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THE TOPOLOGICAL STRUCTURE OF 3-PSEUDOMANIFOLDS

BY

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ABSTRACT

A 3-pseudomanifold (briefly 3-pm) is a finite connected simplicial 3-complex in which the link of every vertex is a closed 2-manifold. Such a link is *singular* if it is not a sphere. It is proved that for a preassigned list Σ of closed 2-manifolds (other than spheres), there is a 3-pm in which the list of singular links is precisely Σ, iff the number of the non-orientable members in Σ with odd genus is even. Close relationship is found between 3-pms and 3-manifolds with boundary. This yields a simple proof for the 2-dimensional case of Pontrjagin–Thom's theorem (i.e., necessary and sufficient condition for a 2-manifold to bound a 3-manifold). The concept of a 3-pm is generalized to higher dimensions.

1. Introduction

A 3-pseudomanifold (briefly: 3-pm), as defined in [1], is a finite connected simplicial 3-complex $\mathcal{K}$, in which every 2-simplex (triangle) belongs to precisely two 3-simplices, the link of every 1-simplex (edge) is a circuit, and the link of every vertex is a connected 2-manifold without boundary.

The main result in [1] is that every finite set Σ of 2-manifolds is pm-realizable. That is, for every finite set Σ of (topologically distinct) 2-manifolds (connected, without boundary), there exists a 3-pm $\mathcal{K}$ such that for every vertex $x \in \mathcal{K}$, link$(x, \mathcal{K})$ is homeomorphic to some $S \in \Sigma$, and for every $S \in \Sigma$ there is some vertex $x \in \mathcal{K}$ such that link$(x, \mathcal{K})$ is homeomorphic to S. In this case we also say that $\mathcal{K}$ pm-realizes Σ. However, very little control was exercized in [1] on the multiplicity of each $S \in \Sigma$ in $\mathcal{K}$, that is, on the number of vertices x in $\mathcal{K}$ such that link$(x, \mathcal{K})$ is homeomorphic to S. Generally speaking, this multiplicity could not be arbitrarily preassigned. Here we show that, at the cost of adding a 2-sphere to the set Σ, the multiplicity of each $S \in \Sigma$ other than the 2-sphere can be arbitrarily predetermined.

Note that if Σ does not contain a sphere, and $\mathcal{K}$ pm-realizes Σ, then the link of

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no vertex in $\mathcal{K}$ is a sphere. Topologically, however, there is no reason to restrict the use of spheres as links of vertices, since if $\mathcal{K}'$ is a subdivision of $\mathcal{K}$, then $\mathcal{K}'$ has the same topological (though not combinatorial) structure as $\mathcal{K}$, and therefore also pm-realizes Σ, except for the fact that the link of every "new" vertex in $\mathcal{K}'$ is a 2-sphere (see Lemma 7 below).

Here we show (Theorem 1), that if there is no restriction on the use of 2-spheres, then not only the set Σ, but also the multiplicity of each 2-manifold in Σ other than 2-sphere can be arbitrarily (up to a certain natural condition) preassigned.

Moreover, we show (Theorem 2) that in this setting the concept of a 3-pm is closely related to the concept of a compact 3-manifold with boundary, and the problem of pm-realizing a set Σ is equivalent to the problem of the structure of the boundary of a 3-manifold. Thus our method yields a new and intrinsic approach to this last problem (Corollary 3), as well as a better understanding of the topological structure of a 3-pm (Theorem 8).

We would have liked to generalize our results to higher dimensions. However, since our investigations base heavily on the classification of 2-manifolds, and no such simple classification is known for higher dimensional manifolds, we do not see how this desired generalization can be carried out. Nevertheless, in Section 4 we generalize (in two ways) the concept of a 3-pm to any dimension n, and investigate some basic properties of those n-pms, in the hope that this will motivate further research in this direction.

2. Notation and main results

Let $\mathcal{K}$ be a simplicial n-complex. The elements of $\mathcal{K}$ are its *faces*, and the 0-elements of $\mathcal{K}$ are also called *vertices*. For every $A \in \mathcal{K}$ define $\mathrm{st}(A, \mathcal{K})$, the *star* of A in $\mathcal{K}$, to be the set $\{B \in \mathcal{K} : A \subset B\}$; define $\mathrm{ast}(A, \mathcal{K})$, the *antistar* of A in $\mathcal{K}$, to be the complex $\{B \in \mathcal{K} : A \cap B = \emptyset\}$; define $\mathrm{clst}(A, \mathcal{K})$ to be the smallest subcomplex of $\mathcal{K}$ which contains $\mathrm{st}(A, \mathcal{K})$, and define $\mathrm{link}(A, \mathcal{K})$, the *link* of A in $\mathcal{K}$, to be $\mathrm{clst}(A, \mathcal{K}) \cap \mathrm{ast}(A, \mathcal{K})$. All the manifolds mentioned in this paper are compact and (unless otherwise specified) connected, and all the 2-manifolds are without boundary.

