphi \in L^1$.*

(2) *(ξ_n) converges in L_X^1.*

(3) *(ξ_n) converges* a.e.

The Banach space X is said to have the Radon–Nikodým property (RNP) provided any $T \in \mathscr{L}(L^1, X)$ is representable. The purpose of this section is to show that we may restrict ourselves to Dunford–Pettis operators. This solves a question raised in [5].

We will prove more precisely the following

THEOREM 5. *Assume $T \in \mathcal{L}(L^1, X)$ not a D–P operator. Then there is a D–P operator $D \in \mathcal{L}(L^1, L^1)$ so that the operator TD is not representable.*

So fix $T: L^1 \to X$ failing the Dunford–Pettis property.

PROPOSITION 5. *There exist a measurable subset Ω of $[0,1]$, a weakly null sequence (φ_r) in L^1 and $\varepsilon > 0$ satisfying*

(1) $m(\Omega) > 0$,

(2) $\|\varphi_r\|_\infty \leqq 1$,

(3) $\overline{\lim}_r \|T(f\varphi_r)\| \geqq \varepsilon \|f\|_1$ *if* $f \in L^1(\Omega)$ *and* $f \geqq 0$.

PROOF. By Proposition 1, there is a weakly null sequence (φ_r) in L^1 and $\varepsilon > 0$ so that $\|\varphi_r\|_\infty \leqq 1$ and $\|T(\varphi_r)\| \geqq 2\varepsilon$ for each r. It is easily verified that the set

$$\mathcal{K} = \left\{ f \in L^1;\ f \geqq 0 \text{ and } \overline{\lim_r} \|T(f\varphi_r)\| \leqq \varepsilon \|f\|_1 \right\}$$

is a closed convex cone. Since $1 \notin \mathcal{K}$, a separation argument provides some $g \in L^\infty$ satisfying $\int g > \int fg$ whenever $f \in \mathcal{K}$. Because $0 \in \mathcal{K}$ we have $\int g > 0$ and consequently $\Omega = [g > 0]$ has positive measure. Clearly $\int fg \leqq 0$ for $f \in \mathcal{K}$. It follows that $f = 0$ or $f \notin \mathcal{K}$ if $f \geqq 0$ and $\operatorname{supp} f \subset \Omega$.

PROPOSITION 6. *Assume $f_1, \cdots, f_d$ a finite set in L^2 and $B < \infty$ such that $\|f_i\|_2 \leqq B$ $(1 \leqq i \leqq d)$ and $|\langle f_i, f_j \rangle| \leqq B^2 d^{-2}$ $(1 \leqq i \neq j \leqq d)$. Then $\sum_i |\langle f_i, g \rangle| \leqq B\sqrt{2d}$ holds, whenever $\|g\|_2 \leqq 1$.*

PROOF. Since

$$\left\| B^2 g - \sum_i \langle g, f_i \rangle f_i \right\|_2^2 = B^4 \|g\|_2^2 - 2B^2 \sum_i |\langle g, f_i \rangle|^2 + \sum_{i,j} \langle g, f_i \rangle \langle g, f_j \rangle \langle f_i, f_j \rangle$$

we have

$$\begin{aligned} B^4 \|g\|_2^2 &\geqq \sum_i (2B^2 - \|f_i\|_2^2) |\langle g, f_i \rangle|^2 - \sum_{i \neq j} \langle g, f_i \rangle \langle g, f_j \rangle \langle f_i, f_j \rangle \\ &\geqq B^2 \sum_i |\langle g, f_i \rangle|^2 - B^2 d^{-2} \sum_{i \neq j} \|g\|_2^2 \|f_i\|_2 \|f_j\|_2 \end{aligned}$$

and thus

$$\sum_i |\langle g, f_i \rangle|^2 \leqq 2B^2.$$

Consequently

$$\sum_i |\langle g, f_i\rangle| \leqq B\sqrt{2d},$$

as required.

