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THE TOPOLOGY OF STATIONARY CURL PARALLEL SOLUTIONS OF EULER'S EQUATIONS†

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ABSTRACT

We study the orbit structure of a vector field V defined on a three-dimensional Riemannian manifold which satisfies $V \wedge \operatorname{curl} V = 0$. Such a vector field represents the velocity of a stationary solution of Euler's equation for a perfect fluid. In addition to several other results, we show that if the vector field admits a first integral, then each level set is toroidal and the induced flow on the level set is either periodic or conditionally periodic.

1. Introduction

The equations of motion of an ideal fluid in a three-dimensional bounded and connected region D are given by Euler's equations which may be expressed in Bernoulli's form as

$$\frac{\partial V}{\partial t} = V \wedge \operatorname{curl} V + \operatorname{grad} \alpha,$$

$$\operatorname{div} V = 0,$$

where V is the velocity and α is a function determined by the conditions that $\operatorname{div} V = 0$ and that V is tangent to the boundary. Using the vorticity equation

$$\frac{\partial \operatorname{curl} V}{\partial t} = [V, \operatorname{curl} V],$$

where [,] is the Lie bracket, Arnold [cf. 2, p. 331] has shown that if the flow is stationary, i.e., $\partial V/\partial t = 0$, and if $V \wedge \operatorname{curl} V$ does not vanish everywhere then the region D can be partitioned into invariant cells which are either tori or

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cylinders. These cells are obtained as level surfaces of α. Moreover, on each torus the flow lines are all periodic or all dense and on each cylinder the flow lines are all periodic. In a remark on this theorem Arnold [2, p. 332] mentions that when $V \wedge \operatorname{curl} V = 0$ everywhere the flow is very complicated. In particular, for the stationary flow given by

$$V_x = A \sin z + C \cos y, \qquad V_y = B \sin x + A \cos z,$$

$$V_z = C \sin y + B \cos x$$

on the three-dimensional torus computer experiments indicate that some flow lines densely fill a three-dimensional region.

Avez and Buzzanca [4] have shown that if V has constant length and if $\operatorname{curl} V = \alpha V$ for some constant α then a connected component of a level surface of a first integral h of V is either a plane, a cylinder or a torus and on the tori all orbits are closed or all orbits are everywhere dense. Their main example is the geodesic flow on the unit tangent bundle T_1M of a Riemannian surface (M^2, g) for which they prove that the geodesic vector field X satisfies

$$\operatorname{curl} X = -X$$

and

$$\|X\| = 1$$

with respect to the natural "Sasaki" metric on T_1M.

In this paper we study vector fields X defined on Riemannian three-manifolds which satisfy $\operatorname{curl} X = \alpha X$ for some function α. In particular, on a not necessarily compact three-manifold we show that a vector field of constant length which has an integral h satisfies the theorem of Avez and Buzzanca when $\operatorname{curl} X = \alpha X$ and α is a nonvanishing function. In addition, we show that X has constant length and $\operatorname{curl} X = \alpha X$ if and only if X is a contact vector field.

2. Contact structures and curl parallel fields

Let (M, g) denote a Riemannian manifold and let Ω denote the associated Riemannian volume form. When M is three dimensional a vector field A on M defines a 1-form $\omega_A(B) = g(A, B)$ and a 2-form $i_A \Omega$ such that both identifications are isomorphisms. With these identifications vector analysis on M is obtained from the calculus of differential forms. In particular, if f is a function and A is a vector field on M

$$df = \omega_{\operatorname{grad} f}, \qquad d\omega_A = i_{\operatorname{curl} A}\,\Omega \quad \text{and} \quad L_A\,\Omega = (\operatorname{div} A)\Omega.$$

With the vector cross product $A \wedge B$ given by

$$\omega_A \wedge \omega_B = i_{A \wedge B}\, \Omega$$

all the familiar formulas of vector analysis can be derived by using the exterior wedge algebra of forms and the properties of the exterior derivative d and interior product i.

