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MONOTONE TRAJECTORIES OF DIFFERENTIAL INCLUSIONS AND FUNCTIONAL DIFFERENTIAL INCLUSIONS WITH MEMORY

BY

GEORGES HADDAD

ABSTRACT

The paper gives a necessary and sufficient condition for the existence of monotone trajectories to differential inclusions $dx/dt \in S[x(t)]$ defined on a locally compact subset X of $\mathbf{R}^p$, the monotonicity being related to a given preorder on X. This result is then extended to functional diffferential inclusions with memory which are the multivalued case to retarded functional differential equations. We give a similar necessary and sufficient condition for the existence of trajectories which reach a given closed set at time $t = 0$ and stay in it with the monotonicity property for $t \geqq 0$.

Introduction

Let X be a given subset of $\mathbf{R}^p$, S a correspondence (set valued map) from X into $\mathbf{R}^p$. The subset X is regarded as the state of a dynamical system and $S(x)$ as the set of feasible velocities of the system when its state is x. A preorder (i.e. a relation both reflexive and transitive) is defined by a set valued map which to any $x \in X$ associates

$$P(x) = \{y \in X / y \succcurlyeq x\}.$$

Let $x_0 \in X$ be given, we say that an absolutely continuous function "u" from $[0, T_0]$ into X, $T_0 > 0$, is a "monotone trajectory for S starting at x_0" if

$$\begin{cases} u(0) = x_0, \\ \dfrac{du}{dt}(t) \in S[u(t)] \text{ for almost all } t \in]0, T_0[, \\ \text{for any } s, t \in [0, T_0],\ s < t \text{ implies } u(t) \in P[u(s)]. \end{cases}$$

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Under reasonable assumptions on S and P, the main theorem of this paper gives when X is locally compact a necessary and sufficient condition for the existence of at least one monotone trajectory starting at $x_0 \in X$. This condition will be that $S(x) \cap T_{P(x)}(x) \neq \emptyset$ for all $x \in X$, when

$$T_{P(x)}(x) = \left\{ v \in \mathbf{R}^p \,/\, \liminf_{\gamma \to 0^+} \frac{d_{P(x)}[x + \gamma v]}{\gamma} = 0 \right\}$$

is the Bouligand's contingent cone.

The theorem generalizes the Nagumo theorem [12] when S is single valued and $P(x) = X$ for all $x \in X$. It generalizes as well different papers by M. Crandall [6], H. Brézis [4], J. M. Bony [3], R. M. Redheffer [13], J. A. Yorke [15], [16], R. H. Martin [11], all in the single valued case. In the multivalued case the theorem appears to be an extension of results due to J. P. Aubin–A. Cellina–J. Nohel [2] with convexity assumptions and techniques from non-linear analysis, to J. P. Aubin–F. Clarke [1] when S is a continuous correspondence and F. Clarke [5] when S is a Lipschitz correspondence, both papers using the Clarke's tangent cone which is included in the Bouligand's cone, and finally to S. Gautier [7] with no monotonicity requirements.

In a second part of the paper we prove the existence of monotone trajectories for functional differential inclusions with memory which are the multivalued extension of retarded functional differential equations as defined by J. Hale [9].

More precisely, for a given closed subset X of $\mathbf{R}^p$ we define $\mathscr{X}$ to be the set of continuous functions from a given interval $(-r, 0]$, $0 < r \leqq +\infty$, into $\mathbf{R}^p$ reaching X at time $t = 0$.

Let F be a given correspondence from $\mathscr{X}$ into $\mathbf{R}^p$ and P a given preorder on X, for any $\phi \in \mathscr{X}$ we define a function $U : (-r, +\infty[$ into $\mathbf{R}^p$ to be a "monotone trajectory with initial value ϕ" if:

$$\begin{cases} U = \phi \text{ on } (-r, 0], \\ U \text{ is monotone with respect to } P \text{ on } [0, +\infty[, \\ U \text{ is absolutely continuous on } [0, +\infty[, \\ \dfrac{dU}{dt}(t) \in F[A(t)U] \text{ for almost all } t \in]0, +\infty[, \end{cases}$$

where $A(t)U$ is defined for $t \geqq 0$ by:

$$[A(t)U](\theta) = U(t + \theta) \qquad \text{for all } \theta \in (-r, 0].$$

Then under reasonable assumptions on F and P, we prove that for any $\phi \in \mathscr{X}$, the existence of at least one monotone trajectory with initial value ϕ is equivalent to $F(\phi) \cap T_{P[\phi(0)]}[\phi(0)] \neq \emptyset$, which is similar to the condition of the first theorem.

Such theorem extents to the multivalued case results by G. Seifert [14], S. Leela–V. Moauro [10] in the case of ordinary retarded functional differential equations with no monotonicity requirements. This theorem generalizes too a previous paper by G. Haddad [8] in the multivalued case with convexity assumptions and techniques from non-linear analysis.

The motivations of this paper could be found in Mechanics where many problems dealing with differential inclusions do appear, in Control Theory when $S(x) = \{f(x, u)\}_{u \in U}$ where u ranges over a subset U of "controls" and where $f(x, u)$ is the velocity of the system when the state is x and the control is u, in planning procedures in Microeconomics where problems are to find monotone trajectories for differential inclusions and in biological evolutions where functional differential inclusions with memory do effectively appear.

I. ORDINARY DIFFERENTIAL INCLUSIONS

1. Introduction

Let X be a non-empty subset of $\mathbf{R}^p$, for all $x \in X$ we define the Bouligand contingent cone to be $T_X(x) = \{v \in \mathbf{R}^p / \lim_{\gamma \to 0^+} \operatorname{Inf} d_X(x + \gamma v)/\gamma = 0\}$, where $d_X(\cdot)$ denotes the distance to X.

We verify easily that $v \in T_X(x)$ if and only if there exist a sequence $\{\gamma_n\}$ of positive numbers with $\gamma_n \to 0^+$ as $n \to +\infty$ and a sequence $\{x_n\}$ in X such that $(x_n - x)/\gamma_n \to v$ as $n \to +\infty$.

The set $T_X(x)$ is then a non-empty closed cone which is larger than the tangent cone introduced by Clarke as:

$$T_X^C(x) = \left\{ v \in \mathbf{R}^p \Big/ \lim_{\substack{\gamma \to 0^+ \\ y \to x}} \operatorname{Sup} \frac{d_X(y + \gamma v) - d_X(y)}{\gamma} \leqq 0 \right\}.$$

Moreover we verify easily that in the case when X is convex these two cones coincide with the natural tangent cone defined by $T_X(x) = \overline{\bigcup_{\lambda \geqq 0} \lambda [X - x]}$.