PROPOSITION 7. *Let Ω, (φ_r) and ε be as in Proposition 5 and take $0<\delta< m(\Omega)$. There is a system $(\psi_{n,k})_{n,1\leqq k\leqq 2^n}$ in $L^\infty(\Omega)$ fulfilling the following conditions*:

(1) $\psi_{n,k}\geqq 0$,

(2) $\delta<\|\psi_{n,k}\|_1<2$,

(3) $\|\psi_{n,k}\|_\infty\leqq(4/3)^n$,

(4) $\|T(\psi_{n+1,2k-1}-\psi_{n+1,2k})\|>\varepsilon\delta/2$,

(5) $2\psi_{n,k}=\psi_{n+1,2k+1}+\psi_{n+1,2k}$,

(6) $\Sigma_k|\langle\psi_{n+1,2k-1}-\psi_{n+1,2k},g\rangle|\leqq 2(4\sqrt{2}/3)^n$ *if* $\|g\|_\infty\leqq 1$.

PROOF. We introduce the $\psi_{n,k}$ $(1\leqq k\leqq 2^n)$ by induction on n. Take $\psi_{0,1}=\chi_\Omega$.

Suppose now $\psi_{n,k}$ $(1\leqq k\leqq 2^n)$ obtained and $\delta+\iota<\|\psi_{n,k}\|_1<2-\iota$ for some $\iota>0$. Combining the facts

$$\overline{\lim_{r\to\infty}}\,\varphi_r=0\ \sigma(L^1,L^\infty);$$

$$\overline{\lim_r}\,\|T(\psi_{n,k}\varphi_r)\|>\varepsilon\delta\qquad\text{for each } k,$$

it is possible to obtain functions η_k $(1\leqq k\leqq 2^n)$ in the sequence (φ_r), satisfying

(i) $|\langle\psi_{n,k},\eta_k\rangle|<\iota$,

(ii) $|\langle\psi_{n,k}\eta_k,\psi_{n,1}\eta_1\rangle|<(2/3)^{2n}$ for $k\neq 1$,

(iii) $\|T(\psi_{n,k}\eta_k)\|>\varepsilon\delta$.

This construction is completely straightforward and we let the reader check the details.

It is clear that the functions

$$\psi_{n+1,2k-1}=\psi_{n,k}\left(1+\frac{1}{3}\eta_k\right),\qquad \psi_{n+1,2k}=\psi_{n,k}\left(1-\frac{1}{3}\eta_k\right)$$

are positive members of $L^\infty(\Omega)$.

(2) follows from (i), (4) from (iii) and (5) holds obviously. Since $|\psi_{n+1,2k-1}|$, $|\psi_{n+1,2k}|$ are bounded by $\frac{4}{3}|\psi_{n,k}|$, we get

$$\|\psi_{n+1,2k-1}\|_\infty\leqq\frac{4}{3}\|\psi_{n,k}\|_\infty\leqq\left(\frac{4}{3}\right)^{n+1}$$

and the same for $\| \psi_{n+1,2k} \|_\infty$. So it remains to verify (6) or

$$\sum_k |\langle \psi_{n,k}\eta_k, g\rangle| \leqq 3\left(\frac{4\sqrt{2}}{3}\right)^n \qquad \text{for } \| g \|_\infty \leqq 1.$$

Using (ii), we see that $f_k = \psi_{n,k}\eta_k$, $B = (\frac{4}{3})^n$ and $d = 2^n$ fulfil the conditions of Proposition 6. This completes the proof.

Proof of Theorem 5. By (2) and (5) of Proposition 7, we may define $D \in \mathscr{L}(L^1, L^1)$ taking $D(h_{n,k}) = \psi_{n,k}$. The martingales (ξ_n) and (ζ_m) corresponding to D and TD respectively are given by

$$\xi_n(t) = 2^{-n}\sum_k h_{n,k}(t)\psi_{n,k} \quad \text{and} \quad \zeta_n(t) = 2^{-n}\sum_k h_{n,k}(t)T(\psi_{n,k}).$$

Fixing n, (6) yields that

$$|||\, \xi_{n+1} - \xi_n \,||| = \sup_{\|g\|_\infty \leqq 1} \int |\langle \xi_{n+1}(t) - \xi_n(t), g\rangle|\, dt$$

$$= 2^{-n-2} \sup_{\|g\|_\infty \leqq 1} \sum_k \| h_{n+1,2k-1} - h_{n+1,2k} \|_1 |\langle \psi_{n+1,2k-1} - \psi_{n+1,2k}, g\rangle| \leqq \left(\frac{2\sqrt{2}}{3}\right)^n.$$

Therefore $\Sigma_n \;|||\, \xi_{n+1} - \xi_n \,||| < \infty$ and hence (ξ_n) is Pettis–Cauchy. By Theorem 1, D is a D–P operator.

