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On binomial and circular binomial distributions of order k for l-overlapping success runs of length k

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The number of l-overlapping success runs of length k in n trials, which was introduced and studied recently, is presently reconsidered in the Bernoulli case and two exact formulas are derived for its probability distribution function in terms of multinomial and binomial coefficients respectively. A recurrence relation concerning this distribution, as well as its mean, is also obtained. Furthermore, the number of l-overlapping success runs of length k in n Bernoulli trials arranged on a circle is presently considered for the first time and its probability distribution function and mean are derived. Finally, the latter distribution is related to the first, two open problems regarding limiting distributions are stated, and numerical illustrations are given in two tables. All results are new and they unify and extend several results of various authors on binomial and circular binomial distributions of order k.

Keywords and phrases: Binomial distributions of order k, circular, success runs, nonoverlapping, overlapping, l-overlapping, recurrence relations, occupancy problem.

1. Introduction

Let $N_{n,k}$ denote the number of nonoverlapping success runs of length k ($k \geq 1$) in n ($n \geq 1$) independent trials with success probability p ($0 < p < 1$). The distribution of $N_{n,k}$ is known as binomial distribution of order k, with parameter vector (n, p). The asymptotic normality of a normalized version of $N_{n,k}$ was first established by von Mises (see Feller (1968 p. 324), where a simpler proof is presented). The exact distribution of $N_{n,k}$ was derived by Hirano (1986) and Philippou and Makri (1986). Since then several papers have appeared on the binomial distribution of order k and its applications, especially on system reliability (see e.g. Philippou (1986), Aki and Hirano (1988), Godbole

(1990), Papastavridis (1990), Hirano and Aki (1993), Antzoulakos and Chadjiconstantinidis (2001) and Balakrishnan and Koutras (2002)). See, also, Eryilmaz (2003) for the distribution and expectation of the number of success runs in nonhomogeneous Markov dependent trials. A different type of binomial distribution of order k, called type II, was introduced and studied by Ling (1988) as the distribution of the number $M_{n,k}$ of overlapping success runs of length k in a sequence of n Bernoulli trials (see, also, Hirano et al. (1991)). When the trials are ordered on a circle, two circular binomial distributions of order k have been introduced and studied by Makri and Philippou (1994) (see, also, Charalambides (1994), Koutras et al. (1994, 1995) and Makri and Philippou (1996)).

Recently, Aki and Hirano (2000) introduced a generalized counting scheme, which includes as special cases the nonoverlapping and the overlapping one, called l-overlapping, where l is a nonnegative integer less than k. The number of l-overlapping success runs of length k is the number of success runs of length k, each of which may have overlapping part of length at most l with the previous success run of length k, that has been enumerated. For $l = 0$ and $l = k - 1$, the nonoverlapping and overlapping cases are obtained respectively. For example, let us assume that $n = 15$ trials are performed, which are numbered from 1 to 15 and that we get the following outcomes

S S S S S S F S S S S F S S S,

where S denotes success and F denotes failure of a specific trial. Then, the nonoverlapping success runs of length $k = 4$ are the outcomes corresponding to the trials numbered by

1 2 3 4 and 8 9 10 11;

the overlapping success runs of length 4, in the sense of Ling, are

1 2 3 4, 2 3 4 5, 3 4 5 6 and 8 9 10 11,

and the 2-overlapping success runs of length 4, are

1 2 3 4, 3 4 5 6 and 8 9 10 11.

Aki and Hirano (2000) introduced a generalized binomial distribution of order k and investigated some of its properties. Han and Aki (2000) derived a recerrence for the probability generating function of the number of l-overlapping success runs in the case of n independent trials, as well as in the case of a higher order Markov chain of length n. See, also, Antzoulakos (2003) for a unified approach for waiting times and number of appearances of runs.

Let us assume that the outcomes are arranged on a circle. Then, if there is (at least) one F in the sequence, we start counting from the first S following an F and if there is no F, we can start counting from any S in the sequence.

If we assume that the above 15 outcomes are arranged on a circle then the 2-overlapping success runs of length 4 are

$$8\ 9\ 10\ 11,\ 13\ 14\ 15\ 1,\ 15\ 1\ 2\ 3 \text{ and } 2\ 3\ 4\ 5.$$

In the present paper, in Section 2, we derive two alternative formulas for the probability distribution function of the random variable $N_{n,k,l}$, representing the number of l-overlapping success runs of length k ($k \geq 1$) in n ($n \geq 1$) independent trials with success probability p ($0 < p < 1$) (see Theorem 2.1 and Theorem 2.2). We also derive the mean of $N_{n,k,l}$ (see Proposition 2.1). In Section 3, we introduce a new circular binomial distribution of order k as the distribution of the random variable $N^c_{n,k,l}$, representing the number of l-overlapping success runs of length k in n independent trials ordered on a circle, and we also derive its mean (see Theorem 3.1 and Proposition 3.1). In Section 4, we establish a recurrence relation for the probability distribution of $N_{n,k,l}$ (see Theorem 4.1) and we relate the two distributions by a recurrence relation (see Theorem 4.2). The usefulness of the recurrences for calculating the respective probabilities is illustrated (see Table 1). Finally, in Section 5 we refer to the known limiting distribution of $N_{n,k,0}$ and $N_{n,k,k-1}$, and we state two open problems regarding $N_{n,k,l}$ ($0 < l < k-1$) and $N^c_{n,k,l}$ ($0 \leq l \leq k-1$). Some numerical results regarding $N_{n,k,l}$ ($0 \leq l \leq k-1$) are also given by means of Theorem 2.2 (see Table 2).

Our proofs employ a result of Riordan (1964) and expand upon some ideas of Aki and Hirano (1988), Ling (1988) and Makri and Philippou (1994) (see, also, Philippou and

Muwafi (1982)).

Throughout the paper, $[x]$ denotes the greatest integer in x, $\delta_{i,j}$ is the Kronecker delta function and $c_i = [\frac{i-l}{k-l}]$ if $i \geq k$ and 0 otherwise.