In the following, each preorder on X will be presented by a set valued map $P: X \to X$ which satisfies the following properties:

$$\begin{cases} \text{reflexivity: for any } x \in X,\ x \in P(x), \\ \text{transitivity: for any } x, y \in X,\ y \in P(x) \text{ implies } P(y) \subset P(x). \end{cases}$$

The preorder is then obviously defined by $y \geqslant x$ if and only if $y \in P(x)$.

A correspondence (or set valued map) S from a topological space X into a topological space Y is said to be lower semi-continuous at $x_0 \in X$ if to any open set ω in Y such that $\omega \cap S(x_0) \neq \emptyset$ we can associate a neighborhood $U(x_0)$ of x_0 such that for any $x \in U(x_0)$, $S(x) \cap \omega \neq \emptyset$.

We say that S is lower semi-continuous on X if S is lower semi-continuous at every point of X.

The correspondence S is said to be upper semi-continuous at point $x_0 \in X$ if to any neighborhood V of $S(x_0)$ we can associate a neighborhood $U(x_0)$ of x_0 such that for any $x \in U(x_0)$, $S(x) \subset V$.

We say that S is upper semi-continuous on X if S is upper semi-continuous at every point of X with compact values.

We shall say that the preorder is continuous if P is a lower semi-continuous correspondence with a closed graph in $X \times X$.

For example, if $X \subset \mathbf{R}^p$ has no isolated points and if $V: X \to \mathbf{R}$ is a continuous function such that every local minimum is a global minimum, then the preorder defined by $P(x) = \{y \in X / V(y) \leqq V(x)\}$ is continuous.

Another example of continuous preorder is when X is convex and $P(x) = \{y \in X / V_i(y) \leqq V_i(x),\ i = 1, \cdots, m\}$ where each V_i is a continuous strictly convex function on X.

2. Existence theorem of monotone solutions in a locally compact subset

THEOREM I-1. *Let X be a locally compact subset of $\mathbf{R}^p$, $P: X \to X$ a given continuous preorder and $S: X \to \mathbf{R}^p$ an upper semi-continuous non-empty convex compact valued correspondence.*

Then the following condition:

$$\text{(C)} \qquad S(x) \cap T_{P(x)}(x) \neq \emptyset, \qquad \textit{for all } x \in X,$$

is equivalent to the existence property:

For any $x_0 \in X$ there exist $T_0 > 0$ and a Lipschitz function $u: [0, T_0] \to X$ such that $u(0) = x_0$ with

$$\text{(I)} \qquad \frac{du}{dt}(t) \in S[u(t)] \qquad \textit{for almost all } t \in]0, T_0[,$$

together with the monotonicity property:

$$\textit{for any } s, t \in [0, T_0],\ s < t \textit{ implies } u(t) \in P[u(s)].$$

Necessity of Condition (C)

Let us suppose that for $x_0 \in X$ there exist $T_0 > 0$ and an absolutely continuous function $u : [0, T_0] \rightarrow X$ such that $u(0) = x_0$ and verifying (I) together with the monotonicity property.

First we deduce that du/dt is bounded on $[0, T_0]$ since S is upper semi-continuous thus bounded on the compact set $u([0, T_0])$.

So u is a Lipschitz function. Let Ω denote the set of limit points of $(u(h) - x_0)/h$ as $h \rightarrow 0^+$, by the remark made above and since we are in a finite dimensional space $\mathbf{R}^p$, it is obvious that $\Omega \neq \emptyset$.

Furthermore the monotonicity of u implies that for all $h > 0$, $u(h) \in P[u(0)] = P[x_0]$.

Thus it is clear that $\Omega \subset T_{P(x_0)}(x_0)$.

From S upper semi-continuous at x_0, for any $\varepsilon > 0$, there exists $\eta > 0$ such that for any $x \in X$, $\|x - x_0\| < \eta$ we have $S(x) \subset S(x_0) + \varepsilon \bar{B}$ where $\bar{B}$ denotes the closed unit ball centered at the origin in $\mathbf{R}^p$.

The continuity of u makes obvious the existence of $\alpha > 0$ such that for any τ, $0 \leqq \tau \leqq \alpha$, $u(\tau) - x_0 < \eta$ and then $S[u(\tau)] \subset S(x_0) + \varepsilon \bar{B}$.

Since u is a solution of (I) then for any h, $0 < h \leqq \alpha$, we have:

$$\frac{u(h) - u(0)}{h} = \frac{1}{h} \int_0^h \frac{du}{dz}(z)dz \subset S(X_0) + \varepsilon \bar{B}$$

which is a convex set since S is convex valued.

Now by the compacity assumption on S, it is obvious that $\Omega \subset S(x_0) + \varepsilon \bar{B}$ and this for any $\varepsilon > 0$, which gives $\Omega \subset S(x_0)$. This proves that $S(x_0) \cap T_{P(x_0)}(x_0) \neq \emptyset$.

For the rest of the proof of the theorem we shall need the preliminary lemma stated below. We denote by $\bar{B}(x_0, R)$ the closed ball of radius $R > 0$ centered at x_0.

LEMMA I-1. *Let all the hypotheses of the preceding theorem be verified. Since X is locally compact, let $R > 0$ be such that $X_0 = X \cap \bar{B}(x_0, R)$ is compact and let $\lambda > 0$ verify $\lambda \geqq \mathrm{Sup}_{v \in S(x), x \in X_0} \|v\|$. Then for any $\alpha > 0$ there exists a finite sequence $0 = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_{m-1} < R/(\lambda + \alpha) \leqq \theta_m$ with $\theta_{k+1} - \theta_k < \alpha$ together with an associate sequence $\{x_0, x_1, \cdots, x_m\}$ in X such that for any $k = 0, 1, \cdots, m-1$ we have $x_k \in X_0$, $x_{k+1} \in P(x_k)$ and the existence of $y_k \in X_0$ depending on x_k and $v_k \in S(y_k)$ verifying*:

$$\text{(1)} \qquad \begin{cases} \|x_k - y_k\| < \alpha, \\ \left\| \dfrac{x_{k+1} - x_k}{\theta_{k+1} - \theta_k} - v_k \right\| < \alpha. \end{cases}$$

PROOF OF LEMMA I-1. Let us first consider the application $(x, y) \to d_{P(x)}(y)$ defined on $X \times \mathbf{R}^p$. It is then immediate to verify that the lower semi-continuity of correspondence P is equivalent to the upper semi-continuity of this application.