As is well known, every abstract finite simplicial n-complex can be rectilinearly realized in the $(2n+1)$-dimensional Euclidean space R^{2n+1}. Thus we may — and usually do — deal with our complexes, which are all simplicial, as abstract complexes. An n-simplex whose vertices are $a_0, a_1, \cdots, a_n$ is denoted by $a_0 a_1 \cdots a_n$. If $\Delta_1 = a_0 a_1 \cdots a_i$ and $\Delta_2 = a_{i+1} \cdots a_n$ are disjoint simplexes, we denote by $\Delta_1 \cdot \Delta_2$ the simplex $a_0 \cdots a_n$. In particular, if $\Delta = a_0 \cdots a_n$ is an n-simplex and x is a vertex not in Δ, $x \cdot \Delta$ is the $(n+1)$-simplex $xa_0 \cdots a_n$ (if $\dim \Delta = -1$, that is, Δ is empty, then $x \cdot \Delta = x$). For a simplicial complex $\mathscr{C}$ and a vertex x not in $\mathscr{C}$, we define $x \cdot \mathscr{C} = \{x \cdot \Delta : \Delta \in \mathscr{C}\}$, and call it the *cone* on $\mathscr{C}$ with *apex* x.

We shall consider a triangulation of a manifold as the manifold itself. Thus we say "$\operatorname{link}(x, \mathscr{K})$ is a manifold" while, strictly speaking, $\operatorname{link}(x, \mathscr{K})$ is a triangulation of a manifold, and $|\operatorname{link}(x, \mathscr{K})|$ is the manifold. Moreover, we assume all the manifolds mentioned in this paper to be endowed with some triangulation so that they form finite simplicial complexes in which the link of every interior vertex is a sphere (for dimensions 2 and 3 this is always the case (see [7]), while for dimensions $n \geqq 5$ there are triangulations which lack this property (see [4])). In particular, the boundary ∂M of a 3-dimensional manifold M is assumed to be endowed with the triangulation induced by that of M.

DEFINITION 1. Let $\mathscr{K}$ be a 3-pm, and let x be a vertex in $\mathscr{K}$. x is *singular* (in $\mathscr{K}$) if the 2-manifold $\operatorname{link}(x, \mathscr{K})$ is not a sphere. (Thus a 3-pm with no singular vertices is simply a 3-manifold without boundary.) A *singular link* (*star*) in $\mathscr{K}$ is the link (star) of a singular vertex. The *multiplicity* in $\mathscr{K}$ of a 2-manifold S is the number of vertices in $\mathscr{K}$ whose links are homeomorphic to S. A simplex Δ in $\mathscr{K}$ is *regular* if none of its vertices is singular in $\mathscr{K}$.

DEFINITION 2. Let $\Sigma = \{S_1^{\alpha_1}, \cdots, S_n^{\alpha_n}\}$ be a finite set of topologically distinct 2-manifolds S_i $(1 \leqq i \leqq n)$ none of which is a sphere, such that to each S_i is adjoined a positive integer α_i, the *multiplicity* of S_i. (If $\alpha_i = 1$, we often omit the superscript α_i.) Σ is said to be *strongly pm-realizable* (briefly: *spm-realizable*) if there exists a 3-pm $\mathscr{K}$ such that the singular links in $\mathscr{K}$ are precisely (homeomorphic to) $S_1, \cdots, S_n$, and each S_i appears in $\mathscr{K}$ with multiplicity α_i. In this case we also say that $\mathscr{K}$ spm-realizes Σ. We say that Σ *bounds* if there is a 3-manifold M whose boundary ∂M is composed of precisely α_1 2-manifolds (each homeomorphic to) S_1, α_2 2-manifolds $S_2, \cdots, \alpha_n$ 2-manifolds S_n, and an arbitrary number (possibly zero) of 2-spheres.

The main results to be proved here are as follows:

THEOREM 1. *A set Σ as in Definition 2 is spm-realizable if and only if $\Sigma\{\alpha_i : S_i \in \Sigma \;\&\; S_i$ is non-orientable and of odd genus$\}$ is even.*

THEOREM 2. *A set Σ as in Definition 2 is spm-realizable if and only if Σ bounds.*

As an immediate result of those two theorems we obtain:

COROLLARY 3. *A (possibly disconnected) 2-manifold N without boundary is (homeomorphic to) the boundary of some 3-manifold iff the number of connected components of N which are non-orientable and of odd genus, is even.*

This corollary is just a particular case ($n = 2$) of the general theorem by Pontrjagin and Thom (see [6, pages 17, 18]) which states that an n-dimensional (differentiable) manifold M is the boundary of an $(n+1)$-manifold iff all the Stiefel–Whitney numbers of M are zero. We could of course reverse the order, and obtain Theorem 1 as a result of Thom's theorem and Theorem 2 above. However, we prefer to prove Theorem 1 independently, since our proof is elementary, and thus obtain a new and simple proof for the 2-dimensional case of the Pontrjagin–Thom theorem.