2. On binomial distribution of order k for l-overlapping success runs of length k

In this section we reconsider the number of l-overlapping success runs of length k in n Bernoulli trials, which was first studied by Aki and Hirano (2000) and Han and Aki (2000) and we derive its probability distribution function in terms of multinomial as well as in terms of binomial coefficients and its mean.

THEOREM 2.1. Let $N_{n,k,l}$ be a random variable (rv) denoting the number of l-overlapping success runs of length k ($l \leq k-1$, $k \geq 1$) in n (≥ 1) independent trials with success probability p ($0 < p < 1$). Then, for $n \leq k-1$, $P(N_{n,k,l} = 0) = 1$, for $n = k$, $P(N_{n,k,l} = 0) = 1 - p^k$ and $P(N_{n,k,l} = 1) = p^k$, and for $n \geq k+1$ and $x = 0, 1, \ldots, [\frac{n-l}{k-l}]$,

$$P(N_{n,k,l} = x) = p^n \sum_{s=0}^{n} \sum \binom{x_1 + \ldots + x_n}{x_1, \ldots, x_n} (q/p)^{x_1 + \ldots + x_n},$$

where the inner summation is over all nonnegative integers $x_1, \ldots, x_n$ satisfying the conditions $\sum_{j=1}^{n} j x_j = n - s$ and $\sum_{i=1}^{[\frac{n-1-l}{k-l}]} i \sum_{j=1}^{m_{i,n}} x_{i(k-l)+l+j} = x - c_s$, and $m_{i,n} = \min\{k - l, n - l - i(k-l)\}$.

PROOF. A typical element of the event $(N_{n,k,l} = x)$ is an arrangement

$$\alpha_1 \alpha_2 \cdots \alpha_{x_1 + \ldots + x_n} \underbrace{SS \ldots S}_{s}, \quad 0 \leq s \leq n,$$

such that x_r of the α's are of the type $e_r = \underbrace{SS \ldots S}_{r-1} F$, $r = 1, \ldots, n$, and there are $x_1 + \ldots + x_k$ e_r's, each of which includes no success run of length k, $x_{k+1} + \ldots + x_{2k-l}$, e_r's each of which includes 1 l-overlapping success run of length k, $x_{2k-l+1} + \ldots + x_{3k-2l}$ e_r's, each of which includes 2 l-overlapping success runs of length k,... . Generally, i l-overlapping success runs of length k are included in each of the

$$x_{ik-(i-1)l+1} + \ldots + x_{(i+1)k-il} = x_{i(k-l)+l+1} + \ldots + x_{i(k-l)+l+(k-l)}$$

e_r's, $i = 1, \ldots, [\frac{n-1-l}{k-l}]$. Thus, the nonnegative integers $x_1, \ldots, x_n$ have to satisfy the conditions

(1) $x_1 + 2x_2 + \ldots + nx_n = n - s,\ 0 \leq s \leq n$

and

(2) $c_s + \sum_{i=1}^{[\frac{n-1-l}{k-l}]} i \sum_{j=1}^{m_{i,n}} x_{i(k-l)+l+j} = x$,

where $m_{i,n}$ is as in the theorem. Fix s and $x_1, \ldots, x_n$. Then, the number of the above arrangements is

$$\binom{x_1 + \ldots + x_n}{x_1, \ldots, x_n}$$

and each one of them has probability

$$P(\alpha_1 \alpha_2 \cdots \alpha_{x_1+\ldots+x_n} \underbrace{SS \ldots S}_{s}) = q^{x_1+\ldots+x_n} p^{n-(x_1+\ldots+x_n)}.$$

But the nonnegative integers $x_1, \ldots, x_n$ may vary subject to the two conditions (1) and (2) and $0 \leq s \leq n$. Therefore, for $n \geq k+1$ and $x = 0, 1, \ldots, [\frac{n-l}{k-l}]$,

$$P(N_{n,k,l} = x) = p^n \sum_{s=0}^{n} \sum \binom{x_1 + \ldots + x_n}{x_1, \ldots, x_n} (q/p)^{x_1+\ldots+x_n},$$

where the inner summation is over $x_1, \ldots, x_n$ satisfying the conditions (1) and (2). For $n \leq k$, $P(N_{n,k,l} = x)$ follows from the definition of the rv. The proof of the theorem is completed. □

For $l = 0$, Theorem 2.1 provides a new formula for the probability distribution of the number of nonoverlapping success runs of length k in n Bernoulli trials, which is alternative to the one given by Hirano (1986) and Philippou and Makri (1986). For $l = k-1$, it reduces to Theorem 3.2 of Ling (1988). For $1 \leq l \leq k-2$, it provides new probability distributions.

Since $N_{n,k,0}$ ($N_{n,k,k-1}$) is distributed as binomial of order k, type I (type II) with parameter vector (n, p) and it is denoted by $B_{k,I}(n,p)$ ($B_{k,II}(n,p)$), we introduce the following definition.

DEFINITION 2.1. A rv X is said to be distributed as binomial of order k, in the l-overlapping case with parameter vector (n,p), to be denoted by $B_{k,l}(n,p)$, if its probability distribution function is given by Theorem 2.1.

Obviously, $B_{k,0}(n,p) = B_{k,I}(n,p)$ and $B_{k,k-1}(n,p) = B_{k,II}(n,p)$.

In the sequel an alternative exact formula for $P(N_{n,k,l} = x)$ is derived in terms of binomial coefficients. We first state a preliminary lemma.

LEMMA 2.1. The number of possible ways of distributing n identical balls into m different urns such that the maximum allowed number of balls in any one urn is r is given by

$$C(n,m,r) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \binom{n+m-j(r+1)-1}{m-1}$$

(see Riordan 1964, p.104). It is noted that $C(0,m,r)$ is considered equal to 1.