Let $\alpha > 0$ be given, then for all $y \in X_0$ condition (C) implies the existence of $v_y \in S(y)$ such that:

$$\lim_{\gamma \to 0^+} \operatorname{Inf} \frac{d_{P(y)}[y + \gamma v_y]}{\gamma} = 0.$$

We deduce the existence of γ_y, $0 < \gamma_y < \alpha$ such that $d_{P(y)}[y + \gamma_y v_y]/\gamma_y < \alpha$. Let us consider $V(y) = \{x \in X_0 \mid d_{P(x)}[x + \gamma_y v_y]/\gamma_y < \alpha\}$; by the upper semi-continuity of the application $x \to d_{P(x)}[x + \gamma_y v_y]$ this set is obviously an open neighborhood of y in X_0. Thus there exists η_y, $0 < \eta_y < \alpha$, such that $B(y, \eta_y) \cap X_0 \subset V(y)$ where $B(y, \eta_y)$ is the open ball of radius η_y centered at y.

Thus we can build an open covering of X_0 compact by these open balls and then extract a finite sub-covering denoted $B(y_1, \eta_1), \cdots, B(y_q, \eta_q)$. To each $B(y_i, \eta_i)$ is associated γ_{y_i} denoted by γ_i and $v_{y_i} \in S(y_i)$ denoted by v_i.

Then we have $x_0 \in B(y_{i_0}, \eta_{i_0})$ for some $i_0 \in \{1, \cdots, q\}$ which gives $d_{P(x_0)}[x_0 + \gamma_{i_0} v_{i_0}]/\gamma_{i_0} < \alpha$.

So there exists $x_1 \in P(x_0) \subset X$ such that $\|(x_1 - x_0)/\gamma_{i_0} - v_{i_0}\| < \alpha$ with $\|x_0 - y_{i_0}\| < \eta_{i_0} < \alpha$ and $v_{i_0} \in S(y_{i_0})$. We define $\theta_1 = \gamma_{i_0}$. Then if $\theta_1 \geqq R/(\lambda + \alpha)$ we stop and the result is proved.

If $\theta_1 < R/(\lambda + \alpha)$ then

$$\|x_1 - x_0\| < \gamma_{i_0}(\alpha + \|v_{i_0}\|) \leqq \frac{R}{\lambda + \alpha} \cdot (\lambda + \alpha) = R,$$

thus $x_1 \in X_0$ and we can continue with x_1. Now by the same argument as before we have $x_1 \in B(y_{i_1}, \eta_{i_1})$ for some $i_1 \in \{1, \cdots, q\}$. We then deduce the existence of $x_2 \in P(x_1)$ such that $\|(x_2 - x_1)/\gamma_{i_1} - v_{i_1}\| < \alpha$ with $\|x_1 - y_{i_1}\| < \eta_{i_1} < \alpha$ and $v_{i_1} \in S(y_{i_1})$. We define $\theta_2 = \theta_1 + \gamma_{i_1} = \gamma_{i_0} + \gamma_{i_1}$. Then if $\theta_2 \geqq R/(\lambda + \alpha)$ we stop and the result is proved.

If $\theta_2 < R/(\lambda + \alpha)$ then $\|x_2 - x_0\| \leqq \|x_2 - x_1\| + \|x_1 - x_0\|$ which gives

$$\|x_2 - x_0\| \leqq (\gamma_{i_0} + \gamma_{i_1})(\alpha + \lambda) < \frac{R}{\lambda + \alpha}(\lambda + \alpha) = R,$$

thus $x_2 \in X_0$ and we can continue.

As we have a finite number of $(\gamma_i)_{i=1,\cdots,q}$ all strictly positive, we are sure that after a finite number of operations we shall get $\theta_m > R/(\lambda + \alpha)$. The lemma is then proved.

We can now end the proof of Theorem I-1.

PROOF. To $\alpha = 1/n$ with n a strictly positive integer we can associate by Lemma I-1 two finite sequences $\{\theta_k^{(n)}\}_{k=1,\cdots,m}$ and $\{x_k^{(n)}\}_{k=1,\cdots,m}$ having the properties given by the lemma. We remark that m depends in fact on n.

We can now define the function $u_n : [0, \theta_m^{(n)}] \to \mathbf{R}^p$ in the following way: on each interval $[\theta_k^{(n)}, \theta_{k+1}^{(n)}]$ the function u_n is the linear function interpolating $x_k^{(n)}$ and $x_{k+1}^{(n)}$.

Thus for all $t \in [\theta_k^{(n)}, \theta_{k+1}^{(n)}]$ we have:

$$u_n(t) = x_k^{(n)} + [t - \theta_k^{(n)}] \frac{x_{k+1}^{(n)} - x_k^{(n)}}{\theta_{k+1}^{(n)} - \theta_k^{(n)}}.$$

Moreover we notice that $u_n(0) = x_0$ and as we have $\theta_m^{(n)} \geqq R/(\lambda + 1/n)$ the function u_n is at least defined on $[0, R/(\lambda + 1/n)]$.

Furthermore as u_n is piecewise linear with

$$\frac{du_n}{dt}(t) = \frac{x_{k+1}^{(n)} - x_k^{(n)}}{\theta_{k+1}^{(n)} - \theta_k^{(n)}} \qquad \text{for } t \in]\theta_k^{(n)}, \theta_{k+1}^{(n)}[,$$

from (1) we have the existence of $y_k^{(n)} \in X_0$ and $v_k^{(n)} \in S(y_k^{(n)})$ such that

$$\|x_k^{(n)} - y_k^{(n)}\| < 1/n \quad \text{and} \quad \left\| \frac{du_n}{dt}(t) - v_k^{(n)} \right\| < \frac{1}{n}.$$

Thus

$$\left\| \frac{du_n}{dt}(t) \right\| < \|v_k^{(n)}\| + \frac{1}{n} \leqq \lambda + \frac{1}{n} \leqq \lambda + 1.$$

So the function u_n is $(\lambda + 1)$-Lipschitz on $[0, \theta_m^{(n)}]$ and then obviously on $[0, R/(\lambda + 1/n)]$. Furthermore, since $\theta_{k+1}^{(n)} - \theta_k^{(n)} < 1/n$ we have for all $t \in [\theta_k^{(n)}, \theta_{k+1}^{(n)}]$ with $k \in \{0, 1, \cdots, m-1\}$,

$$\|u_n(t) - x_k^{(n)}\| \leqq (t - \theta_k^{(n)})(\lambda + 1) \leqq (\theta_{k+1}^{(n)} - \theta_k^{(n)})(\lambda + 1) \leqq (\lambda + 1)/n.$$

Thus as $x_k^{(n)} \in X_0$ for all $k \in \{0, 1, \cdots, m-1\}$ we deduce that $d_{X_0}[u_n(t)] \leqq (\lambda + 1)/n$ and this for all $t \in [0, R/(\lambda + 1/n)] \subset [0, \theta_m^{(n)}]$.

