ON FACTORS OF $C([0, 1])$ WITH NON-SEPARABLE DUAL

BY

HASKELL P. ROSENTHAL†

ABSTRACT

Let C denote the Banach space of scalar-valued continuous functions defined on the closed unit interval. It is proved that if X is a Banach space and $T: C \to X$ is a bounded linear operator with T^*X^* non-separable, then there is a subspace Y of C, isometric to C, such that $T \mid Y$ is an isomorphism. An immediate consequence of this and a result of A. Pelczynski, is that every complemented subspace of C with non-separable dual is isomorphic (linearly homeomorphic) to C.

1.

Let Δ denote the Cantor discontinuum. Our main result is as follows:

THEOREM 1. *Let K be an uncountable compact metric space, X a Banach space, and $T: C(K) \to X$ a bounded linear operator. If T^*X^* is non-separable, there exists a linear subspace Y of $C(K)$, isometric to $C(\Delta)$, such that $T \mid Y$ is an isomorphism. Moreover if T is a quotient map, then given $\varepsilon > 0$, Y may be chosen so that $\| (T \mid Y)^{-1} \| \leqq 1 + \varepsilon$.*

Now let $C = C([0, 1])$. Of course, C is isometric to a subspace of $C(\Delta)$. A special case of a result of A. Pelczynski (Corollary 1 of [9]) asserts that if X is a complemented subspace of C and X contains a subspace isomorphic to C, then X is isomorphic to C. An immediate consequence of this and our main result is

COROLLARY 1. *Every complemented subspace of C with a non-separable dual is isomorphic to C.*

Using results of Milutin, C could be replaced by $C(K)$ for any uncountable compact metric space K. Indeed, fixing such a K, Milutin proved that if K' is

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any compact metric space, there exists an isometric imbedding of $C(K')$ in $C(K)$, which is the range of a contractive projection. He also derived as a consequence of this, that $C(K)$ is isomorphic to C. (See [6] and also [7]). These results suggest that from the isometric viewpoint, there is no reason to prefer one $C(K)$ space over another, as representative of this isomorphism class; for the sake of definiteness, we choose the space C.

The results of Milutin and other recent results allow the conclusion of Theorem 1 to be strengthened somewhat. Suppose that K, T, Y, and X are as in the statement of Theorem 1, with X separable. By a recent result of Hagler and Stegall (Corollary 10 of [3]), there exists a subspace Y_1 of Y with Y_1 isometric to $C(\Delta)$, with TY_1 complemented in X. (This generalizes a previous result of Pelczynski [9], which yields the same assertion with the word "isometric" replaced by the word "isomorphic".) By Milutin's result, there exists a subspace Z of Y_1, isometric to $C(K)$, and complemented in Y_1. Since T is an isomorphism, TZ is complemented in X. Summarizing, we thus have that *if K, X, and T are as in the statement of Theorem 1 with X separable, then there exists a subspace Z of $C(K)$, isometric to $C(K)$, with $T|Z$ an isomorphism and TZ complemented in X.*

The proof of our main result is first outlined and then given in detail in Section 2. In addition to previously known techniques used in Banach space theory, we make essential use of the concept of conditional expectation in the proof of our crucial Lemma 1. Our final result, Theorem 2, applies the proof of Theorem 1 to show that certain subsets of C^* norm subspaces of C isometric to C. We use standard notation; all undefined terms are as given in [11].

We now continue in the present section with more immediate consequences of Theorem 1. Our next result follows directly from the "moreover" assertion of Theorem 1. For its statement, we need the following definition: Banach spaces X and Y are said to be λ-isomorphic if there exists an isomorphism T from X onto Y with $\|T\|\|T^{-1}\| \leqq \lambda$. By a result of Lindenstrauss and Pelczynski [5], given λ, there exists a space X isomorphic to C, such that no subspace of X is λ-isomorphic to C. Our next corollary shows that this cannot happen if X is isometric to a quotient space of C (i.e. X is isometric to C/Y for some $Y \subset C$).

COROLLARY 2. *Let X be isometric to a quotient space of C with X^* non-separable. Then for all $\varepsilon > 0$, there is a subspace Y of X which is $(1+\varepsilon)$-isomorphic to C.*

REMARKS.

1. It follows from the arguments of [9] as well as those of [3], that, in addition, Y may be chosen to be the range of a linear projection defined on X, of norm at most $1+\varepsilon$.

2. We give an example, preceding Theorem 2 below, of a quotient space of C which is isomorphic to C yet contains no subspace isometric to C.