On the other hand, TD is not representable since similar computation yields

$$\int \| \xi_{n+1}(t) - \xi_n(t) \|\, dt = 2^{-n-2} \sum_k \| h_{n+1,2k-1} - h_{n+1,2k} \|_1 \| T(\psi_{n+1,2k-1} - \psi_{n+1,2k}) \|$$

$$> \frac{\varepsilon\delta}{4} \qquad \text{(using (4) of Proposition 7).}$$

Of course the operator TD of Theorem 5 is D–P. This provides a new characterization of the RNP.

Corollary 8. *If a Banach space X fails RNP, then there is a D–P operator $T : L^1 \to X$ which is not representable.*

A factorization problem for Dunford–Pettis operators

For later use, let us show the following property of Pettis–Cauchy martingales.

Proposition 8. *Let X be a Banach space, (ξ_n) a Pettis–Cauchy X-valued martingale and (x_r^*) a* w*-*null sequence in X^*. Then*

$$\lim_{r\to\infty} \sup_n \int |\langle \xi_n(t), x_r^*\rangle|\, dt = 0.$$

PROOF. It is no restriction to assume (x_n^*) bounded by 1.

Suppose $\overline{\lim}_r \sup_n \int |\langle \xi_n(t), x_r^*\rangle|\, dt > \delta > 0$. Applying the Lebesgue theorem on dominated convergence, we see that for a fixed n $\lim_{r\to\infty} \int |\langle \xi_n(t), x_r^*\rangle|\, dt = 0$.

It is now routine to find increasing sequences (r_k) and (n_k) such that $\int |\langle \xi_{n_k}, x^*_{r_{k+1}}\rangle| < \delta/2$ and $\int |\langle \xi_{n_{k+1}}, x^*_{r_{k+1}}\rangle| > \delta$. Consequently

$$|||\, \xi_{n_{k+1}} - \xi_{n_k} \,||| \geqq \int |\langle \xi_{n_{k+1}} - \xi_{n_k}, x^*_{r_{k+1}}\rangle| > \frac{\delta}{2} \qquad \text{for each } k,$$

a contradiction.

PROPOSITION 9. *Let $d \in \mathbf{N}$ be fixed. Then for any r there exists a set $\mathcal{G}_r$ of 4^d functions such that*

(1) $\|g\|_\infty \leqq 1$ *for any r and $g \in \mathcal{G}_r$,*

(2) $E_{r-d}[g] = 0$ *if $g \in \mathcal{G}_r$,*

(3) $\sup_{g\in\mathcal{G}_r} \langle f, g\rangle \geqq \sup_s \int_{I_{r,s}} f - 2^{-d}\|f\|_1$, *whenever $f \in L^1$ and $f \geqq 0$.*

PROOF. For $i = 1, \cdots, 2^d$ and s an integer, write $s \equiv i$ provided $s \equiv i$ (mod 2^d). For each r and $i = 1, \cdots, 2^d$, let $\psi_r^i = 2^{-r}\Sigma_{s\equiv i} h_{r,s}$. Define $\mathcal{G}_r = \{g_r^{ij} = \psi_r^i - \psi_r^j;\ 1 \leqq i, j \leqq 2^d\}$, which has 4^d members. Clearly $\|g_r^{ij}\|_\infty \leqq 1$. Remark that $g_r^{ij} = 2^{-r}\Sigma_{s\equiv i}(h_{r,s} - h_{r,s-i+j})$ and hence $E_{r-d}[g_r^{ij}] = 0$. Now assume f a positive L^1 function and $1 \leqq s \leqq 2^r$. If we take $i = 1, \cdots, 2^d$ so that $s \equiv i$, then $\langle f, \psi_r^i\rangle \geqq \int_{I_{r,s}} f$. Since $\|f\|_1 = \Sigma_j \langle f, \psi_r^j\rangle$, there must be some $j = 1, \cdots, 2^d$ with $\langle f, \psi_r^j\rangle \leqq 2^{-d}\|f\|_1$. Therefore also (3) holds.