We can then build a sequence of such functions u_n defined on increasing intervals $[0, R/(\lambda + 1/n)]$. But by the very properties of the functions u_n, and using Ascoli's theorem, it is possible to extract a subsequence (again denoted u_n) which converges uniformly on every compact subset of $[0, R/\lambda[$ to a function $u : [0, R/\lambda[\to \mathbf{R}^p$ such that $u(0) = x_0$, $u(t) \in X_0$ for all $t \in [0, R/\lambda[$ since we have $d_{X_0}[u(t)] = \lim_{n \to +\infty} d_{X_0}[u_n(t)] = 0$ with X_0 compact.

Furthermore we easily verify that u is λ-Lipschitz on $[0, R/\lambda[$ since each u_n is in fact $(\lambda + 1/n)$ Lipschitz on $[0, R/(\lambda + 1/n)]$. And obviously u can be extended by continuity on $[0, R/\lambda]$ to a λ-Lipschitz function with values in X_0 compact. So $du(t)/dt$ exists almost everywhere on $]0, R/\lambda[$ and $du/dt \in L^\infty([0, R/\lambda], \mathbf{R}^p)$.

We shall now prove that u verifies (I) for almost all $t \in]0, R/\lambda[$. For this let us fix $T, 0 < T < R/\lambda$, then for n large enough we are sure that $T \leqq R/(\lambda + 1/n)$. Thus for all $s, t \in [0, T]$ we have

$$u_n(t) - u_n(s) = \int_s^t \frac{du_n}{dt}(z)dz \quad \text{converges to} \quad u(t) - u(s) = \int_s^t \frac{du}{dt}(z)dz$$

and since the derivatives du_n/dt are equibounded on $[0, T]$ we deduce from $L^\infty([0, T], \mathbf{R}^p) \subset L^1([0, T], \mathbf{R}^p)$ that the sequence du_n/dt converges weakly to du/dt in $L^1([0, T], \mathbf{R}^p)$. Then using Mazur's convexity theorem we can build a sequence of convex combinations of the following type:

$$\mu_p = \sum_{n=p}^{+\infty} a_p^n \frac{du_n}{dt} \qquad \text{with } a_p^n \geqq 0,$$

all but a finite number equal to zero and $\sum_{n=p}^{+\infty} a_p^n = 1$, such that μ_p converges strongly to du/dt in $L^1([0, T], \mathbf{R}^p)$ as $p \to +\infty$.

Then we can extract a subsequence of μ_p (again denoted μ_p for simplicity) which converges pointwise to $du(t)/dt$ for almost all $t \in]0, T[$. Let $t \in]0, T[$ be such a point of convergence. Since S is upper semi-continuous, for any $\varepsilon > 0$ there exists $\eta > 0$ such that for all $x \in X$, $\|x - u(t)\| < \eta$ implies $S(x) \subset S[u(t)] + \varepsilon \bar{B}$. Furthermore for all n, there exists an interval $[\theta_k^{(n)}, \theta_{k+1}^{(n)}]$ such that $t \in [\theta_k^{(n)}, \theta_{k+1}^{(n)}[$ with

$$\frac{du_n}{dt}(t) = \frac{x_{k+1}^{(n)} - x_k^{(n)}}{\theta_{k+1}^{(n)} - \theta_k^{(n)}} \qquad \text{where } \theta_{k+1}^{(n)} - \theta_k^{(n)} < \frac{1}{n}$$

and from (1) we have the existence of $y_k^{(n)} \in X_0 \subset X$ depending on $x_k^{(n)}$ such that

$$\|y_k^{(n)} - x_k^{(n)}\| < 1/n \quad \text{and} \quad \left\| \frac{du_n}{dt}(t) - v_k^{(n)} \right\| < \frac{1}{n} \quad \text{with} \quad v_k^{(n)} \in S(y_k^{(n)}).$$

Thus from the inequality:

$$\|y_k^{(n)} - u(t)\| \leqq \|y_k^{(n)} - u_n(\theta_k^{(n)})\| + \|u_n(\theta_k^{(n)}) - u_n(t)\| + \|u_n(t) - u(t)\|$$

with $u_n(\theta_k^{(n)}) = x_k^{(n)}$ and since $u_n(t)$ converges to $u(t)$, each u_n being $(\lambda + 1)$-Lipschitz which gives:

$$\|u_n(\theta_k^{(n)}) - u(t)\| \leqq (\lambda + 1)(t - \theta_k^{(n)}) \leqq (\lambda + 1)(\theta_{k+1}^{(n)} - \theta_k^{(n)}) < (\lambda + 1)/n$$

we deduce easily that for n large enough we have $\| y_k^{(n)} - u(t)\| < \eta$ which implies that $S(y_k^{(n)}) \subset S[u(t)] + \varepsilon \bar{B}$.

Thus $v_k^{(n)} \in S[u(t)] + \varepsilon \bar{B}$ and from $\| du_n(t)/dt - v_k^{(n)} \| < 1/n$ we deduce that for n large enough we have $du_n(t)/dt \in S[u(t)] + 2\varepsilon \bar{B}$ which is convex since S is convex valued.

Thus for ρ large enough we have $u_\rho(t) \in S[u(t)] + 2\varepsilon \bar{B}$ which is closed since S is compact valued. Now taking the limit as $\rho \to +\infty$ we get $du(t)/dt \in S[u(t)] + 2\varepsilon \bar{B}$.