Our proof of Theorem 1 is actually a little easier in the case where T is a quotient map; Corollaries 1 and 2 are immediate consequences of this case alone. Our last result of Section 1 shows a possible application of Theorem 1 to the case where T is not necessarily a quotient map.

COROLLARY 3. *Let Y be a subspace of C isomorphic to l^1, X a Banach space, and $T: Y \to X$ an isomorphism of Y with some subspace of X. If there exists a bounded linear operator $\tilde{T}: C \to X$ extending T, then X contains an isomorph of C.*

PROOF. By Theorem 1, we need only show that $(\tilde{T})^*$ has non-separable range. Letting $\pi: C^* \to Y^*$ be the natural restriction map, we have that $\pi \circ (\tilde{T})^*$ maps X^* onto Y^*. Since Y^* is isomorphic to the non-separable space l^∞, $\pi \circ (\tilde{T})^*$, has non-separable range, and hence so does $(\tilde{T})^*$.

2.

We shall first outline the proof of our main result. Let X be a Banach space, W a bounded subset of X^*, and Y a linear subspace of X. *We say that W norms Y if there is a constant λ such that*

$$(*) \qquad \| y \| \leqq \lambda \sup_{w \in W} |w(y)| \text{ for all } y \in Y.$$

In the case where W is contained in the unit ball of X^* and $(*)$ holds, we say that W λ-norms Y.

For the sake of convenience in notation, let $K = [0,1]$, and let T and X be as in the statement of Theorem 1. Now let $W = T^* S_{X^*}$ (where for any Banach space B, $S_B = \{b \in B : \| b \| \leqq 1\}$). W is a convex bounded symmetric non-separable subset of C. To prove Theorem 1, it suffices to exhibit a subspace Y of C, isometric to $C(\Delta)$, such that W norms Y (and such that in addition, W $(1+\varepsilon)$-norms Y in the case where T is a quotient map).

Our Lemma 4 reduces the problem of finding this Y, to the case where W is the unit ball of a subspace of C^* isometric to $l^1(\Gamma)$ for some uncountable set Γ.

Although our argument for Lemma 4 is self-contained, it is in reality a variation of arguments in [11].

Our Proposition 3 reduces the problem of finding this Y, to the case where W is the unit ball of a subspace of C^* isometric to L^1. (L^1 denotes the space $L^1(m)$, where m is the Lebesgue measure on the unit interval with respect to the σ-algebra of Lebesgue measurable sets). This reduction is accomplished by showing that if Z is a subspace of C^* isometric to $l^1(\Gamma)$ for some uncountable set Γ, then there is a subspace U of C^* with U isometric to C^*, such that for all $f \in C$,

$$\sup_{u \in S_U} |u(f)| \leqq \sup_{z \in S_Z} |z(f)|.$$

The proof of Proposition 3 is also self-contained; it is in reality a variation of certain arguments of [12] due to Stegall, which are in turn variations of an argument of Pelczynski [8]. In particular, we make use of the notion introduced by Stegall, of subsets of C^* equivalent to the usual basis of l^1 and dense-in-themselves in the weak* topology.

To handle the case where W is the unit ball of a subspace of C^* isometric to L^1, we need an explicit representation of such subspaces. Identifying C^* with the space of scalar-valued Borel measures on $[0,1]$, we show in Proposition 2 that if Z is a subspace of C^* isometric to L^1, there exists a Borel probability measure μ on $[0,1]$ and a σ-algebra $\mathscr{S}$ of Borel subsets of $[0,1]$, such that $([0,1], \mathscr{S}, \mu | \mathscr{S})$ is a purely non-atomic measure space and $Z = L^1(\mu | \mathscr{S})$. (Phrased another way; $\mathscr{S}$ has the property that given $E \in \mathscr{S}$, $\mu(E) > 0$, there is an $F \in \mathscr{S}$ such that $F \subset E$ and $0 < \mu(F) < \mu(E)$. Z consists of the subspace of $L^1(\mu)$ consisting of (equivalence classes of) μ-integrable $\mathscr{S}$-measurable functions; a probability measure μ is simply a positive measure with $\| \mu \| = 1$.) This result is certainly well known; we include a sketch of its proof for the sake of completeness.