THEOREM 2.2. Let $N_{n,k,l}$ be as in Theorem 2.1. Then,

$$\text{(a)}\ \ P(N_{n,k,l} = 0) = \sum_{y=[n/k]}^{n} p^{n-y} q^y C(n-y, y+1, k-1)$$

and for $x = 1, \ldots, [\frac{n-l}{k-l}]$,

$$\begin{aligned}(b)\ \ P(N_{n,k,l} = x) \ &= \sum_{y=[(n+xl)/k]-x}^{n-k-(x-1)(k-l)} p^{n-y} q^y \sum_{i=1}^{[(n-y)/k]} \binom{y+1}{i} \binom{x-1}{i-1} \\ &\quad \times \sum_{\beta_i=m_i}^{M_i} C(\beta_i, y+1-i, k-1) C(\alpha_i - \beta_i, i, k-l-1)\end{aligned}$$

where $\alpha_i = n - y - ik - (x-i)(k-l)$, $m_i = \max\{0, \alpha_i - i(k-l-1)\}$, $M_i = \min\{\alpha_i, (k-1)(y+1-i)\}$.

PROOF. (a) Consider the event $(N_{n,k,l} = 0, Y_n = y)$, where Y_n denotes the number of failures in the n trials. Then, a typical element of the above event is a sequence

$$SS \ldots SFSS \ldots FSS \ldots F$$

of y failures and $n-y$ successes such that at most $k-1$ consecutive successes appear. The probability of any such sequence is $q^y p^{n-y}$ and the number of such sequences is $C(n-y, y+1, k-1)$ by Lemma 2.1, since the y failures create $y+1$ cells and $C(n-y,y+1,k-1)$ is the number of distributing $n-y$ balls (S's) in $y+1$ cells such that each cell contains at most $k-1$ balls. Therefore,

$$P(N_{n,k,l}=0)=\sum_y P(N_{n,k,l}=0, Y_n=y)=\sum_{y=[n/k]}^{n} C(n-y,y+1,k-1)p^{n-y}q^y.$$

We now proceed to prove (b).

(b) Consider the events $A_j = \{\text{at least } k \text{ successes are contained in the } j-\text{th urn}\}$, $j=1,2,\ldots,y+1$, and $\overline{A}=\cap_{j\notin\{j_1,\ldots,j_i\}}A_j^c$, where $\{j_1,\ldots,j_i\}$ is a subset of $\{1,2,\ldots,y+1\}$ and A_j^c denotes the complement of A_j. We observe that for $1\le i\le \min\{y+1,[(n-y)/k]\}$, every element of the event

$$(N_{n,k,l}=x, Y_n=y, A_{j_1}\cap A_{j_2}\cap\ldots\cap A_{j_i}\cap\overline{A})$$

is a sequence

$$SS\ldots SFSS\ldots SFSS\ldots S$$

with y failures and $n-y$ successes such that x l-overlapping success runs of length k appear, which are contained in the j_1-th, j_2-th,..., j_i-th urn, among the $y+1$ created ones by the y failures, and no other urn contains more than $k-1$ successes. Therefore

$$P(N_{n,k,l}=x)=\sum_y\sum_i\sum_{j_1,\ldots,j_i} P(N_{n,k,l}=x, Y_n=y, A_{j_1}\cap A_{j_2}\cap\ldots\cap A_{j_i}\cap\overline{A}). \quad (*)$$

It is clear that every element of the event $(N_{n,k,l}=x, Y_n=y, A_{j_1}\cap A_{j_2}\cap\ldots\cap A_{j_i}\cap\overline{A})$ has probability $q^y p^{n-y}$. So, in order to evaluate its probability we proceed to count its elements, by considering the corresponding occupancy problem. We start by placing k balls (S's) into each of the j_1-th, j_2-th,...,j_i-th urn and we continue by distributing $x-i$ blocks, each consisting of $k-l$ balls into the same urns without any restrictions. It is well known that this is accomplished in

$$\binom{x-i+i-1}{i-1}=\binom{x-1}{i-1}$$

possible ways. Now, there are $a_i = n - y - ik - (x - i)(k - l)$ remaining balls to be placed into the $y + 1$ urns under the following restrictions: Every one of the above i specified urns (the j_1-th, j_2-th,..., j_i-th) is allowed to contain no more than $k - l - 1$ balls and every one of the remaining $y + 1 - i$ urns is allowed to contain no more than $k - 1$ balls. If β_i of the α_i balls are to be distriduted in all the specified $y + 1 - i$ urns then $\alpha_i - \beta_i$ are to be placed in the i specified urns. According to Lemma 2.1 the distribution of the β_i balls can be accomplished in $C(\beta_i, y + 1 - i, k - 1)$ different ways. For every distribution of the β_i balls into the $y+1-i$ urns there are $C(\alpha_i - \beta_i, i, k-l-1)$ different ways of distributing the remaining $\alpha_i - \beta_i$ balls into the i urns. Observing that $\max\{0, \alpha_i - i(k - l - 1)\} \leq \beta_i \leq \min\{\alpha_i, (k - 1)(y + 1 - i)\}$, we conclude that the total number of ways of distributing the α_i balls into the $y+1$ urns under the above restrictions is given by

$$\sum_{\beta_i=m_i}^{M_i} C(\beta_i, y + 1 - i, k - 1)C(\alpha_i - \beta_i, i, k - l - 1),$$

so that the number of the elements of the event $(N_{n,k,l} = x, Y_n = y, A_{j_1} \cap A_{j_2} \cap \ldots \cap A_{j_i} \cap \overline{A})$ is

$$\binom{x-1}{i-1} \sum_{\beta_i=m_i}^{M_i} C(\beta_i, y + 1 - i, k - 1)C(\alpha_i - \beta_i, i, k - l - 1).$$

Therefore,

$$\begin{aligned} P(N_{n,k,l} = x, Y_n = y, A_{j_1} \cap A_{j_2} \cap \ldots \cap A_{j_i} \cap \overline{A}) = p^{n-y} q^y \\ \times \binom{x-1}{i-1} \sum_{\beta_i=m_i}^{M_i} C(\beta_i, y + 1 - i, k - 1)C(\alpha_i - \beta_i, i, k - l - 1), \end{aligned}$$

and the result follows from $(*)$ by noting that there are $\binom{y+1}{i}$ i-combinations of the set $\{1, 2, \ldots, y + 1\}$ and $[(n + xl)/k] - x \leq y \leq n - k - (x - 1)(k - l)$. □