PROPOSITION 10. *If (ξ_n) is a uniformly bounded positive L^1-valued Pettis–Cauchy martingale, then*

$$\lim_{r\to\infty} \sup_n \int \left(\sup_s \int_{I_{r,s}} \xi_n(t)\right) dt = 0.$$

PROOF. Assume $\sup_n \|\xi_n\|_\infty \leqq B$ and fix $\varepsilon > 0$. Take an integer d so that $2^d\varepsilon \leqq B$ and let the $\mathcal{G}_r$ satisfy the conditions of Proposition 9. We deduce from (3) that for each r and t

$$\sup_s \int_{I_{r,s}} \xi_n(t) \leqq \sup_{g\in\mathcal{G}_r} \langle \xi_n(t), g\rangle + 2^{-d}\|\xi_n(t)\|_1$$

$$\leqq \sum_{g\in\mathcal{G}} |\langle \xi_n(t), g\rangle| + 2^{-d}B.$$

Therefore

$$\int \left(\sup_s \int_{I_{r,s}} \xi_n(t)\right) dt \leqq 4^d \max_{g \in \mathcal{G}_r} \int |\langle \xi_n(t), g\rangle| \, dt + \varepsilon.$$

It follows easily from (1), (2) of Proposition 9 and Proposition 8 that the first term of the second member in the above inequality tends to 0 uniformly over n for $r \to \infty$. Since $\varepsilon > 0$ was choosen arbitrarily, the proof is complete.

$M[0,1]$ is the space of Radon measures on $[0,1]$. If μ is a Radon probability measure on $[0,1]$, then $L^1(\mu)$ will be viewed as a subspace of $M[0,1]$. In fact, any separable subspace of $M[0,1]$ is contained in some $L^1(\mu)$-subspace.

Consider the family $\mathcal{I}$ of all subintervals of $[0,1]$. If we let $\|\mu\|_{\mathcal{I}} = \sup_{I \in \mathcal{I}} |\mu(I)|$, then $\| \; \|_{\mathcal{I}}$ is a weaker norm on $M[0,1]$. Let M_0 be the normed space $M[0,1]$, $\| \; \|_{\mathcal{I}}$ and $W: M[0,1] \to M_0$ the identity map.

The next result will be used to show a more general fact.

Proposition 11. *If $T \in \mathcal{L}(L^1, L^1)$ is a D–P operator, then WT is representable.*

Proof. It follows from Theorem 2 that we may assume T a positive D–P operator. Let (ξ_n, Σ_n) be the L^1-valued martingale corresponding to T. Fix p, q and r.

For any $t \in [0,1]$ and $I \in \mathcal{I}$,

$$\begin{aligned} &|\langle \xi_p(t) - \xi_q(t), \chi_I\rangle| \\ &\quad \leqq |\langle \xi_p(t) - \xi_q(t), E_r[\chi_I]\rangle| + |\langle \xi_p(t), \chi_I - E_r[\chi_I]\rangle| + |\langle \xi_q(t), \chi_I - E_r[\chi_I]\rangle| \\ &\quad \leqq \|(E_r \circ \xi_p)(t) - (E_r \circ \xi_q)(t)\|_1 + 2\sup_s \int_{I_{r,s}} \xi_p(t) + 2 \sup_s \int_{I_{r,s}} \xi_q(t). \end{aligned}$$

Hence

$$\int \|\xi_p(t) - \xi_q(t)\|_{\mathcal{I}} \, dt \leqq \|(E_r \circ \xi_p) - (E_r \circ \xi_q)\|_1 + 4 \sup_n \int \left(\sup_s \int_{I_{r,s}} \xi_n(t)\right) dt.$$

Since (ξ_n) is Pettis–Cauchy, Proposition 10 asserts that r can be taken big enough to make the second term small. For fixed r, the first term tends to 0 for $p, q \to \infty$, since $(E_r \circ \xi_n)_n$ ranges in a finite dimensional subspace of L^1.