But since this is verified for any $\varepsilon > 0$ and since $S[u(t)]$ is compact we get $du(t)/dt \in S[u(t)]$, and this for almost all $t \in]0, T[$. But since this is true for any $T < R/\lambda$, we are sure that (I) is verified for almost all $t \in]0, R/\lambda[$. For the theorem we take $T_0 = R/\lambda$. To finish we must now verify the monotonicity of u. Let $s, t \in [0, R/\lambda[$, $s < t$ be given. Then for n large enough we have $s \in [\theta_k^{(n)}, \theta_{k+1}^{(n)}]$ and $t \in [\theta_\rho^{(n)}, \theta_{\rho+1}^{(n)}]$ with $k + 1 \leqq \rho$. Then by the definition of u_n and by the transitivity of P we deduce from Lemma I-1 that $u_n(\theta_\rho^{(n)}) = x_\rho^{(n)} \in P[x_{k+1}^{(n)}] = P[u_n(\theta_{k+1}^{(n)})]$. It is then obvious to verify that $u_n(\theta_\rho^{(n)}) \to u(t)$ and $u_n(\theta_{k+1}^{(n)}) \to u(s)$ as $n \to +\infty$ and as the graph of P is closed we get $u(t) \in P[u(s)]$. For the same reason and thanks to the continuity of u at R/λ we have also $u(R/\lambda) \in P[u(s)]$. The proof is then complete.

As an immediate consequence of this theorem we can give the following corollary.

COROLLARY I-1. *Let X be a locally compact subset of $\mathbf{R}^p$, $S : X \to \mathbf{R}^p$ an upper semi-continuous non-empty convex compact valued correspondence. Then the following condition:*

$$\text{(C')} \qquad S(x) \cap T_X(x) \neq \varnothing, \qquad \textit{for all } x \in X$$

is equivalent to the existence property:

For any $x_0 \in X$ there exist $T_0 > 0$ and a Lipschitz function $u : [0, T_0] \to X$ such that $u(0) = x_0$ with:

$$\text{(I)} \qquad \frac{du}{dt}(t) \in S[u(t)] \qquad \textit{for almost all } t \in]0, T_0[.$$

PROOF. The proof is immediate by taking $P(x) = X$ for all $x \in X$.

REMARK 1. We have considered solutions for initial time $t = 0$; the result is in fact the same for any initial time $t_0 \in \mathbf{R}$, the solution being defined on the interval $[t_0, t_0 + T_0]$.

REMARK 2. Since locally compact subsets of $\mathbf{R}^p$ are all the subsets which are the intersection of an open and of a closed subset, Theorem I-1 and its Corollary are particularly applicable to open or closed subsets of $\mathbf{R}^p$.

REMARK 3. If X is closed we can choose $R>0$ independently from x_0 to get $X\cap\bar{B}(x_0,R)$ compact. Then if we suppose that S is bounded on X by $\lambda>0$, since all solutions of (I) can be defined on $[t_0,t_0+R/\lambda]$ for any $t_0\in\mathbf{R}$ and any initial value in X, it becomes obvious that the solution u of Theorem I-1 can be extended on $[0,+\infty[$ and will be λ-Lipschitz.

If X is compact we are exactly in that situation.

3. The time dependent version

We can now give a time dependent version when the subset depends on time.

THEOREM I-2. *Let $t\to K(t)\subset\mathbf{R}^p$ be a non-empty valued correspondence defined on $[0,+\infty[$ with a locally compact graph $\mathcal{K}$. Let $S:\mathcal{K}\to\mathbf{R}^p$ be an upper semi-continuous non-empty convex compact valued correspondence such that for all $t\geqq 0$ and all $x\in K(t)$ there exists $v\in S(t,x)$ verifying*:

$$(\mathrm{C}_1)\qquad \liminf_{h\to 0^+}\frac{d_{K(t+h)}[x+hv]}{h}=0.$$

Then this is equivalent to:

For all $x_0\in K(0)$, there exist $T_0>0$ and a Lipschitz function $u:[0,T_0]\to\mathbf{R}^p$ such that $u(0)=x_0$, $u(t)\in K(t)$ for all $t\in[0,T]$ and

$$(\mathrm{I}')\qquad \frac{du}{dt}(t)\in S[t,u(t)]\qquad \textit{for almost all } t\in\,]0,T_0[.$$

PROOF. Since

$$\frac{du}{dt}(t)=\lim_{h\to 0^+}\frac{u(t+h)-u(t)}{h}\qquad \text{with } u(t+h)\in K(t+h)$$

we see that condition (C_1) appears to be necessary with a proof similar to the one of Theorem I.1.

For the sufficiency let us consider the correspondence $H:\mathcal{K}\to\mathbf{R}\times\mathbf{R}^p$ defined by $H(t,x)=\{1\}\times S(t,x)$.

We see that H is obviously semi-continuous non-empty convex compact valued. Moreover, for $(t,x)\in\mathcal{K}$ let us consider $(1,v)\in H(t,x)$ with v given by condition (C_1). Then from (C_1) we have the existence of $h_n \to 0^+$ and $x_n \in K(t+h_n)$ such that $\|(x_n - x)/h_n - v\| \to 0$ as $n \to +\infty$.

It becomes obvious to verify that we have:

$$\left\| \frac{(t+h_n, x_n) - (t, x)}{h_n} - (1, v) \right\| \to 0 \qquad \text{as } n \to +\infty,$$

the norm being taken on the product space $\mathbf{R} \times \mathbf{R}^p$. Then we have

$$\lim_{h \to 0^+} \operatorname{Inf} \frac{d_{\mathcal{K}}[(t, x) + h(1, v)]}{h} = 0$$

and since $\mathcal{K}$ is locally compact, H verifies all the hypotheses of Theorem I-1.

Thus there exist $T_0 > 0$ and a Lipschitz function $\xi \to (t(\xi), u(\xi)) \in \mathcal{K}$ defined on $[0, T_0]$ such that $t(0) = 0$, $u(0) = x_0$ and verifying:

$$\begin{cases} \dfrac{dt(\xi)}{d\xi} = 1, \\ \\ \dfrac{du}{d\xi}(\xi) \in S[t(\xi), u(\xi)] \qquad \text{for almost all } \xi \in]0, T_0[. \end{cases}$$

And now the end of the proof becomes obvious.

REMARK 4. If S is assumed to be bounded and $\mathcal{K}$ closed, by Remark 3 we can extend the solution u on $[0, +\infty[$.

4. Solutions on a subset defined by constraints

The following theorem will give for a subset Z defined by $Z = \{x \in X / Lx \in Y\}$, a tangential condition involving X, Y and L rather than Z itself.