3. Proofs

PROOF OF THEOREM 1. For every simplicial complex $\mathcal{M}$, we denote by $f_i(\mathcal{M})$ the number of i-faces of $\mathcal{M}$. For every 2-manifold S let $\chi(s)$ denote the Euler characteristic of S, and let $q(s)$, the connectivity of S, be defined by $q(s) = 2 - \chi(s)$. Thus $q(s)$ is the genus of S if S is not orientable, and twice the genus of S if S is orientable. For a 3-pm $\mathcal{K}$, we have the following analog of the Euler–Poincare relation:

LEMMA 4. *For every 3-pm $\mathcal{K}$ we have $\sum_{i=0}^{3}(-1)^i f_i(\mathcal{K}) = \frac{1}{2}\sum_x q(\mathrm{link}(x, \mathcal{K}))$, where the right sum ranges over all the vertices x of $\mathcal{K}$.*

PROOF. See [1, lemma 3].

From Lemma 4 it follows that the number of singular vertices in a 3-pm $\mathcal{K}$, whose links are non-orientable and of odd genus, is even. Thus the "only if" part of Theorem 1 is proved.

In the proof of the "if" part of Theorem 1, we will often use the following construction: Starting with two disjoint 3-pm's $\mathcal{K}_1, \mathcal{K}_2$, we will identify some 3-simplex $\Delta_1 \in \mathcal{K}_1$ with some 3-simplex $\Delta_2 \in \mathcal{K}_2$ (we may also start with two 3-pm's $\mathcal{K}_1, \mathcal{K}_2$ whose intersection is a unique 3-simplex Δ and its faces), and remove the simplex $\Delta = \Delta_1 = \Delta_2$ (but none of its proper faces). The resulting complex $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2 \setminus \{\Delta\}$ is said to be obtained from $\mathcal{K}_1$ and $\mathcal{K}_2$ by *assembling* them at $\Delta = \Delta_1 = \Delta_2$, and its structure is given in the following lemma, which is a particular case of [1, lemma 7]:

LEMMA 5. *Let $\mathcal{K}_1, \mathcal{K}_2$ be 3-pm's whose intersection is a unique 3-simplex Δ and its faces. Then the complex $\mathcal{K}$ obtained from $\mathcal{K}_1$ and $\mathcal{K}_2$ by assembling them at Δ is a 3-pm, and*

(1) *for every vertex* $x \in \mathcal{K}_i \backslash \{\Delta\}$ *we have* $\operatorname{link}(x, \mathcal{K}) = \operatorname{link}(x, \mathcal{K}_i)$ $(i = 1, 2)$;

(2) *for every vertex* $x \in \Delta$ *we have* $q(\operatorname{link}(x, \mathcal{K})) = q(\operatorname{link}(x, \mathcal{K}_1)) + q(\operatorname{link}(x, \mathcal{K}_2))$ *and* $\operatorname{link}(x, \mathcal{K})$ *is orientable iff both* $\operatorname{link}(x, \mathcal{K}_1)$ *and* $\operatorname{link}(x, \mathcal{K}_2)$ *are orientable.*

For a set Σ as in Definition 2, let $\Sigma_1 = \{S_{i_1}^{\beta_1}, \cdots, S_{i_k}^{\beta_k}\}$, $\Sigma_2 = \{S_{j_1}^{\gamma_1}, \cdots, S_{j_l}^{\gamma_l}\}$ be similar sets such that $\{S_1, \cdots, S_n\} = \{S_{i_1}, \cdots, S_{i_k}\} \cup \{S_{j_1}, \cdots, S_{j_l}\}$, and such that the multiplicity of each S_i in Σ is the sum of its multiplicities in Σ_1 and Σ_2. Then we say that Σ_1, Σ_2 form a *partition* of Σ into two parts, and we write $\Sigma = \Sigma_1 \cup \Sigma_2$. The partition of Σ into any finite number of parts is defined similarly.

In order to construct the 3-pm $\mathcal{K}$ spm-realizing the set Σ of Theorem 1, we first partition Σ into sets $\Sigma_1, \cdots, \Sigma_m$, each of which is of the form dealt with in Lemma 6. Lemma 6 will enable us to spm-realize each of these Σ_i by some 3-pm, and the desired $\mathcal{K}$ will be obtained by assembling together all those 3-pm's.