Theorem 7. *If $T \in \mathcal{L}(L^1, M[0,1])$ is a D–P operator, then WT is representable.*

Proof. Since the range of T is separable, T ranges in some $L^1(\mu)$ subspace of $M[0,1]$. Decomposing μ in its atomic and diffuse part, $L^1(\mu)$ can be viewed as a subspace of $l^1 \oplus L^1(\nu)$, where ν is a diffuse Radon probability measure and for convenience $m \ll \nu$. Now l^1 has the Radon–Nikodým property (cf. [5]) and therefore we may assume T ranging in the $L^1(\nu)$-component. It is easily verified that the map $\iota : [0,1] \to [0,1]$, $\iota(x) = \nu([0,x])$ is a homeomorphism and moreover $m = \iota(\nu)$. Thus the map $U : L^1 \to L^1(\nu)$, $U(f) = f \circ \iota$ is an isometry. Because ι maps intervals on intervals, also the map $V : M_0 \to M_0$, $V(\lambda) = \iota^{-1}(\lambda)$ is an isometry. We obtain clearly the following scheme:

$$\begin{array}{ccccc} L^1 & \xrightarrow{T} & L^1(\nu) & \xrightarrow{W} & M_0 \\ & & {\scriptstyle U^{-1}}\downarrow & & \uparrow{\scriptstyle v} \\ & & L^1 & \xrightarrow{W} & M_0 \end{array}$$

Since $U^{-1}T$ is D–P, $WU^{-1}T$ is representable by Proposition 11 and hence also WT. This completes the proof.

Theorem 7 has an interesting aspect which we will discuss later. Let us come to the purpose of this section. It is well known (and not difficult to see) that the following property is true. Assume X a Banach space, Y a subspace of X and $T : L^1 \to Y^*$ an operator. Then there exists a factorizing operator $T' : L^1 \to X^*$, such that $T = i^* T'$ where $i : Y \to X$ is the canonical injection. (T' is generally not uniquely determined.)

It seems to us a natural question whether or not T' can be made D–P, if we assume that T is a D–P operator. As we will show, this problem however has a negative solution. Thus the lifting property for Pettis–Cauchy martingales in dual spaces does not hold.

For convenience we replace the Lebesgue space $[0,1]$ by the Cantor set Δ with the Haar measure. Take $X_1 = C(\Delta) = X_2$ and $X = X_1 \oplus X_2$. Denote $i_\alpha : X_\alpha \to X$ the injection and $\pi_\alpha : X \to X_\alpha$ the projection ($\alpha = 1,2$). The Haarfunctions $h_{n,k}$ on Δ will be viewed as members of X_1. Consider a system $(e_{n,k})_{n,1 \leq k \leq 2^n}$ in X_2 equivalent to the l^2-basis, i.e.

$$\Big\| \sum_{n,k} a_{n,k} e_{n,k} \Big\| = \Big(\sum_{n,k} a_{n,k}^2 \Big)^{1/2} \qquad \text{for scalars } a_{n,k}.$$

Consider the closed aubspace Y of X spanned by the vectors

$$y_{n,k} = 2^{-n} h_{n,k} + e_{n,k} \qquad (n \in \mathbf{N},\ 1 \leqq k \leqq 2^n).$$