THEOREM I-3. *Let $X \subset \mathbf{R}^p$ and $Y \subset \mathbf{R}^q$, two non-empty closed subsets, and $L \in \mathcal{L}(\mathbf{R}^p, \mathbf{R}^q)$ be given.*

If $S : X \to \mathbf{R}^p$ is an upper semi-continuous non-empty convex compact valued correspondence such that for all $x \in X$ there exists $v \in S(x)$ verifying:

$$(\mathrm{C}_2) \qquad \lim_{\gamma \to 0^+} \operatorname{Inf} \frac{1}{\gamma} \operatorname{Max}\{d_X(x + \gamma v), d_Y(L(x + \gamma v)) - d_Y(Lx)\} = 0.$$

Then for any $x_0 \in X$ such that $Lx_0 \in Y$, there exist $T_0 > 0$ and a Lipschitz function $u : [0, T_0] \to \mathbf{R}^p$ which verifies: $u(0) = x$, $u(t) \in X$ and $Lu(t) \in Y$ for all $t \in [0, T_0]$ together with:

$$\text{(I)} \qquad \frac{du}{dt}(t) \in S[u(t)] \qquad \textit{for almost all } t \in]0, T_0[.$$

Proof. The proof is very similar to the proof of Theorem I-1, when no preorder is considered. We shall give only the main features.

For $R > 0$ we consider the compact set $X_0 = X \cap \bar{B}(x_0, R)$ and $\lambda > 0$ such that $\lambda \geqq \operatorname{Sup}_{v \in S(x), x \in X_0} \| v \|$.

Then as in Lemma I-1, let $\alpha > 0$ be given. By (C_2), for any $y \in X_0$ there exist $v_y \in S(y)$ and γ_y, $0 < \gamma_y < \alpha$ such that:

$$\operatorname{Max} \frac{1}{\gamma_y} \{ d_X(y + \gamma_y v_y), d_Y[L(y + \gamma_y v_y)] - d_Y(Ly) \} < \alpha.$$

Then we consider $V(y)$ defined by:

$$V(y) = \{ x \in X_0 / \frac{1}{\gamma_y} \operatorname{Max} \{ d_X(x + \gamma_y v_y), d_Y[L(x + \gamma_y v_y)] - d_Y(Lx) \} < \alpha \}.$$

This set is obviously an open neighborhood of y in X_0. Thus if we continue as in the proof of Lemma I-1, we prove the existence of $0 = \theta_0 < \theta_1 < \cdots < \theta_{m-1} < R/(\lambda + \alpha) \leqq \theta_m$ with $\theta_{k+1} - \theta_k < \alpha$ together with an associate sequence $\{x_0, x_1, \cdots, x_m\}$ in X such that for any $k = 0, 1, \cdots, m-1$ we have $x_k \in X_0$ and the existence of $y_k \in X_0$ depending on x_k with $v_k \in S(y_k)$ such that:

$$\text{(1)} \qquad \begin{cases} \| x_k - y_k \| < \alpha, \\ \left\| \dfrac{x_{k+1} - x_k}{\theta_{k+1} - \theta_k} - v_k \right\| < \alpha, \end{cases}$$

with, in addition:

$$\text{(2)} \qquad \frac{d_Y[L(x_k + (\theta_{k+1} - \theta_k) v_k)] - d_Y(Lx_k)}{\theta_{k+1} - \theta_k} < \alpha.$$

Thus we get:

$$\frac{d_Y(Lx_{k+1}) - d_Y(Lx_k)}{\theta_{k+1} - \theta_k} = \frac{d_Y(Lx_{k+1}) - d_Y[L(x_k + (\theta_{k+1} - \theta_k) v_k)]}{\theta_{k+1} - \theta_k} + \frac{d_Y[L(x_k + (\theta_{k+1} - \theta_k) v_k)] - d_Y(Lx_k)}{\theta_{k+1} - \theta_k}.$$

Then, using (2), the linearity and continuity of L together with the fact that the function $d_Y(\cdot)$ is Lipschitz gives:

$$\frac{d_Y(Lx_{k+1})-d_Y(Lx_k)}{\theta_{k+1}-\theta_k} \leqq \frac{\|Lx_{k+1}-L(x_k+(\theta_{k+1}-\theta_k)v_k)\|}{\theta_{k+1}-\theta_k}+\alpha$$

$$= \left\| L\left(\frac{x_{k+1}-x_k}{\theta_{k+1}-\theta_k}-v_k\right)\right\| + \alpha$$

$$\leqq \|L\| \left\| \frac{x_{k+1}-x_k}{\theta_{k+1}-\theta_k}-v_k\right\| + \alpha$$

$$\leqq \|L\|\alpha+\alpha \qquad \text{by (1).}$$

Thus we have:

$$d_Y(Lx_{k+1})-d_Y(Lx_k) \leqq [\theta_{k+1}-\theta_k]\cdot[\|L\|\alpha+\alpha]. \tag{3}$$

Summing these inequalities from $j=0$ to $k+1$ and using the fact that $d_Y(Lx_0)=0$ we get:

$$d_Y(Lx_{k+1}) \leqq \theta_{k+1}[\|L\|\alpha+\alpha]. \tag{4}$$

Then as in the proof of Theorem I-1 we build the sequence of functions u_n defined on $[0, R/(\lambda+1/n)]$ and prove the existence of a λ-Lipschitz function $u:[0,R/\lambda]\to X_0\subset X$ such that $u(0)=x_0$ verifying (I) for almost all $t\in]0,R/\lambda[$. The only difference is to prove that $Lu(t)\in Y$ for all $t\in[0,R/\lambda]$.

Let first $t\in[0,R/\lambda]$ be given, for n large enough we have $t<R/(\lambda+1/n)$ and $t\in[\theta_k^{(n)},\theta_{k+1}^{(n)}]\subset[0,R/\lambda]$. Moreover by the properties of u_n and u we know that $u_n(\theta_{k+1}^{(n)})=x_{k+1}^{(n)}\to u(t)$ as $n\to+\infty$. Then by (4) we have:

$$d_Y[Lx_{k+1}^{(n)}] \leqq \theta_{k+1}^{(n)}\left(\frac{\|L\|}{n}+\frac{1}{n}\right) \leqq \frac{R}{\lambda}\left(\frac{\|L\|}{n}+\frac{1}{n}\right).$$

Taking the limit as $n\to+\infty$ we get $d_Y[Lu(t)]=0$, thus $Lu(t)\in Y$ which is closed. By continuity we also have $Lu(R/\lambda)\in Y$ and the proof is complete.