LEMMA 6. *Let* S, S_1, S_2 *be* 2*-manifolds such that* $q(S)$ *is positive and even and* $q(S_1), q(S_2)$ *are odd. That is,* S *is a* 2*-manifold which is either orientable and not a* 2*-sphere, or non-orientable with even genus,* S_1, S_2 *are non-orientable* 2*-manifolds with odd genus. If* Σ *is either* $\{S\}$, *or* $\{S_1, S_2\}$ *or* $\{S_1^2\}$, *then* Σ *is spm-realizable.*

PROOF. *Case* (a) $\Sigma = \{S\}$, S *is orientable.*

S can be embedded in R^3 and such an embedding, which we denote again by S, separates R^3 into two components — one of them bounded — with common boundary S. Denote by $\bar{S}$ the bounded component. Then $\bar{S}$ is a 3-manifold, and $\partial\bar{S} = S$. We endow $\bar{S}$ with some triangulation so that it be a simplicial complex. Let x be a point (vertex) not in $\bar{S}$. We claim that $x \cdot S \cup \bar{S}$ is a 3-pm in which the link of every vertex other than x is a 2-sphere, and the link of x is S. Thus $x \cdot S \cup \bar{S}$ spm-realizes $\Sigma = \{S\}$.

Indeed, clearly $\operatorname{link}(x, x \cdot S \cup \bar{S}) = S$. If $y \neq x$ is a vertex in $x \cdot S \cup \bar{S}$, then y is either an inner vertex in $\bar{S}$, in which case $\operatorname{link}(y, x \cdot S \cup \bar{S}) = \operatorname{link}(y, \bar{S}) =$ 2-sphere, or $y \in S$, in which case $\operatorname{link}(y, x \cdot S \cup \bar{S}) = \operatorname{link}(y, \bar{S}) \cup x \cdot \operatorname{link}(y, S)$, and this is the union of two discs — disjoint except for their common boundary which is a 1-sphere — and therefore it is a 2-sphere.

Case (b) $\Sigma = \{S\}$, S *is non-orientable with even genus* $2p$

Let B_1 be a solid Klein bottle, i.e., a 3-manifold whose boundary ∂B_1 is a Klein bottle. (For the existence of such B_i see, e.g., [2, page 133].) Let x be a vertex not in B_1. Then, as in Case (a), $K_1 = x \cdot \partial B_1 \cup B_1$ is a 3-pm in which the only singular vertex is x, and $\operatorname{link}(x, \mathcal{K}_1) = \partial B_1$. Thus, in case $p = 1$, $\mathcal{K}_1$ pm-realizes Σ. If $p > 1$, let $B_2, \cdots, B_p$ be solid Klein bottles (or solid toruses) which are mutually disjoint, except that for each $1 \leqq i \leqq p$, ∂B_i shares with ∂B_{i+1} a common 2-simplex Δ_{i+1} (and its faces), such that $\Delta_2, \cdots, \Delta_p$ are mutually disjoint. We claim that $\mathcal{K} = \bigcup_{i=1}^{p} (x \cdot \partial B_i \cup B_i) \setminus \bigcup_{i=2}^{p} (x \cdot \Delta_i)$ is a 3-pm which spm-realizes Σ. Namely, $\mathcal{K}$ is a 3-pm in which the only singular vertex is x, and $\operatorname{link}(x, \mathcal{K})$ is homeomorphic to S.

Indeed, for each $1 \leqq j \leqq p$, let $\mathcal{K}_j = \bigcup_{i=1}^{j} (x \cdot \partial B_i \cup B_i) \setminus \bigcup_{i=2}^{j} (x \cdot \Delta_i)$. We already know that $\mathcal{K}_1$ is a 3-pm in which the only singular vertex is x, and $\operatorname{link}(x, \mathcal{K}_1)$ is a Klein bottle. We proceed by induction on i. Assume that for some $1 \leqq i < p$ $\mathcal{K}_i$ is indeed a 3-pm in which the only singular vertex is x, and $\operatorname{link}(x, \mathcal{K}_i)$ is a non-orientable 2-manifold with genus $2i$. Define $\mathcal{K}_i^* = x \cdot \partial B_{i+1} \cup B_{i+1}$. Then $\mathcal{K}_i^*$, like $\mathcal{K}_i$, is a 3-pm in which the only singular vertex is x, and $\operatorname{link}(x, \mathcal{K}_i^*) = \partial B_{i+1}$. The complex $\mathcal{K}_{i+1}$ is then obtained by assembling $\mathcal{K}_i$ and $\mathcal{K}_i^*$ at their common 3-simplex $x \cdot \Delta_{i+1}$, and it therefore follows from Lemma 5 that $\mathcal{K}_{i+1}$ is a 3-pm with a unique singular vertex, namely x, and $\operatorname{link}(x, \mathcal{K}_{i+1})$ is a non-orientable 2-manifold of genus $2(i+1)$. Since $\mathcal{K} = \mathcal{K}_p$, we are done.