PROPOSITION 2.1. Let $N_{n,k,l}$ be a rv as in Theorem 2.1. Then, for $n \leq k - 1$, $E(N_{n,k,l}) = 0$ and for $n \geq k > l \geq 0$,

$$E(N_{n,k,l}) = p^l \sum_{j=1}^{[\frac{n-l}{k-l}]} \{1 + (1 - p)\{n - l - j(k - l)\}\} p^{j(k-l)}.$$

PROOF. Let $X_1, \ldots, X_n$ be independent rvs with probability distribution

$$P(X_i = x) = p^x(1-p)^{1-x}, \;\; x = 0,1, \;\; 0 < p < 1, \;\; 1 \le i \le n.$$

For $i = 1, k-l+1, 2(k-l)+1, \ldots, [\frac{n-k}{k-l}](k-l)+1$, let E_i be the event that "$X_1 X_2 \cdots X_{i+k-1} = 1$", and for $i = 2, \ldots, n-k+1$ and $j = 0, 1, \ldots, [\frac{i-2}{k-l}]$, let E_{ij} be the event that "$X_{i-j(k-l)-1} = 0$ and $X_{i-j(k-l)} \cdots X_i \cdots X_{i+k-1} = 1$". Next, let $\{A_1, A_2, A_3\}$ be a partition of the index set $I = \{1, 2, \ldots, n-k+1\}$ where $A_1 = \{1\}$, $A_2 = \{k-l+1, 2(k-l)+1, \ldots, [\frac{n-k}{k-l}](k-l)+1\}$ and $A_3 = I - (A_1 \cup A_2)$. We define rvs Y_i, $1 \le i \le n-k+1$, as follows:

$$Y_1 = \begin{cases} 1, & \text{if } E_1 \text{ occurs,} \\ 0, & \text{otherwise.} \end{cases}$$

For $i \in A_2$,

$$Y_i = \begin{cases} 1, & \text{if } (\cup_{j=0}^{[\frac{i-2}{k-l}]} E_{ij}) \cup E_i \text{ occurs,} \\ 0, & \text{otherwise,} \end{cases}$$

and for $i \in A_3$,

$$Y_i = \begin{cases} 1, & \text{if } \cup_{j=0}^{[\frac{i-2}{k-l}]} E_{ij} \text{ occurs,} \\ 0, & \text{otherwise.} \end{cases}$$

Then, $N_{n,k,l} = \sum_{i=1}^{n-k+1} Y_i$, so that

$$\begin{aligned} E(N_{n,k,l}) &= P(Y_1 = 1) + \sum_{i \in A_2} P(Y_i = 1) + \sum_{i \in A_3} P(Y_i = 1) \\ &= p^k \sum_{i=0}^{n-k} \sum_{j=0}^{[\frac{i}{k-l}]} p^{j(k-l)} - p^{k+1} \sum_{i=0}^{n-k-1} \sum_{j=0}^{[\frac{i}{k-l}]} p^{j(k-l)} \\ &= p^k \{ \sum_{j=0}^{[\frac{n-k}{k-l}]} \{n+1-(j+1)k+jl\} p^{j(k-l)} - \sum_{j=0}^{[\frac{n-1-k}{k-l}]} \{n-(j+1)k+jl\} p^{j(k-l)+1} \}, \end{aligned}$$

and the proof follows by simplification. □

For $l = 0$, Proposition 2.1 readily gives

$$E(N_{n,k,0}) = \sum_{j=1}^{[\frac{n}{k}]} \{1 + (1-p)(n-jk)\} p^{jk},$$

which is Proposition 2.4 of Aki and Hirano (1988) (see also Antzoulakos and Chadjikonstantinidis (2001)). For $l = k - 1$, it gives

$$E(N_{n,k,k-1}) = (n - k + 1)p^k,$$

which is Theorem 4.1(i) of Ling (1988).

REMARK 2.1. Another formula for the mean of $N_{n,k,l}$ may be obtained from its probability generating function $g_{n,k,l}(t)$, derived explicitly by Inoue and Aki (2003) in terms of restricted multiple sums involving multinomial coefficients (see their Propositions 3 and 4). Alternatively, $E(N_{n,k,l})$ may be evaluated recursively from (or explicitly from the solution of) a recurrence formula for it, which readily follows from a recurrence formula for $g_{n,k,l}(t)$ of Han and Aki (2000).

3. Circular binomial distribution of order k for l-overlapping success runs of length k

In this section we consider the number of l-overlapping success runs of length k in n Bernoulli trials ordered circularly and we derive its probability distribution function and mean.

THEOREM 3.1. Let $N^c_{n,k,l}$ be a random variable denoting the number of l-overlapping success runs of length k ($l \leq k-1, k \geq 1$) in n ($n \geq 1$) independent trials with success probability p ($0 < p < 1$) ordered circularly. Then,

(a) for $n \leq k - 1$, $P(N^c_{n,k,l} = 0) = 1$;

(b) for $n = k$, $P(N^c_{n,k,l} = 0) = 1 - p^k$ and $P(N^c_{n,k,l} = [\frac{k}{k-l}]) = p^k$;

(c) for $n = k+1$, $P(N^c_{n,k,l} = 0) = 1-(k+1)qp^k - p^{k+1}$ and $P(N^c_{n,k,l} = x) = (k+1)qp^k\delta_{x,1} + p^{k+1}\delta_{x,[\frac{k+1}{k-l}]}$, $x = 1, [\frac{k+1}{k-l}]$;

(d) for $n \geq k+2$ and $x = 0, 1, \ldots, [\frac{n-1-l}{k-l}], [\frac{n}{k-l}]$,

$$P(N^c_{n,k,l} = x) = qp^{n-1} \sum_{s=1}^{M_{x,n}} s \sum \binom{x_1 + \ldots + x_{n-1}}{x_1, \ldots, x_{n-1}} (q/p)^{x_1+\cdots+x_{n-1}} + p^n \delta_{x,[\frac{n}{k-l}]},$$

where the inner summation is over all nonnegative integers $x_1, \ldots, x_{n-1}$ satisfying the conditions $\sum_{j=1}^{n-1} jx_j = n - s$ and $\sum_{i=1}^{[\frac{n-2-l}{k-l}]} i \sum_{j=1}^{m_{i,n-1}} x_{ik-(i-1)l+j} = x - c_{s-1}$, $M_{x,n} = \min\{x(k -$

$l) + k, n\}$ and $m_{i,n}$ is as in Theorem 2.1.