Lemma 1 is the crucial step in the proof of our main result. In combination with Proposition 2 and an easy extension theorem, it yields that if W is the unit ball of a subspace of C^* isometric to L^1, then for all $\varepsilon > 0$, there is a subspace Y of C isometric to $C(\Delta)$ such that W $(1 + \varepsilon)$-norms Y. Since the statement of Lemma 1 (to be given shortly) is rather technical, we wish first to motivate it. It is known that if B denotes the linear span of the characteristic functions of half-open intervals of $[0,1]$ with dyadic-rational end points, then the closure of B in the L^∞-norm is isometric to $C(\Delta)$, while the closure in the L^1-norm is isometric to L^1. Given $\mathscr{S}$ a σ-algebra of Borel subsets of $[0,1]$ and μ a probability measure with

$\mu|\mathcal{S}$ purely non-atomic, Lemma 1 yields that there is a compact subset K of $[0,1]$ and a linear subspace A of $C(K)$ with the following properties: first, A is very close, in the $L^1(\mu)$ norm, to a subspace of $L^1(\mu|\mathcal{S})$; secondly, the behavior of A in both the $L^1(\mu)$ and $L^\infty(\mu)$ norms is almost like the behavior of B in both the $L^1(m)$ and $L^\infty(m)$ norms. The crucial use of the concept of conditional expectation is contained in the "Sublemma" part of the proof of the lemma.

LEMMA 1. *Let Ω be a compact Hausdorff space, μ a regular Borel probability measure on Ω, $\mathcal{S}$ a σ-subalgebra of the Borel subsets of Ω such that $\mu|\mathcal{S}$ is purely non-atomic, and $\varepsilon > 0$. Then there exist sets $F_i^n \in \mathcal{S}$ and compact subsets K_i^n of Ω satisfying the following properties for all $1 \leqq i \leqq 2^n$ and $n = 0,1,2,\cdots$:*

(i) $K_i^n \cap K_{i'}^n = F_i^n \cap F_{i'}^n = \emptyset$ for any $i' \neq i$.

(ii) $K_i^n = K_{2i-1}^{n+1} \cup K_{2i}^{n+1}$

and $F_i^n = F_{2i-1}^{n+1} \cup F_{2i}^{n+1}$.

(iii) $K_i^n \subset F_i^n$.

(iv) $\dfrac{1-\varepsilon}{2^n} \leqq \mu(K_i^n)$ and $\mu(F_i^n) \leqq \dfrac{1}{2^n}$.

REMARKS. Before passing to the proof, we wish to point out some immediate consequences of the statement. Let $K = K_1^0$ and let A denote the closure of the linear span of $\{\chi_{K_i^n}: 1 \leqq i \leqq 2^n; n = 0,1,2,\cdots\}$ in $C(K)$. Then A is a subalgebra of $C(K)$ algebraically isometric to $C(\Delta)$. Fix n and let

$$\phi = \sum_{i=1}^{n} c_i \chi_{K_i^n}$$

for some scalars $c_1,\cdots,c_n$; such functions ϕ are of course dense in A. Suppose $\|\phi\|_\infty = 1$ and suppose $\tilde{\varphi} \in C(\Omega)$ is such that $\|\tilde{\varphi}\|_\infty = 1$ and $\tilde{\varphi}|K = \phi$. Choose i so that $|c_i| = 1$ and put $f = \chi_{F_i^n}/\mu(F_i^n)$. Then of course $\|f\|_{L^1(\mu)} = 1$; we have that by (iii) and (iv),

$$\begin{aligned}\left|\int f\tilde{\varphi}d\mu\right| &\geqq \left|\int_{K_i^n} f\tilde{\varphi}d\mu\right| - \int_{F_i^n \sim K_i^n} |f\tilde{\varphi}|d\mu \\ &= \frac{\mu(K_i^n)}{\mu(F_i^n)} - \int_{F_i^n \sim K_i^n} |f\tilde{\varphi}|d\mu \\ &\geqq \frac{\mu(K_i^n)}{\mu(F_i^n)} - \frac{\mu(F_i^n) - \mu(K_i^n)}{\mu(F_i^n)} \\ &\geqq 1 - 2\varepsilon.\end{aligned}$$

Thus if W denotes the unit ball of $L^1(\mu | \mathscr{S})$, then

$$\| \tilde{\varphi} \| \leqq (1 - 2\varepsilon)^{-1} \sup_{f \in W} \left| \int f \tilde{\varphi} d\mu \right| . \tag{1}$$

PROOF OF LEMMA 1. For E a Borel subset of Ω, we denote $\mu(E)$ by $|E|$. We first need the following:

DENSITY SUBLEMMA. *Let $E \in \mathscr{S}$, $\tau > 0$, and $\delta > 0$ be given. Then there exist subsets F and K of E with $F \in \mathscr{S}$ and K compact, such that*

(a) $|E \sim F| \leqq \tau |E|$

(b) $|H \sim K| \leqq \delta |H|$ *for all* $H \in \mathscr{S}$ *with* $H \subset F$.