REMARK 5. Always as in Remark 3, if S is bounded, then u can be extended on $[0,+\infty[$ with the same properties.

II. FUNCTIONAL DIFFERENTIAL INCLUSIONS WITH MEMORY

We define $\mathscr{C}=\mathscr{C}[(-r,0],\mathbf{R}^p]$ the space of continuous functions from $(-r,0]$ into $\mathbf{R}^p$. If not specified, the interval $(-r,0]$ can be either $[-r,0]$ when $0<r<+\infty$ or $]-r,0]$ when $0<r\leqq+\infty$. The topology on $\mathscr{C}$ will be the topology of uniform convergence in the first case and the topology of uniform convergence on compact subsets in the second case. For any $t\geqq 0$, let $A(t)$ be the

operator which to any continuous function U, defined at least on the interval $(-r, t]$, associates the function $A(t)U \in \mathscr{C}$ such that:

$$[A(t)U](\theta) = U(t+\theta) \qquad \text{for all } \theta \in (-r, 0].$$

Let Ω be a non-empty subset of $\mathscr{C}$, $F:\Omega \to \mathbf{R}^p$ a given correspondence with non-empty values, we call an autonomous functional differential inclusion with memory the differential inclusion:

$$\text{(M)} \qquad \frac{dU}{dt}(t) \in F[A(t)U].$$

A solution to (M) is a continuous function U from an interval $(-r, T)$ into $\mathbf{R}^p$ with $0 < T \leqq +\infty$ such that:

$$\begin{cases} U \text{ is absolutely continuous on every compact subset of } [0, T), \\ A(t)U \in \Omega \text{ for all } t \in [0, T), \\ \dfrac{dU}{dt}(t) \in F[A(t)U] \text{ for almost all } t \in]0, T). \end{cases}$$

We shall say that U is a solution to (M) with initial value $\phi \in \mathscr{C}$ if U is a solution of (M) and verifies $A(0)U = \phi$, which is equivalent to $U = \phi$ on $(-r, 0]$.

For simplicity we only consider solutions for initial time $t = 0$ and this with no loss of generality.

Let now X be a non-empty closed subset of $\mathbf{R}^p$; we define by $\mathscr{X}$ the set of all $\phi \in \mathscr{C}$ such that $\phi(0) \in X$. The topology on $\mathscr{X}$ will be induced by the topology of $\mathscr{C}$.

We can now state the following theorem:

THEOREM II-1. *Let $F:\mathscr{X} \to \mathbf{R}^p$ be an upper semi-continuous correspondence with non-empty convex compact values. We suppose that F is bounded on $\mathscr{X}$, which implies the existence of $\lambda > 0$ such that $\sup_{v \in F(\phi), \phi \in \mathscr{X}} \|v\| \leqq \lambda$. Then the following condition:*

$$(\mathrm{C}_3) \qquad F(\phi) \cap T_X(\phi(0)) \neq \emptyset \qquad \textit{for any } \phi \in \mathscr{X}$$

is equivalent to the existence property:

For any given $\phi \in \mathscr{X}$ there exists a continuous function $U:(-r, +\infty[\to \mathbf{R}^p$ such that $A(0)U = \phi$, $U(t) \in X$ for all $t \geqq 0$, U is λ-Lipschitz on $[0, +\infty[$ and verifies:

$$\text{(M)} \qquad \frac{dU}{dt}(t) \in F[A(t)U] \qquad \textit{for almost all } t > 0.$$

Necessity of Condition (C_3)

The proof of the necessity of (C_3) is similar to the proof in the first part. For the sufficiency we shall need a preliminary lemma stated below.

LEMMA II-1. *Let F be given exactly as in the theorem and $\phi \in \mathscr{X}$. Then for any strictly positive integer n, there exists a function $U_n : (-r, +\infty[\to \mathbf{R}^p$ such that $A(0)U_n = \phi$, $U_n(t) \in X$ for all $t \geqq 0$, U_n is λ-Lipschitz on $[0, +\infty[$ and verifies the following property:*

$$\frac{dU_n}{dt}(t) \in F\left[A\left(\frac{k+1}{n}\right)\psi_{k/n, U_n(t)}\right] \tag{5}$$
$$\text{for any } k \in \mathbf{N} \text{ and for almost all } t \in [k/n, (k+1)/n].$$

The function $\psi_{k/n,x} : (-r, (k+1)/n] \to \mathbf{R}^p$ is defined for any $k \in \mathbf{N}$ and $x \in X$ by $\psi_{k/n,x} = U_n$ on the interval $(-r, k/n]$ and as the linear function which interpolates $U_n(k/n)$ and x on $[k/n, (k+1)/n]$.

PROOF OF LEMMA II-1. We shall build U_n by induction on each $[k/n, (k+1)/n]$, $k \in \mathbf{N}$. For any $x \in X$ let us define $\psi_{0,x} : (-r, 1/n] \to \mathbf{R}^p$ by $\psi_{0,x} = \phi$ on $(-r, 0]$ and

$$\psi_{0,x}(t) = \phi(0) + nt(x - \phi(0)) \qquad \text{for } t \in [0, 1/n].$$

Let us then consider the correspondence $S_0 : X \to \mathbf{R}^p$ defined by $S_0(x) = F[A(1/n)\psi_{0,x}]$ for any $x \in X$. Since we verify easily that the mapping $x \to A(1/n)\psi_{0,x}$ is continuous from X into $\mathscr{C}$, the correspondence S_0 is upper semi-continuous convex compact valued. Furthermore, since F is bounded by λ, the same is true for S_0 on X.

At last from (C_3) and from $[A(1/n)\psi_{0,x}](0) = \psi_{0,x}(1/n) = x$ we see that:

$$S_0(x) \cap T_X(x) \neq \varnothing \qquad \text{for any } x \in X. \tag{C'}$$

Thus S_0 verifies all the hypotheses of Theorem I-1 and by its boundedness on X we have the existence of a λ-Lipschitz function $u_0 : [0, +\infty[\to X$ solution of the differential inclusion

$$\frac{du}{dt}(t) \in S_0[u(t)] = F[A(1/n)\psi_{0,u(t)}], \tag{I_0}$$

for the initial value $u_0(0) = \phi(0)$.