PROOF. Obviously, for $n \leq k-1$, $n = k$ and $n = k+1$ (a), (b) and (c) of the theorem hold. For $n \geq k+2$, we first observe that for $[\frac{n-1-l}{k-l}] < x < [\frac{n}{k-l}]$, $P(N^c_{n,k,l} = x) = P(\emptyset)$. Let $x = 0, 1, \ldots, [\frac{n-1-l}{k-l}], [\frac{n}{k-l}]$. An element of the event $(N^c_{n,k,l} = x)$ which includes at least one F is a cyclic arrangement

$$\underbrace{SS\ldots S}_{\alpha} F \alpha_1 \alpha_2 \cdots \alpha_{x_1+\ldots+x_{n-1}} \underbrace{SS\ldots S}_{\beta}$$

such that x_j of the α's are of the type $e_j = \underbrace{SS\ldots S}_{j-1} F$ $(0 \leq j \leq n-1)$, i l-overlapping success runs of length k are included in each of the

$$x_{ik-(i-1)l+1} + \ldots + x_{ik-(i-1)l+1+k-l-1}$$

e_r's, $i = 1, \ldots, [\frac{n-2-l}{k-l}]$, and

(1) $0 \leq \alpha, \beta, \alpha+\beta \leq \min\{x(k-l)+k-1, n-1\} (= M_{x,n} - 1)$,

(2) $\sum_{j=1}^{n-1} j x_j = n - 1 - (\alpha+\beta)$, $\;c_{\alpha+\beta} + \sum_{i=1}^{[\frac{n-2-l}{k-l}]} i(x_{ik-(i-1)l+1} + \ldots + x_{ik-(i-1)l+1+k-l-1}) = x$,

where $c_{\alpha+\beta}$ represents the number of l-overlapping success runs included in the $\alpha+\beta$ successes of the element. If x_j $(1 \leq j \leq n-1)$, α and β are kept fixed, the number of the above arrangements is

$$\binom{x_1 + \ldots + x_{n-1}}{x_1, \ldots, x_{n-1}}$$

and each one of them has probability

$$q q^{x_1+\ldots+x_{n-1}} p^{n-(x_1+\ldots+x_{n-1}+1)}.$$

But α, β may vary subject to (1) and the nonnegative integers $x_1, \ldots, x_{n-1}$ may vary subject to (2). Therefore, observing that there are $[n/(k-l)]$ l-overlapping success runs of length k in an element with no F's, and denoting by $\sum'$ the summation over all nonnegative integers $x_1, \ldots, x_{n-1}$ satisfying (2), we have

$$\begin{aligned} P(N^c_{n,k,l} = x) &= qp^{n-1} \sum_{\alpha=0}^{M_{x,n}-1} \sum_{\substack{\beta=0 \\ 0\le\alpha+\beta\le M_{x,n}-1}}^{M_{x,n}-1} \sum{}' \binom{x_1 + \ldots + x_{n-1}}{x_1, \ldots, x_{n-1}} (q/p)^{x_1+\ldots+x_{n-1}} + p^n \delta_{x,[\frac{n}{k-l}]} \\ &= qp^{n-1} \sum_{\alpha+\beta=0}^{M_{x,n}-1} (\alpha+\beta+1) \sum{}' \binom{x_1 + \ldots + x_{n-1}}{x_1, \ldots, x_{n-1}} (q/p)^{x_1+\ldots+x_{n-1}} + p^n \delta_{x,[\frac{n}{k-l}]} \end{aligned}$$

The theorem follows. □

For $l = 0$, Theorem 3.1 provides a formula for the probability distribution of the number of nonoverlapping success runs of length k in n Bernoulli trials ordered circularly, which is alternative to the one given in Makri and Philippou (1994). For $l = k-1$, it reduces to Theorem 2.2 of Makri and Philippou (1994). For $1 \le l \le k-2$, it provides new probability distributions.

Since $N^c_{n,k,0}$ ($N^c_{n,k,k-1}$) is distributed as circular binomial of order k, type I (type II) with parameter vector (n, p) and it is denoted by $B^c_{k,I}(n,p)$ ($B^c_{k,II}(n,p)$), we introduce the following definition.

DEFINITION 3.1. A rv X is said to be distributed as circular binomial of order k, in the l-overlapping case with parameter vector (n,p), to be denoted by $B^c_{k,l}(n,p)$, if its probability distribution function is given by Theorem 3.1.

Obviously, $B^c_{k,0}(n,p) = B^c_{k,I}(n,p)$ and $B^c_{k,k-1}(n,p) = B^c_{k,II}(n,p)$.

PROPOSITION 3.1. Let $N^c_{n,k,l}$ be a rv as in Theorem 3.1. Then, for $n \le k-1$, $E(N^c_{n,k,l}) = 0$ and for $n \ge k > l \ge 0$,

$$E(N^c_{n,k,l}) = nqp^k \frac{1 - p^{[\frac{n-1-l}{k-l}](k-l)}}{1 - p^{k-l}} + [\frac{n}{k-l}]p^n.$$

PROOF. Obviously, for $n \le k-1$, $E(N^c_{n,k,l}) = 0$ and for $n = k$, $E(N^c_{n,k,l}) = [\frac{k}{k-l}]p^k$. For $n \ge k+1$ and $0 \le l < k$, let E_{ij} ($1 \le i \le n$, $1 \le j \le [\frac{n-1-l}{k-l}]$) be the event that the i-th trial results in the $l + j(k-l)$-th success of a success run of length $l + j(k-l)$

preceded by a failure and $E_{i,[\frac{n}{k-l}]}$ be the event that the i-th trial ($1 \leq i \leq [\frac{n}{k-l}]$) results in the n-th success of a success run of length n.