PROOF. By the regularity of μ, we may choose K a compact subset of E with $(1 - \tau\delta) |E| \leqq |K|$. By the Radon-Nikodym theorem, there is an $\mathscr{S}$-measurable function k supported on E, with values in $[0,1]$, so that for all $S \in \mathscr{S}$,

$$\int_S k d\mu = |S \cap K| .$$

(The function k is of course the conditional expectation of the function χ_K with respect to the σ-subalgebra $\mathscr{S}$.) Now let $F = \{t : k(t) \geqq 1 - \delta\}$. It follows that (b) holds. Indeed if $H \in \mathscr{S}$ and $H \subset F$, then $|H \cap K| = \int_H k d\mu \geqq (1 - \delta) |H|$. To see that (a) holds, we have that

$$\begin{aligned} (1 - \tau\delta) |E| &\leqq |K| \\ &= \int_{E \sim F} k d\mu + \int_F k d\mu \\ &\leqq (1 - \delta)(|E| - |F|) + |F| \\ &= (1 - \delta)|E| + \delta |F| \end{aligned}$$

which yields (a) immediately. This concludes the proof of the Sublemma.

Now choose $\varepsilon' > 0$ such that

$$1 - \varepsilon < (1 - \varepsilon') \prod_{j=0}^{\infty} \left(1 - \frac{\varepsilon'}{2^{j+1}} \right) . \tag{2}$$

By applying the Sublemma, we may choose by induction sets $\tilde{F}_i^n \in \mathscr{S}$ and compact subsets $\tilde{K}_i^n$ of Ω satisfying the following conditions for all $1 \leqq i \leqq 2^n$ and $n = 0, 1, 2, \cdots$:

(i′) $\tilde{K}_i^n \cap \tilde{K}_{i'}^n = \tilde{F}_i^n \cap \tilde{F}_{i'}^n = \varnothing$ for any $i' \neq i$.

(ii′) $\tilde{K}_i^n \supset \tilde{K}_{2i-1}^{n+1} \cup \tilde{K}_{2i}^{n+1}$

and $\tilde{F}_i^n \supset \tilde{F}_{2i-1}^{n+1} \cup \tilde{F}_{2i}^{n+1}$.

(iii′) $\tilde{F}_i^n \supset \tilde{K}_{2i-1}^{n+1} \cup \tilde{K}_{2i}^{n+1}$.

(iv′) $|H \sim \tilde{K}_i^n| \leqq \sum_{j=0}^{n} \frac{\varepsilon'}{2^{j+1}} |H|$ for any $H \in \mathscr{S}$ with $H \subset \tilde{F}_i^n$.

(v′) $\frac{1}{2^n} \prod_{j=0}^{n} \left(1 - \frac{\varepsilon'}{2^{j+1}}\right) < |\tilde{F}_i^n| \leqq \frac{1}{2^n}$

To see this, let τ_n and δ_n be defined for $n = 0, 1, 2, \cdots$ by

$$\frac{\tau_n}{1 - \tau_n} = \frac{\varepsilon'}{2^{n+1}} = \delta_n, \tag{3}$$

and let $\tilde{K}_1^0 = \Omega = \tilde{F}_1^0$.

Suppose the sets $\tilde{K}_i^n$ and $\tilde{F}_i^n$ have been chosen satisfying (i′), (iv′), and (v′) for all $1 \leqq i \leqq 2^n$. Now fix $i, 1 \leqq i \leqq 2^n$; since $\mu | \mathscr{S}$ is purely non-atomic, there exist disjoint subsets E_1 and E_2 of $\tilde{F}_i^n$, each belonging to $\mathscr{S}$, with

$$|E_1| = |E_2| = |\tilde{F}_i^n|/2. \tag{4}$$

Now by the Sublemma, we may choose for each $j = 1, 2$, subsets F_j and K_j of E_j so that $F_j \in \mathscr{S}$, K_j is compact,

$$|E_j \sim F_j| \leqq \tau_{n+1} |E_j| \text{ and} \tag{5a}$$

$$|H \sim K_j| \leqq \delta_{n+1} \; H| \text{ for any } H \in \mathscr{S} \text{ with } H \subset F_j. \tag{5b}$$