Case (*c*) $\Sigma = \{S_1, S_2\}$

S_1, S_2 are non-orientable 2-manifolds, $q(S_1) = 2m + 1$, $q(S_2) = 2n + 1$ $(m, n \geqq 0)$.

Let P be a projective plane, and let x, y be two distinct vertices not in P. The complex $\mathcal{K}_1 = x \cdot P \cup y \cdot P$, i.e., the suspension of P, is easily seen to be a 3-pm with precisely two singular vertices, namely x and y, and $\operatorname{link}(x, \mathcal{K}_1) = \operatorname{link}(y, \mathcal{K}_1) = P$. Now let $B_1, \cdots, B_m$, $B'_1, \cdots, B'_n$ be solid Klein bottles (or toruses) which are mutually disjoint and disjoint to $\mathcal{K}_1$, except that $P \cap \partial B_1$ is a 2-simplex Δ_1 (and its faces), $P \cap \partial B'_1$ is a 2-simplex Δ'_1 (and its faces) and for each $1 \leqq i < m$, $1 \leqq j < n$, $\partial B_i \cap \partial B_{i+1}$ is a 2-simplex Δ_{i+1} (and its faces), and $\partial B'_j \cap \partial B'_{j+1}$ is a 2-simplex Δ'_{i+1} (and its faces). Define

$$\mathcal{K} = \mathcal{K}_1 \cup \left(\bigcup_{i=1}^{m} (x \cdot \partial B_i \cup B_i) \right)$$
$$\cup \left(\bigcup_{j=1}^{n} (y \cdot \partial B'_j \cup B'_j) \right) \setminus \left(\bigcup_{i=1}^{m} (x \cdot \Delta_i) \cup \bigcup_{j=1}^{n} (y \cdot \Delta'_i) \right).$$

Then, as in Case (b), it follows easily from Lemma 5 that $\mathcal{K}$ is a 3-pm with precisely two singular vertices, namely x and y, $\operatorname{link}(x, \mathcal{K})$ is a non-orientable 2-manifold of genus $2m + 1$ and $\operatorname{link}(y, \mathcal{K})$ is a non-orientable 2-manifold of genus $2n + 1$. Thus $\mathcal{K}$ spm-realizes Σ.

Case (d) $\Sigma = \{S_1^2\}$

Proceed as in Case (c), just replacing S_2 by S_1.

In order to complete the proof of Theorem 1 we need one more lemma:

LEMMA 7. *If $\mathcal{K}$ is a 3-pm and $\mathcal{K}'$ is a subdivision of $\mathcal{K}$, then $\mathcal{K}'$ is a 3-pm and has the same singularities as $\mathcal{K}$, i.e., for every vertex $x \in \mathcal{K}'$, if x is not a vertex in $\mathcal{K}$ then* $\mathrm{link}(x, \mathcal{K}')$ *is a sphere, and if x is a vertex in $\mathcal{K}$ then* $\mathrm{link}(x, \mathcal{K}')$ *is homeomorphic to* $\mathrm{link}(x, \mathcal{K})$.

PROOF. See [5, lemma 1.14].

Now let Σ be a set as in Definition 2, satisfying the condition stated in Theorem 1. Then Σ can clearly be partitioned into $\Sigma_1, \cdots, \Sigma_m$, where each Σ_i $(1 \leqq i \leqq m)$ is of one of the types dealt with in Lemma 6, and therefore can be spm-realized by some $\mathcal{K}'_i$.

Define $\mathcal{K}_1 = \mathcal{K}'_1$ and inductively define $\mathcal{K}_i$ $(1 < i \leqq m)$ as follows: Using Lemma 7, we may assume that $\mathcal{K}_{i-1}$ and $\mathcal{K}'_i$ contain regular 3-simplices Δ_1 and Δ_2 respectively (otherwise replace $\mathcal{K}_{i-1}$ and $\mathcal{K}'_i$ by their second barycentric subdivisions). We first identify Δ_1 with Δ_2, and then let $\mathcal{K}_i$ be the complex obtained by assembling $\mathcal{K}_{i-1}$ and $\mathcal{K}'_i$ at $\Delta_1 = \Delta_2$. By Lemma 5 and the induction hypothesis it is easily seen that $\mathcal{K}_i$ is a 3-pm which spm-realizes the set $\Sigma_1 \cup \cdots \cup \Sigma_i$. Thus the complex $\mathcal{K} = \mathcal{K}_m$ is the desired 3-pm which spm-realizes Σ, and the proof of Theorem 1 is therefore complete. □