We define rvs Y_i, $1 \leq i \leq n$, as follows:

For $i = 1, 2, \ldots, [\frac{n}{k-l}]$,

$$Y_i = \begin{cases} 1, & \text{if } (\cup_{j=1}^{[\frac{n-1-l}{k-l}]} E_{ij}) \cup E_{i,[\frac{n}{k-l}]} \text{ occurs,} \\ 0, & \text{otherwise.} \end{cases}$$

For $i = [\frac{n}{k-l}] + 1, \ldots, n$,

$$Y_i = \begin{cases} 1, & \text{if } \cup_{j=1}^{[\frac{n-1-l}{k-l}]} E_{ij} \text{ occurs,} \\ 0, & \text{otherwise.} \end{cases}$$

Then, $N^c_{n,k,l} = \sum_{i=1}^{n} Y_i$, so that

$$\begin{aligned} E(N^c_{n,k,l}) &= \sum_{i=1}^{n} \sum_{j=1}^{[\frac{n-1-l}{k-l}]} P(E_{ij}) + \sum_{i=1}^{[\frac{n}{k-l}]} P(E_{i,[\frac{n}{k-l}]}) \\ &= q \sum_{i=1}^{n} \sum_{j=1}^{[\frac{n-1-l}{k-l}]} p^{l+j(k-l)} + \sum_{i=1}^{[\frac{n}{k-l}]} p^n \\ &= nqp^l \sum_{j=1}^{[\frac{n-1-l}{k-l}]} (p^{k-l})^j + [\frac{n}{k-l}] p^n. \end{aligned}$$

The proof of the proposition follows. □

For $l = 0$, Proposition 3.1 readily gives

$$E(N^c_{n,k,0}) = nqp^k \frac{1 - p^{[\frac{n-1}{k}]k}}{1 - p^k} + [\frac{n}{k}] p^n,$$

which coincides with a result of Charalambides (1994) and Makri and Philippou (1994). For $l = k - 1$, it gives

$$E(N^c_{n,k,k-1}) = np^k,$$

which is Proposition 2.2 of Makri and Philippou (1994).

4. Recurrence relations for l-overlapping success runs of length k in n Bernoulli trials

In the following we derive a recurrence relation concerning the probability distribution of the rv $N_{n,k,l}$ and a recurrence relation relating the respective probability distributions of $N_{n,k,l}$ and $N^c_{n,k,l}$. These results are useful for the calculation of the probabilities. A numerical illustration is given.

THEOREM 4.1. Let $N_{n,k,l}$ be as in Theorem 2.1 and set $m_{x,n,j} = \min\{x, [\frac{n-1-l-j}{k-l}]\}$. Then,

(a) for $n \geq k+1$ and $x = 1, 2, \ldots, [\frac{n-l}{k-l}]$,

$$\begin{aligned} P(N_{n,k,l} = x) &= P(N_{n-1,k,l} = x) - qp^k \sum_{i=0}^{m_{x,n,k-l}} p^{i(k-l)} P(N_{n-1-k-i(k-l),k,l} = x - i) \\ &+ qp^l \sum_{i=1}^{m_{x,n,0}} p^{i(k-l)} P(N_{n-1-l-i(k-l),k,l} = x - i) \\ &+ p^n \{\delta_{x,[\frac{n-l}{k-l}]} - \delta_{x,[\frac{n-1-l}{k-l}]}\}; \end{aligned}$$

(b) for $n \geq k+1$, $P(N_{n,k,l} = 0) = P(N_{n-1,k,l} = 0) - qp^k P(N_{n-k-1,k,l} = 0)$;

(c) for $n = k$, $P(N_{n,k,l} = 0) = 1 - p^k$, $P(N_{n,k,l} = 1) = p^k$;

(d) for $0 \leq n \leq k-1$, $P(N_{n,k,l} = 0) = 1$.

We shall first establish a preliminary lemma.

LEMMA 4.1. Let $N_{n,k,l}$ and $m_{x,n,j}$ be as in Theorem 4.1. Then,

(a) for $n \geq k+1$ and $x = 1, 2, \ldots, [\frac{n-l}{k-l}]$,

$$\begin{aligned} P(N_{n,k,l} = x) &= p^n \delta_{x,[\frac{n-l}{k-l}]} + q \sum_{j=0}^{k-1} p^j P(N_{n-1-j,k,l} = x) \\ &+ qp^l \sum_{j=0}^{k-l-1} p^j \sum_{i=1}^{m_{x,n,j}} p^{i(k-l)} P(N_{n-1-l-j-i(k-l),k,l} = x - i); \end{aligned}$$

(b) for $n \geq k$, $P(N_{n,k,l} = 0) = q \sum_{j=0}^{k-1} p^j P(N_{n-1-j,k,l} = 0)$.

PROOF. (a) Let $n \geq k+1$, $x = 1, \ldots, [\frac{n-l}{k-l}]$, and $m_{x,n,j}$ be as in the lemma. For $j = 0, \ldots, n-1$, we define the events A_j="j S's precede the first F in the sequence of n Bernoulli trials" and B="there is no F in the sequence of n Bernoulli trials", so that

$$(N_{n,k,l} = x) = \cup_{j=0}^{n-1} [(N_{n,k,l} = x) \cap A_j] \cup [(N_{n,k,l} = x) \cap B].$$

Obviously, $(N_{n,k,l} \neq [\frac{n-l}{k-l}]) \cap B = \emptyset$. Then, since A_j $(j = 0, 1, \ldots, n-1)$ and B are disjoint events, we have

$$\begin{aligned} P(N_{n,k,l} = x) &= \sum_{j=0}^{n-1} P[(N_{n,k,l} = x) \mid A_j]P(A_j) + P[(N_{n,k,l} = x) \mid B]P(B) \\ &= \sum_{j=0}^{k-1} qp^j P(N_{n-1-j,k,l} = x) + \sum_{j=k}^{n-1} qp^j P(N_{n-1-j,k,l} = x - [\frac{j-l}{k-l}]) + p^n \delta_{x,[\frac{n-l}{k-l}]}, \end{aligned}$$

which implies part (a) of the lemma.