We then set $\tilde{F}_{2i-1}^{n+1} = F_1$, $\tilde{F}_{2i}^{n+1} = F_2$, and $\tilde{K}_{2i-1}^{n+1} = \tilde{K}_i^n \cap K_1$, $\tilde{K}_{2i}^{n+1} = \tilde{K}_i^n \cap K_2$. Again letting $j = 1$ or 2, we have that

$$|F_j| > (1-\tau_{n+1})|E_j| \qquad \text{by (5a)}$$

$$= (1-\tau_{n+1})\frac{|\tilde{F}_i^n|}{2} \qquad \text{by (4)}$$

$$> \frac{1}{2^{n+1}}\prod_{k=0}^{n+1}\left(1-\frac{\varepsilon'}{2^{k+1}}\right) \text{ by (v') and the definition}$$

of τ_{n+1}. Of course, also

$$|F_j| \leqq |E_j| = \frac{|\tilde{F}_i^n|}{2} \leqq \frac{1}{2^{n+1}} \text{ by (4) and (v').}$$

Finally, suppose $H \in \mathscr{S}$ is a subset of F_j. Then $H \sim (K_j \cap \tilde{K}_i^n) \subset (H \sim K_j) \cup (H \sim \tilde{K}_i^n)$; hence

$$|H \sim (K_j \cap \tilde{K}_i^n)| \leqq |H \sim K_j| + |H \sim \tilde{K}_i^n|$$

$$\leqq \delta_{n+1}|H| + \sum_{k=0}^{n}\frac{\varepsilon'}{2^{k+1}}|H| = \sum_{k=0}^{n+1}\frac{\varepsilon'}{2^{k+1}}|H|;$$

the last inequality following from (5b) and (iv'), the last equality from (3).

This completes the definition of the $\tilde{F}_i^n$'s and $\tilde{K}_i^n$'s by induction, as well as the verification that these objects satisfy (i')–(v'). We now define

$$K = K_1^0 = \bigcap_{n=0}^{\infty}\bigcup_{i=1}^{2^n}\tilde{K}_i^n \text{ and } F = F_1^0 = \bigcap_{n=0}^{\infty}\bigcup_{i=1}^{2^n}\tilde{F}_i^n;$$

then we put $K_i^n = K \cap \tilde{K}_i^n$ and $F_i^n = F \cap \tilde{F}_i^n$. Now fix i and n. It is immediate that (i) and (ii) hold. To see that (iii) holds, for $m \geqq 0$ put $\tilde{K}_i^{n+m} = \bigcup_j \tilde{K}_j^{n+m}$ and $\tilde{F}_i^{n+m} = \bigcup_j \tilde{F}_j^{n+m}$, where both unions are extended over all j with $\tilde{F}_j^{n+m} \subset \tilde{F}_i^n$ (i.e., $(i-1)2^m + 1 \leqq j \leqq i \cdot 2^m$). It follows from (iii') that $\tilde{K}_i^{n+m+1} \subset \tilde{F}_i^{n+m}$ for any $m \geqq 0$. Hence taking (ii') into account,

$$K_i^n = \bigcap_{m=0}^{\infty} K_i^{n+m+1} \subset \bigcap_{m=0}^{\infty} F_i^{n+m} = F_i^n.$$

Since $\tilde{F}_i^n \supset F_i^n$, it follows immediately from (v') that $|F_i^n| \leqq 1/2^n$.

It remains to show that $|K_i^n| \geqq (1-\varepsilon)/2^n$. We first estimate $|F_i^n|$ from below.

We have that $|F_i^n| = \lim_{m\to\infty}\Sigma|\tilde{F}_j^{n+m}|$, the sum extended over all j with $\tilde{F}^{n+m} \subset \tilde{F}_j^n$; since there are 2^m such j's, we have by (v′) that

$$
\begin{aligned}
|F_i^n| &\geqq \lim_{m\to\infty} 2^m \frac{1}{2^{n+m}} \prod_{j=0}^{n+m}\left(1 - \frac{\varepsilon'}{2^{j+1}}\right) \\
&= \frac{1}{2^n}\prod_{j=0}^{\infty}\left(1 - \frac{\varepsilon'}{2^{j+1}}\right).
\end{aligned} \tag{6}
$$