PROOF OF THEOREM 2. Let $\Sigma = \{S_1^{\alpha_1}, \cdots, S_n^{\alpha_n}\}$ be as in Definition 2. First assume that Σ bounds, i.e., there is a 3-manifold M whose boundary ∂M is composed of α_1 copies $S_{1,1}, \cdots, S_{1,\alpha_1}$, of S_1, α_2 copies $S_{2,1}, \cdots, S_{2,\alpha_2}$ of $S_2, \cdots, \alpha_n$ copies $S_{n,1}, \cdots, S_{n,\alpha_n}$ of S_n, and an arbitrary number, α_0 say, of 2-spheres $S_{0,1}, \cdots, S_{0,\alpha_0}$. Constructing cones on the connected components of ∂M such that the apexes of the cones are different from each other and none of them is in M, and adding those cones to M, yields the desired 3-pm $\mathcal{K}$ which spm-realizes Σ. More precisely, let $A = \{x_i^1, \cdots, x_i^{\alpha_i} : 0 \leqq i \leqq n\}$ be a set of $\Sigma_{i=0}^n \alpha_i$ distinct points none of which is in M. Define $\mathcal{K} = M \cup (\bigcup_{i=0}^n (\bigcup_{j=1}^{\alpha_i} x_i^j \cdot S_{i,j}))$. Then $\mathcal{K}$ is easily seen to be a 3-pm, its set of singular vertices is $A \setminus \{x_0^j : 1 \leqq j \leqq \alpha_0\}$, and for each $1 \leqq i \leqq n$ and $1 \leqq j \leqq \alpha_i$, $\mathrm{link}(X_i^j, \mathcal{K}) = S_{i,j}$. Thus $\mathcal{K}$ spm-realizes Σ.

Next assume that Σ is spm-realizable and $\mathcal{K}$ spm-realizes Σ. We may assume that all the singular stars in $\mathcal{K}$ are mutually disjoint, since otherwise (see Lemma 7) we can replace $\mathcal{K}$ by its second barycentric subdivision. Now the removal from $\mathcal{K}$ of all the singular stars in $\mathcal{K}$ clearly yields a 3-manifold M such that ∂M is composed of precisely α_i copies of S_i for each $1 \leqq i \leqq n$ (and no spheres at all). Thus Σ bounds. □

The proof of Theorem 2 yields also the following reformulation of Theorem 2:

THEOREM 8. *Every 3-manifold M yields a 3-pm by coning all the connected components of ∂M, and every 3-pm $\mathcal{K}$ yields a 3-manifold by removing from the second barycentric subdivision of $\mathcal{K}$ all the singular stars.*

4. Higher dimensional pseudomanifolds

The concept of a 3-pm can be generalized into higher dimensions in two natural ways:

DEFINITION 9 I. An *n-pseudomanifold* (briefly: n-pm; $n \geqq 2$) *of type* 1 is a connected simplicial n-complex in which the link of every vertex is a connected $(n-1)$-manifold without boundary.

DEFINITION 9 II. Inductively: A *1-pm of type* 2 is a 1-sphere. An *n-pm of type* 2 ($n \geqq 2$) is a connected simplicial n-complex in which the link of every vertex is an $(n-1)$-pm of type 2.

Clearly, every n-pm of type 1 is also an n-pm of type 2, and both definitions coincide for $n = 2, 3$. The suspension of a 3-pm (which is not a 3-manifold) is a 4-pm of type 2, but not of type 1. Similarly, for every $n \geqq 4$ there are n-pms of type 2 which are not of type 1. Recall that all the manifolds mentioned here are assumed to possess a triangulation in which the link of every interior vertex is a sphere.

At first glance it seems that for $n = 3$, Definition 9 I does not coincide with the definition of a 3-pm as given in [1, page 213] and in the beginning of the present article, since for a 3-pm it was required also that the link of every i-simplex ($i = 1, 2$) be a $(2-i)$-sphere. However, the next theorem and its corollary show that this is not the case.

THEOREM 10. *Let $\mathcal{M}$ be a connected simplicial n-complex. Fix i, $0 \leqq i < n-1$. If for every i-simplex Δ_i in $\mathcal{M}$ $\mathrm{link}(\Delta_i, \mathcal{M})$ is an $(n-i-1)$-manifold, then for every $i < j < n$, the link of every j-simplex in $\mathcal{M}$ is an $(n-j-1)$-sphere.*

PROOF. Let $i < j < n$, and let $\Delta_j \in \mathcal{M}$ be a j-simplex. Let Δ_i be some i-face of Δ_j. Then $\Delta_j = \Delta_i \cdot \Delta_{j-i-1}$, where Δ_{j-i-1} is the $(j-i-1)$-face of Δ_j disjoint to Δ_i. Now $\mathrm{link}(\Delta_j, \mathcal{M}) = \{\Delta \in \mathcal{M} : \Delta \cdot \Delta_j \in \mathcal{M}, \Delta \cap \Delta_j = \emptyset\} = \{\Delta \in \mathcal{M} : \Delta \cdot \Delta_i \cdot \Delta_{j-i-1} \in \mathcal{M}, \Delta \cap \Delta_i \cdot \Delta_{j-i-1} = \emptyset\} = \mathrm{link}(\Delta_{j-i-1}, \mathrm{link}(\Delta_i, \mathcal{M})) = (n-j-1)$-sphere, since $\mathrm{link}(\Delta_i, \mathcal{M})$ is an $(n-i-1)$-manifold. □

COROLLARY 11. *In an n-pm of type* 1, *the link of every i-simplex, $0 < i < n$, is an $(n - i - 1)$-sphere.*

Thus, the requirements (i) and (ii) in the definition of a 3-pm in [1, page 213] are superfluous.