(b) When $x = 0$ and $n \geq k$ we observe again that

$$(N_{n,k,l} = 0) = \cup_{j=0}^{k-1} [(N_{n,k,l} = 0) \cap A_j],$$

so that

$$P(N_{n,k,l} = 0) = \sum_{j=0}^{k-1} [P(N_{n,k,l} = 0) \mid A_j]P(A_j)$$

from which we get part (b) of the lemma.

Proof of Theorem 4.1. For $n \geq k+1$ and $x = 1, 2, \ldots, [\frac{n-l}{k-l}]$, Lemma 4.1(a) gives

$$\begin{aligned} \frac{1}{p}P(N_{n+1,k,l} = x) - P(N_{n,k,l} = x) &= \frac{q}{p}P(N_{n,k,l} = x) - qp^{k-1}P(N_{n-k,k,l} = x) \\ &+ qp^{l-1} \sum_{i=1}^{\min\{x,[\frac{n-l}{k-l}]\}} p^{i(k-l)} P(N_{n-l-i(k-l),k,l} = x - i) \\ &- qp^{k-1} \sum_{i=1}^{\min\{x,[\frac{n-k}{k-l}]\}} p^{i(k-l)} P(N_{n-k-i(k-l),k,l} = x - i) \\ &+ p^n \{\delta_{x,[\frac{n+1-l}{k-l}]} - \delta_{x,[\frac{n-l}{k-l}]}\} \end{aligned}$$

from which we get part (a) of the theorem. Using the same argument and Lemma 4.1(b) we get part (b). Finally, (c) and (d) are implied by the definition of $N_{n,k,l}$. □

Theorem 4.2. Let $N_{n,k,l}$ be as in Theorem 2.1, and $N^c_{n,k,l}$ and $M_{x,n}$ be as in Theorem 3.1. Then, for $n \geq k$ and $x = 0, 1, \ldots, [\frac{n-1-l}{k-l}], [\frac{n}{k-l}]$, we have

$$\begin{aligned} P(N^c_{n,k,l} = x) &= q^2 \sum_{i=1}^{M_{x,n-1}} ip^{i-1} P(N_{n-1-i,k,l} = x - c_{i-1}) \\ &\quad + nqp^{n-1}\delta_{x,[\frac{n-1-l}{k-l}]} + p^n \delta_{x,[\frac{n}{k-l}]}. \end{aligned}$$

PROOF. Let $x = 0, 1, \dots, [\frac{n-1-l}{k-l}], [\frac{n}{k-l}]$. We define the events A_j = "j S's precede the first F in the sequence of n Bernoulli trials", ($j = 0, 1, \dots, M_{x,n-1} - 1$) and B_r = "r S's follow the last F in the sequence of n Bernoulli trials", ($r = 0, 1, \dots, M_{x,n-1} - 1$), $0 \le j + r \le M_{x,n-1} - 1$. Furthermore, if $X_1, \dots, X_n$ are as in the proof of Proposition 2.1, we set

$C = \cup_{i=1}^{n} \{X_i = 0, X_j = 1 \ (1 \le j \ne i \le n)\}$ and $D = \{X_1 = X_2 = \dots = X_n = 1\}$. Obviously, $(N^c_{n,k,l} \ne [\frac{n-1-l}{k-l}]) \cap C = (N^c_{n,k,l} \ne [\frac{n}{k-l}]) \cap D = \emptyset$ and $P(C) = nqp^{n-1}$, $P(D) = p^n$. Then we have

$$\begin{aligned}(N^c_{n,k,l} = x) &= \bigcup_{\substack{j=0 \\ 0 \le j+r \le M_{x,n-1}-1}}^{M_{x,n-1}-1} \bigcup_{r=0}^{M_{x,n-1}-1} \{(N^c_{n,k,l} = x) \cap (A_j \cap B_r)\} \\ &\quad \cup \{(N^c_{n,k,l} = x) \cap C\} \cup \{(N^c_{n,k,l} = x) \cap D\}\end{aligned}$$

from which we get

$$\begin{aligned}P(N^c_{n,k,l} = x) &= \sum_{\substack{j=0 \\ 0 \le j+r \le M_{x,n-1}-1}}^{M_{x,n-1}-1} \sum_{r=0}^{M_{x,n-1}-1} P(A_j)P(B_r)P(N^c_{n,k,l} = x \mid A_j \cap B_r) \\ &\quad + P(C)P(N^c_{n,k,l} = x \mid C) + P(D)P(N^c_{n,k,l} = x \mid D) \\ &= \sum_{j+r=0}^{M_{x,n-1}-1} (j + r + 1)p^j qqp^r P(N_{n-2-(j+r),k,l} = x - c_{j+r}) \\ &\quad + nqp^{n-1}\delta_{x,[\frac{n-l-1}{k-l}]} + p^n \delta_{x,[\frac{n}{k-l}]}.\end{aligned}$$

The theorem follows. □

For $l = 0$ and $l = k - 1$, respectively, Theorem 4.2 reduces to Theorems 3.1 and 3.2 of Makri and Philippou (1994).

In Table 1 we give the distributions and means of the rvs $N_{15,5,l}$ and $N^c_{15,5,l}$ for $l = 0, 1, 2, 3, 4$ for a sequence of 15 Bernoulli trials with success probability $p = 0.9$.

We end this paper with a few words on limiting distributions and open problems.