We only consider u_0 on $[0, 1/n]$ and define $U_n = u_0$ on $[0, 1/n]$. Condition (5) is then verified by U_n on $[0, 1/n]$.

Now, for any $x \in X$ we define $\psi_{1/n,x}:(-r,2/n]\to \mathbf{R}^p$ by $\psi_{1/n,x}=\phi$ on $(-r,0]$, $\psi_{1/n,x}=u_0$ on $[0,1/n]$ and $\psi_{1/n,x}(t)=u_0(1/n)+n(t-1/n)(x-u(1/n))$ for any $t\in[1/n,2/n]$.

Let us then consider the correspondence $S_1:X\to\mathbf{R}^p$ defined by $S_1(x)=F[A(2/n)\psi_{1/n,x}]$ for any $x\in X$. Then by exactly the same argument as for S_0 we verify the existence of a λ-Lipschitz function $u_1:[1/n,+\infty[\to X$ solution of the differential inclusion

$$\text{(I)}\qquad \frac{du}{dt}(t)\in S_1[u(t)]=F[A(2/n)\psi_{1/n,u(t)}],$$

for the initial value $u_1(1/n)=u_0(1/n)$.

We only consider u_1 on $[1/n,2/n]$ and define $U_n=u_1$ on $[1/n,2/n]$. Condition (5) is then verified by U_n on $[1/n,2/n]$.

By induction we complete the construction of U_n on $[0,+\infty[$ and the lemma is proved.

We can now give a proof of Theorem II-1.

PROOF. Let us consider the sequence $(U_n)_{n\geqq 1}$ as given by the preceding lemma.

Each U_n is equal to ϕ on $(-r,0]$, λ-Lipschitz on $[0,+\infty[$. Thus using Ascoli's theorem there exists a function $U:(-r,+\infty[\to\mathbf{R}^p$, $A(0)U=\phi$, and a subsequence (again denoted U_n) which converges uniformly to U on every compact subset of $[0,+\infty[$.

As each U_n, the function U is λ-Lipschitz on $[0,+\infty[$ and since X is closed we have $U(t)\in X$ for any $t\geqq 0$.

We must now prove that U is a solution to (M). For this it is sufficient to prove that for any strictly positive integer T we have:

$$\frac{dU}{dt}(t)\in F[A(t)U]\qquad \text{for almost all } t\in\,]0,T[.$$

The arguments of the proof are in fact the same as those of the proof of Theorem I-1. The only difference is to prove that for some $t\in\,]0,T[$ chosen as in the proof of Theorem I-1 we have:

$$\frac{dU_n}{dt}(t)\in F[A(t)U]+\varepsilon\bar{B}\qquad \text{for } n \text{ large enough.}$$

Since F is upper semi-continuous on $\mathscr{X}$, for any $\varepsilon>0$ there exist $\eta>0$ and $[-r_\varepsilon,0]\subset(-r,0]$ such that for any $\psi\in\mathscr{X}$, $\|\psi-A(t)U\|_{[-r_\varepsilon,0]}\leqq\eta$ implies $F(\psi)\subset F[A(t)U]+\varepsilon\bar{B}$.

Let us consider $A((k+1)/n)\psi_{k/n, U_n(t)}$ with $t \in [k/n, (k+1)/n]$, as defined in Lemma II-1.

We have:

$$\left\|A\left(\frac{k+1}{n}\right)\psi_{k/n, U_n(t)} - A(t)U\right\|_{[-r_\varepsilon, 0]} \leqq \left\|A\left(\frac{k+1}{n}\right)\psi_{k/n, U_n(t)} - A\left(\frac{k+1}{n}\right)U\right\|_{[-r_\varepsilon, 0]} + \left\|A\left(\frac{k+1}{n}\right)U - A(t)U\right\|_{[-r_\varepsilon, 0]}.$$

But by the uniform continuity of U on the compact interval $[-r_\varepsilon, T]$ we deduce easily from the definition of $A((k+1)/n)U$ and $A(t)U$, that for n large enough:

$$\left\|A\left(\frac{k+1}{n}\right)U - A(t)U\right\|_{[-r_\varepsilon, 0]} \leqq \eta/2.$$

At last by the definition of $\psi_{k/n, U_n(t)}$ we have:

$$\psi_{k/n, U_n(t)} = U_n \qquad \text{on } (-r, k/n]$$

and

$$\psi_{k/n, U_n(t)}(s) = U_n(k/n) + n(s - k/n)(U_n(t) - U_n(k/n))$$

for all $s \in [k/n, (k+1)/n]$.

This together with the fact that $U_n = U = \phi$ on $(-r, 0]$, that U_n converges uniformly to U on every compact subset of $[0, +\infty[$, being λ-Lipschitz leads obviously to:

$$\left\|A\left(\frac{k+1}{n}\right)\psi_{k/n, U_n(t)} - A\left(\frac{k+1}{n}\right)U\right\|_{[-r_\varepsilon, 0]} \leqq \eta/2$$

for n large enough.

Thus we have $F[A((k+1)/n)\psi_{k/n, U_n(t)}] \subset F[A(t)U] + \varepsilon\bar{B}$, $dU_n(t)/dt \in F[A(t)U] + \varepsilon\bar{B}$ which is the desired property.

THEOREM II-2. *Let $F: \mathscr{X} \to \mathbf{R}^p$ be a bounded upper semi-continuous correspondence with non-empty convex compact values and $P: X \to X$ a given continuous preorder.*

Then the following condition:

$$(\mathrm{C}_3') \qquad F(\phi) \cap T_{P[\phi(0)]}(\phi(0)) \neq \varnothing \qquad \textit{for any } \phi \in \mathscr{X}$$

is equivalent to the existence property:

For any given $\phi \in \mathscr{X}$ there exists a continuous function $U: (-r, +\infty[\to \mathbf{R}^p$ such *that* $A(0)U = \phi$, $U(t) \in X$ *for all* $t \geqq 0$, U *is Lipschitz and monotone with respect to* P *on* $[0, +\infty[$ *and verifies*:

$$\text{(M)} \qquad \frac{dU}{dt}(t) \in F[A(t)U] \qquad \text{for almost all } t > 0.$$

PROOF. The proof is exactly the same as the proof of Theorem II-1; the monotonicity is verified since the functions u_n used in the proof are monotone with respect to P thanks to condition (C_3').

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CEREMADE
UNIVERSITÉ DE PARIS — DAUPHINE
PARIS, FRANCE