For an n-pm of type 2 it is of course not true that the link of every i-simplex, $1 \leqq i < n$, is a sphere, or even a manifold. However:

THEOREM 12. *In an n-pm of type* 2, *the link of every i-simplex, $1 \leqq i < n$, is an $(n - i - 1)$-pm of type* 2.

PROOF. By induction on n. The assertion holds trivially for $n = 2, 3$. Assume the theorem has been proved for $n \geqq 3$, and let $\mathcal{K}$ be an $(n + 1)$-pm of type 2. Let Δ_i be an i-simplex in $\mathcal{K}$, $1 \leqq i \leqq n$, and let x be a vertex in Δ_i. Write $\Delta_i = x \cdot \Delta_{i-1}$. Then $\text{link}(\Delta_i, \mathcal{K}) = \{\Delta \in \mathcal{K} : \quad \Delta \cdot \Delta_i \in \mathcal{K}, \quad \Delta \cap \Delta_i = \emptyset\} = \{\Delta \in \text{link}(x, \mathcal{K}) : \Delta \cap \Delta_{i-1} = \emptyset, \ \Delta \cdot \Delta_{i-1} \in \text{link}(x, \mathcal{K})\} = \text{link}(\Delta_{i-1}, \text{link}(x, \mathcal{K})) = \mathcal{K}'$. Now, if $i = 1$ then $\mathcal{K}'$ is an n-pm of type 2 by Definition 9 II, and if $i > 1$ then $\mathcal{K}'$ is an $(n - (i - 1) - 1)$-pm of type 2 by the induction hypothesis, since $\text{link}(x, \mathcal{K})$ is an n-pm of type 2 by Definition 9 II, and $(n - (i - 1) - 1) = (n + 1) - i - 1$. □

As in the 3-dimensional case, we call a vertex in an n-pm (of either type) *singular* if its link is not a sphere. Singular links an stars are defined similarly.

Lemma 7, modified for n-pm's of types 1 and 2, still holds. (See again [5, lemma 1.14.) Together with Corollary 11 it implies:

THEOREM 13. *If $\mathcal{K}$ is an n-pm of type* 1 *which is not a manifold, and $\mathcal{K}'$ is the second barycentric subdivision of $\mathcal{K}$, then the removal of all the singular stars* (*of vertices*) *in $\mathcal{K}'$ yields an n-manifold with boundary.*

PROOF. Let x be a singular vertex in $\mathcal{K}'$ and let y be a vertex in $\text{link}(x, \mathcal{K}')$. We have to show that $\text{link}(y, \text{ast}(x, \mathcal{K}'))$ is an $(n - 1)$-ball. Since $\text{link}(y, \mathcal{K}')$ is an $(n - 1)$-sphere (by the modified Lemma 7), it is sufficient to show that $\text{link}(xy, \mathcal{K}')$ is an $(n - 2)$-sphere. Now, by Corollary 11, $\text{link}(xy, \mathcal{K}')$ is indeed an $(n - 2)$-sphere. □

Clearly, Theorem 13 does not hold for n-pms of type 2, and it might be interesting to investigate the topological object obtained by removing from the second barycentric subdivision of an n-pm of type 2, the stars of all the singular vertices.

The converse of Theorem 12 is also true. That is, if M is an n-manifold with boundary, and $\mathcal{K}$ is obtained from M by coning all the components of ∂M on distinct apexes, then $\mathcal{K}$ is an n-pm of type 1. Other simple ways of constructing n-pms are given in the next theorem.

THEOREM 14. (a) *If* $\mathcal{K}_1$, $\mathcal{K}_2$ *are* n*-pms of type* i $(i = 1, 2)$, *then the connected sum of* $\mathcal{K}_1$ *and* $\mathcal{K}_2$ *is again an* n*-pm of type* i.

(b) *If* $\mathcal{K}$ *is an* n*-pm of type* i $(i = 1, 2)$, *then any subdivision of* $\mathcal{K}$ *yields an* n*-pm of type* i.

(c) *If* M *is a connected* $(n-1)$*-manifold without boundary and* $\mathcal{K}$ *is the suspension of* M *on two vertices* x, y, *then* $\mathcal{K}$ *is an* n*-pm of type* 1.