5. Poisson and compound Poisson convergence: Open problems

Set $N_{n,k} = N_{n,k,0}$ and $M_{n,k} = N_{n,k,k-1}$. For large k, the probability distribution function $P(N_{n,k}=x)$ is approximated well by an appropriate Poisson distribution and the probability distribution function $P(M_{n,k}=x)$ is approximated well by an appropriate compound Poisson distribution. More precicely, if $k=k_n\to\infty$ and $nqp^k\to\lambda>0$, as $n\to\infty$, then

$$\lim_{n\to\infty} P(N_{n,k}=x)=P(X_P=x)=e^{-\lambda}(\lambda^x/x!), x=0,1,\dots,$$

and

$$\lim_{n\to\infty} P(M_{n,k}=x)=P(X_{cP}=x)=\begin{cases} e^{-\lambda}, & x=0,\\ e^{-\lambda}p^x\sum_{j=1}^{x}\binom{x-1}{j-1}\frac{(\lambda q/p)^j}{j!}, & x=1,2,\dots, \end{cases}$$

See von Mises (1921) and Feller (1968) for the Poisson convergence and Geske et al. (1995) for the compound Poisson convergence. For a different pair of conditions, which imply Poisson convergence when k is constant, namely $p=p_n\to 0$ and $np_n^k\to\lambda>0$, as $n\to\infty$, we refer to Godbole (1990). We also refer to Barbour, Chryssaphinou and Vaggelatou (2001) for an alternative compound Poisson approximation.

In Table 2 we present the Poisson and compound Poisson approximation to $P(N_{n,k,l}=x)$ for $p=0.9$, $l=0,1,\dots,k-1$ and various values of (n,k) by means of the total variation distances $d(N_{n,k,l},X_P)=(1/2)\sum_{x=0}^{\infty}\mid P(N_{n,k,l}=x)-P(X_P=x)\mid$ and $d(N_{n,k,l},X_{cP})=(1/2)\sum_{x=0}^{\infty}\mid P(N_{n,k,l}=x)-P(X_{cP}=x)\mid$, after truncation. The exact probabilities $P(N_{n,k,l}=x)$, $P(X_P=x)$ and $P(X_{cP}=x)$ were computed by using Theorem 2.2 and the above two formulas, respectively.

The derivation of the limiting distributions of $N_{n,k,l}$, $0<l<k-1$, is an open problem. It is also an open problem to derive the limiting distributions of $N^c_{n,k,l}$, $0\le l\le k-1$.

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Table 1. The exact distributions and means of $N_{15,5,l}$ and $N^c_{15,5,l}$

	$N_{15,5,l}$					$N^c_{15,5,l}$				
$x \backslash l$	0	1	2	3	4	0	1	2	3	4
0	.02823	.02823	.02823	.02823	.02823	.01298	.01298	.01298	.01298	.01298
1	.23955	.18845	.13753	.08803	.04153	.15351	.09540	.06896	.04381	.02058
2	.52633	.43255	.28815	.14609	.06020	.62762	.30445	.20333	.07146	.02898
3	.20589	.35078	.29445	.22968	.07215	.20589	.58717	.16569	.19514	.03503
4			.25164	.20294	.07888			.34315	.08944	.03915
5				.09913	.08160			.20589	.38128	.13437
6				.20589	.19807					.05230
7					.07218				.20589	.04707
8					.06213					.04236
9					.05338					.03813
10					.04575					.34315
11					.20589					
12										
13										
14										
15										.20589
mean	1.9099	2.1059	2.6038	3.6120	6.4954	2.0264	2.4658	3.3748	4.4775	8.8574

Table 2. Poisson and Compound Poisson Approximations to $P(N_{n,k,l} = x)$ by means of total variation distance for $p = 0.9, l = 0, 1, \ldots, k-1$ and $(n,k) = (15,5), (17,6), (19,7), (21,8), (23,9), (25,10), (28,11), (31,12), (35,13)$

l	$d(N_{15,5,l}, X_P)$	$d(N_{15,5,l}, X_{cP})$	$d(N_{17,6,l}, X_P)$	$d(N_{17,6,l}, X_{cP})$	$d(N_{19,7,l}, X_P)$	$d(N_{19,7,l}, X_{cP})$
0	.52269	.86816	.42775	.87532	.38232	.84369
1	.57379	.86816	.45640	.84264	.33191	.81099
2	.61413	.83747	.51000	.84264	.40056	.81099
3	.67716	.78132	.55394	.81180	.44382	.78010
4	.80951	.66764	.62477	.75531	.49934	.75097
5			.76183	.62167	.57783	.69764
6					.71650	.57164

l	$d(N_{21,8,l}, X_P)$	$d(N_{21,8,l}, X_{cP})$	$d(N_{23,9,l}, X_P)$	$d(N_{23,9,l}, X_{cP})$	$d(N_{25,10,l}, X_P)$	$d(N_{25,10,l}, X_{cP})$
0	.34736	.80764	.31136	.76863	.27527	.72785
1	.34736	.80764	.31136	.76863	.27527	.72785
2	.29750	.77496	.26299	.73604	.27527	.72785
3	.35340	.77496	.26299	.73604	.22910	.69542
4	.39690	.74412	.31349	.73604	.22910	.69542
5	.45344	.71503	.35673	.70531	.26959	.66485
6	.53570	.66180	.41328	.67634	.32227	.66485
7	.67355	.51940	.49768	.59920	.37779	.60899
8			.63291	.46649	.46320	.55957
9					.59448	.41415

l	$d(N_{28,11,l}, X_P)$	$d(N_{28,11,l}, X_{cP})$	$d(N_{31,12,l}, X_P)$	$d(N_{31,12,l}, X_{cP})$	$d(N_{35,13,l}, X_P)$	$d(N_{35,13,l}, X_{cP})$
0	.24816	.70245	.22199	.67552	.20395	.66089
1	.24816	.70245	.22199	.67552	.20395	.66089
2	.24816	.70245	.22199	.67552	.17892	.63586
3	.20120	.66996	.17538	.64305	.16687	.62831
4	.20120	.66996	.17538	.64305	.15574	.62831
5	.21679	.66996	.17538	.64305	.15574	.62831
6	.25229	.63933	.19283	.61245	.15087	.59772
7	.30085	.61048	.23656	.61245	.18847	.59759
8	.35583	.58332	.28302	.58363	.22816	.56863
9	.44684	.51117	.33583	.53099	.27381	.54136
10	.57235	.36424	.43009	.46443	.32672	.49439
11			.55017	.32132	.42474	.42240
12					.53924	.28589

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