We regard $L^1(\Delta)$ as a subspace of X_1^* and let $j : L^1(\Delta) \to X_1^*$ be the injection. Consider the operators $R : X_2^* \to l^2$, $R(x_2^*) = (x_2^*(e_{n,k}))_{n,k}$ and $S : Y^* \to l^\infty$, $S(y^*) = (y^*(y_{n,k}))_{n,k}$. The operator $W : X_1^* \to M_0(\Delta)$ is as previously (the intervals in subspace are the intersections of intervals in $[0, 1]$ with Δ). We obtain the following diagram:

$$\begin{array}{ccccccccc}
L^1(\Delta) & \xrightarrow{j} & X_1^* & \xrightarrow{\pi_1^*} & X^* & \xrightarrow{i^*} & Y^* & \xrightarrow{\quad S \quad} & l^\infty \\
 & \searrow^{T'} & & & & & \uparrow i^* & & \uparrow U \\
 & & & & & & X^* & \xrightarrow{(i_1^*, i_2^*)} X_1^* \oplus X_2^* \xrightarrow{(W,R)} & M_0(\Delta) \oplus l^2.
\end{array}$$

The existence of the factorizing U is clear from the fact that for $x^* = (x_1^*, x_2^*)$ in X^* we have

$$|x^*(y_{n,k})| \leqq |x_1^*(2^{-n}h_{n,k})| + |x_2^*(e_{n,k})| \leqq \| W(x_1^*)\| + \| R(x_2^*)\|,$$

since $2^{-n}h_{n,k}$ is the characteristic function of an interval in Δ.

Proposition 12. *If $T = i^*\pi_1^* j$, then*
(1) ST is not representable,
(2) T is Dunford–Pettis.

Proof. T and ST are represented by the respective martingales

$$\xi_n = 2^{-n} \sum_k h_{n,k} T(h_{n,k}), \qquad \xi'_n = 2^{-n} \sum_k h_{n,k} ST(h_{n,k}).$$

So for $y \in Y$ we obtain

$$\int |\langle \xi_{n+1}(t) - \xi_n(t), y\rangle|\, dt = 2^{-n-1} \sum_k |\langle T(h_{n+1,2k-1} - h_{n+1,2k}), y\rangle|$$

and also

$$\int \|\xi'_{n+1}(t) - \xi'_n(t)\|\, dt = 2^{-n-1} \sum_k \| ST(h_{n+1,2k-1} - h_{n+1,2k})\|.$$

(1) For each k

$$\begin{aligned}\| ST(h_{n+1,2k-1} - h_{n+1,2k})\| &\geqq \langle T(h_{n+1,2k-1} - h_{n+1,2k}), y_{n+1,2k-1}\rangle \\ &\geqq 2^{-n-1}\langle h_{n+1,2k-1} - h_{n+1,2k}, h_{n+1,2k-1}\rangle = 1.\end{aligned}$$

Hence $\| \xi'_{n+1} - \xi'_n \|_1 \geqq \frac{1}{2}$.

(2) If $\| y \| \leqq 1$ and $y = \Sigma_{r,s} a_{r,s} y_{r,s}$, then $\Sigma_{r,s} a_{r,s}^2 = \| \pi_2(y)\|^2 \leqq 1$. For each k, we obtain

$$\langle T(h_{n+1,2k-1}-h_{n+1,2k}),y\rangle=\sum_{r,s}2^{-r}a_{r,s}\langle h_{n+1,2k-1}-h_{n+1,2k},h_{r,s}\rangle$$

$$=\sum_{r>n,s}2^{-r}a_{r,s}\langle h_{n+1,2k-1}-h_{n+1,2k},h_{r,s}\rangle.$$

Thus

$$\int|\langle\xi_{n+1}(t)-\xi_n(t),y\rangle|\,dt\leqq 2^{-n-1}\sum_{r>n,s}2^{-r}|a_{r,s}|\sum_k|\langle h_{n+1,2k-1}-h_{n+1,2k},h_{r,s}\rangle|$$

$$\leqq 2^{-n}\sum_{r>n,s}2^{-r}|a_{r,s}|\sum_k\langle h_{n,k},h_{r,s}\rangle$$

$$=\sum_{r>n,s}2^{-r}|a_{r,s}|$$

$$\leqq\Big(\sum_{r>n}\sum_{s=1}^{2^r}4^{-r}\Big)^{1/2}\Big(\sum_{r,s}a_{r,s}^2\Big)^{1/2}\leqq(\sqrt{2})^{-n}.$$

Since the ball of Y is w*-dense in the ball of Y^{**}, we find that $|||\,\xi_{n+1}-\xi_n\,|||\leqq(\sqrt{2})^{-n}$. Consequently (ξ_n) is Pettis–Cauchy and thus T is D–P.