If M *is not a sphere, then the only singular vertices in* $\mathcal{K}$ *are* x *and* y, *and* $\operatorname{link}(x, \mathcal{K}) = \operatorname{link}(y, \mathcal{K}) = M$.

(d) *If* $\mathcal{K}_1$ *is an* $(n-1)$*-pm of type* 2 *and* $\mathcal{K}$ *is a suspension of* $\mathcal{K}_1$, *then* $\mathcal{K}$ *is an* n*-pm of type* 2.

PROOF. (a) We use here the term "connected sum" as a natural modification of that term as used in [8, page 46]. Namely, we assume $\mathcal{K}_1, \mathcal{K}_2$ to be disjoint, we then identify some n-simplex Δ in $\mathcal{K}_1$ with some n-simplex in $\mathcal{K}_2$, and define the *connected sum* of $\mathcal{K}_1, \mathcal{K}_2$ to be $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2 \setminus \{\Delta\}$.

Let x be a vertex in Δ. It is sufficient to show that $\operatorname{link}(x, \mathcal{K})$ is a connected $(n-1)$-manifold in the case $i = 1$, and an $(n-1)$-pm of type 2 in the case $i = 2$. The connectedness of $\operatorname{link}(x, \mathcal{K})$ is easily shown as in the proof of Lemma 7 in [1]. We proceed by induction on n. Write $\Delta = x \cdot \Delta'$. Then $\operatorname{link}(x, \mathcal{K}) = \operatorname{link}(x, \mathcal{K}_1) \cup \operatorname{link}(x, \mathcal{K}_2) \setminus \{\Delta'\}$, and this, by the induction hypothesis, is indeed an $(n-1)$-manifold in the case $i = 1$, and an $(n-1)$-pm of type 2 in the case $i = 2$.

(b) This is just the modification of Lemma 7 mentioned above.

(c) Obvious.

(d) Let $\mathcal{K}$ be the suspension of $\mathcal{K}_1$ on the two vertices x, y, and let $z \in \operatorname{vert} \mathcal{K}$. We have to show that $\mathcal{K}' = \operatorname{link}(z, \mathcal{K})$ is an $(n-1)$-pm of type 2. We do this by induction on n. If $z = x$ or $z = y$, then clearly $\mathcal{K}' = \mathcal{K}_1$ and we are done. If $z \in \operatorname{vert} \mathcal{K}_1$, then $\mathcal{K}'$ is easily seen to be the suspension of $\operatorname{link}(z, \mathcal{K}_1)$ on x and y. Since, by Theorem 12, $\operatorname{link}(z, \mathcal{K}_1)$ is an $(n-2)$-pm of type 2, the induction hypothesis yields that $\mathcal{K}'$ is indeed an $(n-1)$-pm of type 2. □

REMARKS. (A) The proof of theorem 15 in [1] is incomplete, since only nine of the eleven existing 2-neighborly 5-polytopes were considered there. However, the two missing cases show the same phenomena as the other nine cases, and therefore that Theorem 15 still holds. The two missing lines in Table 5 there are as follows:

10. 1, 2, 3, 0, 0, 4, 5, 0, 6, 0, 7, 8, 0, 0 {123, 178, 345, 456, 678} — · —
11. 1, 2, 0, 3, 4, 5, 0, 0, 6, 7, 8, 0 {123, 128, 345, 456, 567, 678} /\/ /

(b) The existence of 3-pms with vertices having non-orientable 2-manifolds as links, established both in [1] and in the present paper, shows that lemma 6 on page 163 of [3] is false. S.S. Cairns points out (private communication) that the statement in the proof that $\mathcal{H}_{m-1}(\mathcal{B}^{m-1})$ is a free cyclic group is the source of the error.

REFERENCES

1. A. Altshuler, *3-pseudomanifolds with preassigned links*, Trans. Amer. Math. Soc. **241** (1978), 213–237.

2. A. Altshuler and L. Steinberg, *Neighbourly combinatorial 3-manifolds with 9 vertices*, Discrete Math. **8** (1974), 113–137.

3. S. S. Cairns, *Introductory Topology*, Ronald Press, New York, 1968.

4. R. D. Edwards, *The double suspension of a certain homology 3-sphere is S^5*, Amer. Math. Soc. Notices **22** (February 1975), A-334.

5. J. F. P. Hudson, *Piecewise Linear Topology*, University of Chicago Lecture Notes, W. A. Benjamin, New York, 1969.

6. J. Milnor, *Lectures on Characteristic Classes*, Mimeographed notes by J. Stasheff, Princeton University, Princeton, N.J., 1957.

7. E. E. Moise, *Affine structures in 3-manifolds*, Ann. of Math. **56** (1952), 96–114.

8. C. P. Rourke and B. J. Sanderson, *Introduction to Piecewise-Linear Topology*, Springer-Verlag, Berlin, Heidelberg, New York, 1972.

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