Recall that a 1-form λ defines a contact structure on a manifold of dimension $2n+1$ if $\lambda \wedge (d\lambda)^n$ is a volume form. A vector field X is a contact vector field if $\lambda(X)=1$ and $i_X d\lambda = 0$. It follows easily that X preserves the volume $\lambda \wedge (d\lambda)^n$ and that X is nonvanishing. Every contact structure has a unique associated contact vector field and it is a classical fact that the geodesic flow on the unit tangent bundle of a Riemannian manifold is a contact vector field with respect to the contact structure given by the Liouville 1-form.

When M is the unit sphere bundle of an orientable Riemannian 2-manifold, M is parallelizable. In particular, if X is the geodesic vector field, Y is the perpendicular geodesic vector field and A is the fiber rotation field one has the bracket relations (cf. [5])

$$[X, Y] = kA, \quad [X, A] = -Y \quad \text{and} \quad [Y, A] = X$$

where k is the Gauss curvature of the base manifold. Define the Sasaki metric S on M by declaring that $\langle X, Y, A\rangle$ is an orthonormal oriented frame field and let Ω be the associated volume form defined by $\Omega(X, Y, A) = 1$. Then, with respect to S, a coordinate free computation yields the equation

$$d\omega_X = -i_X \Omega$$

which is equivalent to the statement that $\operatorname{curl} X = -X$.

Since $\operatorname{curl} X = -X$ for the geodesic field X and since X is a contact vector field it is natural to ask for the relationship between contact vector fields and curl parallel fields, i.e. vector fields such that $\operatorname{curl} X = \alpha X$. To find this relationship we will need the following theorem.

THEOREM 2.1. *Let X be the contact vector field for the contact structure λ on the $(2n+1)$-dimensional manifold M. Then, there exists a Riemannian metric g such that $g(\cdot, X) = \lambda$ and such that the Riemannian volume $\Omega = \lambda \wedge (d\lambda)^n$.*

PROOF. The proof will proceed in two steps.

(a) There exists a Riemannian metric h on M such that $h(\cdot, X) = \lambda$.

(b) Let Ω_1 be the Riemannian volume of h and f be the positive function such that

$$f\Omega_1 = \Lambda = \lambda \wedge (d\lambda)^n.$$

Then, there exists an h positive definite bundle homomorphism $B: TM \to TM$ such that $BX = X$ and $\det B = f^2$.

Given (a) and (b), define

$$g(U, V) = h(U, BV).$$

We have $g(\cdot, X) = \lambda$ and the Riemannian volume Ω of g is given by

$$\Omega = (\det B)^{1/2}\Omega_1 = \Lambda$$

as required.

To prove (a) note that X is nonvanishing and, therefore, by Darboux's theorem there is a locally finite cover $\{U_\alpha\}$ of M such that in the coordinates $(x_1, \cdots, x_{2n+1})$ of U_α

$$X = \partial/\partial x_1$$

and

$$\lambda = dx_1 + x_2 dx_{n+2} + \cdots + x_{n+1} dx_{2n+1}.$$

Define a Riemannian metric h_α on U_α by assigning a smooth positive definite symmetric matrix of functions h_{ij} in the variables $(x_1, \cdots, x_{2n+1})$ to each point of U_α such that the first row and first column of h_{ij} is $(1, 0, \cdots, x_2, \cdots, x_{n+1})$. If M is three dimensional such an assignment would be

$$h_{ij} = \begin{pmatrix} 1 & 0 & x_2 \\ 0 & 1 & \\ x_2 & & 1 + x_2^2 \end{pmatrix}.$$

Let $\{\phi_\alpha\}$ be a partition of unity subordinate to the cover $\{U_\alpha\}$ and define

$$h = \Sigma\, \phi_\alpha h_\alpha.$$

Let $p \in M$ and compute

$$\begin{aligned} h(\cdot, X_p) &= \sum \phi_\alpha(p) h_\alpha(\cdot, \partial/\partial x_1) \\ &= \sum \phi_\alpha(p)\lambda_p \\ &= \lambda_p \end{aligned}$$

as required.