Proposition 13. *If $T'\in\mathscr{L}(L^1,X^*)$ is a D–P operator, then Si^*T' is representable. Consequently, T does not admit a D–P factorization.*

Proof. Since l^2 has RNP, Ri_2^*T' is representable for any $T'\in\mathscr{L}(L^1,X^*)$. If now T' is D–P, then also i_1^*T' is D–P. Regarding $M(\Delta)$ as subspace of $M[0,1]$, it is an easy exercise to deduce from Theorem 7 that Wi_1^*T' is representable. Taking the above diagram into account, this clearly completes the proof.

Remarks. (I) From Theorem 1, it is obvious that for a space X the following properties are equivalent:

(1) Any $T\in\mathscr{L}(L^1,X)$ is D–P.

(2) Any uniformly bounded X-valued martingale (ξ_n) is Pettis–Cauchy.

They are fulfilled for three well-known classes of spaces:

(A) the RNP spaces,

(B) the Schur spaces,

(C) duals X^* such that $l^1\not\subset X$ (cf. [6]).

(II) Theorem 7 has the following consequence, which is in fact a reformulation of Proposition 11.

Corollary 8. *Consider the operator $O:L^1\to C[0,1]$ given by $O(f)(s)=\int_0^s f(t)dt$. If now $T:L^1\to L^1$ is D–P, then OT is representable.*

PROOF. If $B[0,1]$ is the space of the bounded functions on $[0,1]$, we may clearly consider the operator $\tilde{O}: M_0 \to B[0,1]$, $\tilde{O}(\mu)(s) = \mu[0,s]$.

The following scheme together with Proposition 11 will provide us the required result:

$$\begin{array}{ccccccc} L^1 & \xrightarrow{T} & L^1 & \xrightarrow{O} & C[0,1] & \hookrightarrow & B[0,1] \\ \downarrow & & & & & & \uparrow \tilde{O} \\ M & & & \xmapsto{w} & & & M_0 \end{array}$$

(III) Let us call a tree in a Banach space X a bounded system $(x_{n,k})_{n,1 \le k \le 2^n}$ in X such that

(1) $2x_{n,k} = x_{n+1,2k-1} + x_{n+1,2k}$,

(2) $\inf_{n,k} \| x_{n+1,2k-1} - x_{n+1,2k} \| > 0$.

It is clear that if we take $(\psi_{n,k})_{n,k}$ as in Proposition 7 and $x_{n,k} = T\psi_{n,k}$, then $(x_{n,k})_{n,k}$ is a tree in X. This proves that X has the tree property whenever there exists a non-Dunford–Pettis operator $T: L^1 \to X$ (in fact, this can be obtained a bit more directly). Recently [3], it was shown that the failure of RNP does not imply the existence of a tree in general. (For other partial results, we refer to [1] and [2].)

REFERENCES

1. J. Bourgain, *Sets with the Radon–Nikodým property in conjugate Banach spaces*, Studia Math. **663** (1978).

2. J. Bourgain, *A non-dentable set without the tree-property*, Studia Math. **682** (1980), to appear.

3. J. Bourgain and H. P. Rosenthal, *Martingales valued in certain subspaces of L^1*, Israel J. Math. **37** (1980), 54–75.

4. J. Diestel, *Topics in Geometry of Banach Spaces*, No. 485, Springer Verlag, 1975.

5. J. Diestel and J. J. Uhl, *The Theory of Vector Measures*, Mathematical Surveys, Vol. 15, Amer. Math. Soc., 1977.

6. K. Musial, *The weak Radon–Nikodým property in Banach spaces*, to appear.

7. H. Rosenthal, *Convolution by a biased coin*, The Altgeld Book 1975/76, University of Illinois, Functional Analysis Seminar.

8. J. J. Uhl, *Applications of Radon–Nikodým theorems to martingale convergence*, Trans. Amer. Math. Soc. **145** (1969), 271–285.

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