Now suppose that $F \in \mathscr{S}$ with $F \subset F_i^n$. Then

$$|F \sim K_i^n| = \lim_{m\to\infty}\Sigma|(F_j^{n+m}\cap F) \sim \tilde{K}_j^{n+m}|,$$

the sum extended over all j so that $\tilde{F}_j^{n+m} \subset \tilde{F}_i^n$. But by (iv′), each term of this sum is dominated by

$$\sum_{k=0}^{n+m}\frac{\varepsilon'}{2^{k+1}}|F \cap \tilde{F}_j^{n+m}|;$$

since F equals the disjoint union of $F \cap \tilde{F}_j^{n+m}$ over these j's, the entire sum is dominated by

$$\sum_{k=0}^{n+m}\frac{\varepsilon'}{2^{k+1}}|F|,$$

which is in turn dominated by $\varepsilon'|F|$. Thus for any such F, $|F \sim K_i^n| \leqq \varepsilon'|F|$. In particular, $|F_i^n \sim K_i^n| \leqq \varepsilon'|F_i|$. Hence

$$
\begin{aligned}
|K_i^n| &\geqq (1-\varepsilon')|F_i^n| \\
&\geqq (1-\varepsilon')\frac{1}{2^n}\prod_{j=0}^{\infty}\left(1 - \frac{\varepsilon'}{2^{j+1}}\right) && \text{by (6)} \\
&\geqq \frac{1-\varepsilon}{2^n} && \text{by (2).}
\end{aligned}
$$

Q.E.D.

PROPOSITION 2. *Let Ω be a compact Hausdorff space, let $C(\Omega)^*$ be identified with the space of all regular scalar-valued Borel measures on Ω, and let Z be a subspace of $C(\Omega)^*$ isometric to L^1. Then there exists a regular Borel probability measure μ on Ω and a σ-algebra $\mathscr{S}$ of the Borel subsets of Ω, such that $(\Omega, \mathscr{S}, \mu|\mathscr{S})$ is a purely non-atomic measure space and $Z = L^1(\mu|\mathscr{S})$.*

PROOF. Let $T: L^1 \to C(\Omega)^*$ be an into-isometry. It is easily seen that there exists a positive Borel measure ν such that the range of T is contained in $L^1(\nu)$. Now it is known (see [10]) that if f and g are members of $L^1(\nu)$ such that $\| af + bg \|_{L^1(\nu)} = |a| + |b|$ for all scalars a and b, then there are disjoint Borel sets F and G with $F \cup G = \Omega$ and ν-integrable Borel-measurable functions $\tilde{f}$ and $\tilde{g}$ representing f and g, with $\tilde{f}$ supported on F and $\tilde{g}$ supported on G. Proposition 2 is now proved simply by iterating this observation. Let

$$E_i^n = \left[\frac{i-1}{2^n}, \frac{i}{2^n}\right) \text{ for } 1 \leqq i \leqq 2^n,\ n = 0, 1, 2, \cdots,$$

and let $S_1^0 = \Omega$. We may choose by induction Borel subsets S_i^n and Borel-measurable ν-integrable functions f_i^n satisfying for all $1 \leqq i \leqq 2^n$ and $n = 0, 1, 2, \cdots$;

(i) $S_i^n \cap S_{i'}^n = \emptyset$ for all $i' \neq i$.

(ii) $S_i^n = S_{2i-1}^{n+1} \cup S_{2i}^{n+1}$.

(iii) $f_i^n = f_{2i-1}^{n+1} + f_{2i}^{n+1}$.

(iv) f_i^n is supported on S_i^n.

(v) f_i^n is a representative of $T\chi_{E_i^n}$ in $L^1(\nu)$.

For each n, let $\mathscr{A}_n$ equal the family of all finite unions of the sets S_i^n, $1 \leqq i \leqq 2^n$, let $\mathscr{A} = \cup_{n=1}^{\infty} \mathscr{A}_n$, let $\mathscr{S}$ be the σ-algebra of Borel sets generated by $\mathscr{A}$ and let μ be the measure on Ω such that $d\mu = f_1^0 d\nu$. As measures on Ω, we have that for all i and n, $f_i^n d\nu = \chi_{S_i^n} d\mu$. Indeed, $f_1^0 = \Sigma_{j=1}^{2^n} f_j^n = \Sigma_{j=1}^{2^n} f_j^n \chi_{S_j^n}$; hence for any Borel set S,

$$\int_S f_i^n d\nu = \int_{S \cap S_i} f_i^n d\nu = \int_{S \cap S_i} f_1^0 d\nu = \mu(S_i^n \cap S).$$

It follows that if B denotes the linear span of the $\chi_{E_i^n}$'s, then $TB \subset L^1(\mu \mid \mathscr{S})$; so also $TL^1 \subset L^1(\mu \mid \mathscr{S})$ since B is dense in L^1. On the other hand, $\mathscr{A}$ is itself an algebra of sets; by elementary facts from measure theory, it follows that for all $\varepsilon > 0$ and $S \in \mathscr{S}$, there exists an $E \in \mathscr{A}$ with $\mu(E \sim S) + \mu(S \sim E) < \varepsilon$. This implies that $TL^1 \supset L^1(\mu \mid \mathscr{S})$, from which $TL^1 = L^1(\mu \mid \mathscr{S})$; the fact that $\mu \mid \mathscr{S}$ is purely non-atomic now follows immediately. *Q.E.D.*

For our next result, we recall that a sequence (f_n) of vectors is said to be isometrically equivalent to the usual l^1 basis if $\|\Sigma a_j f_j\| = \Sigma |a_j|$ for all finite sequences of scalars $a_1, a_2, \cdots$.