To prove (b) observe that $TM = [X] \oplus E$ where $[X]$ denotes the line bundle generated by X and E is the h orthocomplement of $[X]$ in TM. Let I_0 be the identity bundle homomorphism on $[X]$ and I_1 the identity bundle homomorphism on E. Finally, define

$$B = I_0 \oplus f^{1/n} I_1.$$

Clearly, B is positive definite and at each point

$$\det B = f^2. \qquad \text{Q.E.D.}$$

THEOREM 2.2. *Let X be a vector field on M^3. The following statements are equivalent.*

(a) *There exists a Riemannian metric g on M such that $g(X,X) = c$ and* $\operatorname{curl} X = \alpha X$ *where α is a nonvanishing function and c is a nonzero constant.*

(b) *There exists a contact form λ such that X is the contact vector field for λ.*

(c) *There exists a Riemannian metric g such that $g(X,X) = 1$ and* $\operatorname{curl} X = X$.

PROOF. (a) $\Rightarrow$ (b). Let $\lambda = c^{-1}\omega_X$. Then,

$$d\lambda = c^{-1} d\omega_X = c^{-1}\, i_{\operatorname{curl} X}\, \Omega = \alpha c^{-1}\, i_X\, \Omega$$

where Ω is the associated Riemannian volume. We have

$$\lambda(X) = 1$$

and

$$i_X d\lambda = \alpha c^{-1} i_X i_X \Omega = 0,$$

hence, we need only show that $\lambda \wedge d\lambda$ is a volume. But,

$$\lambda \wedge d\lambda = \alpha c^{-2} \omega_X \wedge i_X \Omega$$

and

$$0 = i_X(\omega_X \Omega) = c\Omega - \omega_X \wedge i_X \Omega$$

imply

$$\lambda \wedge d\lambda = \alpha c^{-1} \Omega.$$

(b) $\Rightarrow$ (c). If λ is a contact form and X is the associated contact vector field, then by Theorem 2.1 there is a Riemannian metric g such that $g(\cdot, X) = \lambda$ and such that the associated Riemannian volume is $\lambda \wedge d\lambda$. Then,

$$g(X,X) = 1$$

and

$$i_X(\lambda \wedge d\lambda) = d\lambda = d\omega_X = i_{\mathrm{curl}\,X}\,\Omega$$
$$= i_{\mathrm{curl}\,X}(\lambda \wedge d\lambda).$$

Hence, as required $X = \mathrm{curl}\,X$.

The implication (c) $\Rightarrow$ (a) is clear. Q.E.D.

Theorem 2.2 leads to a generalization of the result of Avez and Buzzanca.

THEOREM 2.3. *Let X be a vector field on a three-dimensional oriented Riemannian manifold (M, g) such that $g(X, X) = c$ and $\mathrm{curl}\,X = \alpha X$ where c is a constant and α is a nonvanishing function. If h is a first integral of X then a component of a regular level set N of h is an X invariant plane, cylinder or torus. Moreover, if N is a cylinder all orbits are closed and if N is a torus either all orbits are dense or all orbits are closed.*

PROOF. The fact that h is a first integral of X does not depend on the choice of Riemannian metric. Hence, applying Theorem 2.2 there is a Riemannian metric g' such that $g'(X, X) = 1$ and such that $\mathrm{curl}\,X = X$ with respect to g'. This reduces the result to the theorem of Avez and Buzzanca. In effect, since X and $X \wedge \mathrm{grad}\,h$ are tangent to the regular level set of h the theorem follows from the easily proved fact that

$$[X, X \wedge \mathrm{grad}\,h] = 0. \qquad \text{Q.E.D.}$$

REMARK. Theorem 2.2 also shows that the theorem of Avez and Buzzanca is equivalent to the reduction of the phase space of a contact vector field to a level set of an integral of the motion. Of course, this is the situation frequently encountered in the Hamiltonian formulation of particle mechanics.

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