PROPOSITION 3. *Let X be a separable Banach space and (f_n) a sequence in X^* isometrically equivalent to the usual basis of l^1, such that $\{f_n : n = 1,2,\cdots\}$ is dense-in-itself in the weak* topology. Then there exists a subspace U of X^*, isometric and w*-isomorphic to $C(\Delta)^*$, such that for all $x \in X$,*

$$\sup_{u \in S_U} |u(x)| \leqq \sup_n |f_n(x)|. \tag{7}$$

PROOF. Let $D = \{f_1, f_2, \cdots\}$ and let K equal the closure of D in the weak* topology. We shall construct a subset Ω of K, homeomorphic to the Cantor set Δ, so that the map $T: X \to C(\Omega)$ defined by $(Tx)(\omega) = \omega(x)$ for all $x \in X$, $\omega \in \Omega$, is a surjective quotient map; i.e., if we let Y be the kernel of T, π the natural map from X to X/Y, and $\tilde{T}: X/Y \to C(\Omega)$ the map so that $T = \tilde{T}\pi$, then $\tilde{T}$ is a surjective isometry. It then follows easily that $U = T^*(C(\Omega))^*$ has the desired properties.

Indeed, if $x \in X$, then

$$\sup_{u \in S_U} |u(x)| = \| Tx \| = \sup_{\omega \in \Omega} |\omega(x)| \leqq \sup_n |f_n(x)|.$$

Let K be endowed with a suitable metric, inducing the weak* topology on K; we now restrict ourselves, topologically, to the space K; let $K_1^0 = K$.

Let $n \geqq 0$, and supppose $K_1^n, \cdots, K_{2^n}^n$ have been chosen, with the K_i^n's disjoint, and each K_i^n a compact non-empty neighborhood in K. Let F_{n+1} be a finite $1/2^{n+1}$ dense subset of the surface of the unit ball of $l^\infty_{2^{n+1}}$, so that F_{n+1} contains the usual basis of $l^\infty_{2^{n+1}}$. (Thus, given $x \in l^\infty_{2^{n+1}}$ with $\|x\| = 1$, there is an $f \in F_{n+1}$ with $\|f - x\| < 1/2^{n+1}$.) Since D is dense-in-itself, for each i we may choose d_{2i-1} and d_{2i} in D, which are distinct elements of K_i^n. (Of course the d_i's depend on n also). Since the linear span of the d_i's is isometric to $l^1_{2^{n+1}}$, for each $f \in F_{n+1}$, we may choose $x_f \in X$ with $\|x_f\| \leqq 1 + 1/2^{n+1}$, with $d_j(x_f) = f(j)$ for all j, $1 \leqq j \leqq 2^{n+1}$. Now for each i and $j = 2i-1$ or $2i$, let K_j^{n+1} be a compact neighborhood of d_j, of diameter at most $1/2^{n+1}$, contained in

$$\left\{ \bigcap_{f \in F_{n+1}} \{k \in K_i^n : |k(x_f) - d_j(x_f)| < \frac{1}{2^{n+1}}\right\}. \tag{8}$$

(The set described in (8) is a non-empty open set in K.) We also note that since F_{n+1} contains the usual basis of $l^\infty_{2^{n+1}}$, the sets K_{2i-1}^{n+1} and K_{2i}^{n+1} are disjoint.

This completes the definition of the K_i^n's and F_n's. We now set

$$\Omega = \bigcap_{n=0}^{\infty} \bigcup_{i=1}^{2^n} K_i^n.$$

Then Ω is homeomorphic to Δ. Given n, scalars $c_1, \cdots, c_{2^n}$, and

$$\phi = \sum_{j=1}^{2^n} c_j \chi_{K_j^n \cap \Omega}$$

with $\| \phi \| = 1$, we may choose $f \in F_n$ with $|f(j) - c_j| < 1/2^n$ for all j. Fixing j, it follows by (8) that if $k \in K_j^n \cap \Omega$, then $|k(x\ ) - f(j)| < 1/2^n$. Thus defining T as at the beginning of this proof, we have that $\| Tx_f - \phi | \Omega \| \leqq 1/2^{n-1}$. Standard arguments now complete the proof that T is a quotient map. *Q.E.D.*

REMARKS.

1. A tiny variation in the above proof yields that if the f_n's are assumed to be equivalent to the usual basis of l^1, and dense in themselves in the weak* topology then there is a subset Ω of x^*, homeomorphic in the ω^* topology to Δ, so that the natural map of X into $C(\Omega)$ is surjective. The variation: there is a constant λ (depending on the constant of basis-equivalence) so that the x_f's may be chosen with $\| x_f \| \leqq \lambda$.

2. Let X be separable, and suppose X^* contains a subspace Z isometric to $l^1(\Gamma)$ for some uncountable set Γ, with $\{e_\gamma : \gamma \in \Gamma\}$ isometrically equivalent to the usual basis of $l^1(\Gamma)$. Then as pointed out by C. Stegall in [12], since any uncountable subset of a separable metric space contains a countable set dense in itself, there exist $\gamma_1, \gamma_2, \cdots$ in Γ, so that setting $f_n = e_{\gamma_n}$ for all n, the sequence (f_n) satisfies the hypotheses of Proposition 3. In particular, $C(\Delta)$ is then isometric to a quotient space of X. Similarly, by the first remarks, if X^* contains a subspace isomorphic to $l^1(\Gamma)$ for some uncountable set Γ, then $C(\Delta)$ is isomorphic to a quotient space of X. This last result was first proved by A. Pelczynski [8] under certain restrictive hypotheses which were later removed by J. Hagler [2]. A different argument for this was given by Stegall in [12], and as we mentioned earlier, our argument for Proposition 3 itself is a variation of the arguments of [12]. The main difference in our approach, is that by using ε-dense subsets of the unit ball of l_n, we are able to exhibit $C(\Delta)$ directly as a continuous linear image of X.

3. We wish finally to note that, in this paper, we are only concerned with applying Proposition 3 to the case where $X = C$. In this case, if the f_n's satisfy the hypotheses of Proposition 3, their closed linear span is the range of a contractive projective defined on C^*. One may reach the conclusion of Proposition 3 by then applying directly the arguments of [3].

Our final lemma requires some preliminary notations and definitions. Given K a compact metric space, we identify $C(K)^*$ with the space of all scalar-valued Borel measures on K. Given μ and $\nu \in C(K)^*$, we write $\mu \ll \nu$ if μ is absolutely continuous with respect to ν, and $\mu \perp \nu$ if μ is singular with respect to ν. If $\mu \in C(K)^*$ and f is a μ-integrable Borel measurable function, $f \cdot \mu$ denotes the measure defined by $(f \cdot \mu)(E) = \int_E f d\mu$ for all Borel sets E. $d\mu/d\nu$ denotes the Radon-Nikodym derivative of μ with respect to ν; thus $\mu \ll \nu$ if and only if $\mu = d\mu/d\nu \cdot \nu$. Given $\mu_1, \mu_2, \cdots$ in $C(K)^*$, $\bigvee \{\mu_n : n = 1, 2, \cdots\}$ denotes the measure defined by

$$\mu = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|\mu_n|}{1 + \|\mu_n\|},$$

(where $|\mu_n|$ denotes the total variation of the measure μ_n for all n). Of course the definition of μ depends on the particular enumeration of the μ_n's. However if $\nu \in C(K)^*$, the measure $d\nu/d\mu \cdot \mu$ is independent of the particular enumeration; i.e., if σ is a permutation of the positive integers and

$$\mu' = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|\mu_{\rho(n)}|}{1 + \|\mu_{\rho(n)}\|}$$

then $d\nu/d\mu \cdot \mu = d\nu/d\mu' \cdot \mu'$. This follows simply becasue $\mu' \ll \mu$ and $\mu \ll \mu'$. Since we shall work only with measures of the form $d\nu/d\mu \cdot \mu$ for μ as above, we shall treat $\bigvee \{\mu_n : n = 1, 2, \cdots\}$ as if it were defined independently of the particular enumeration. (Of course $\mu_j \ll \bigvee \{\mu_n : n = 1, 2, \cdots\}$ for all j.)

LEMMA 4. *Let K be a compact metric space and let L be a convex bounded symmetric non-norm-separable subset of $C(K)^*$. Then there is a $\delta > 0$, such that for all $\varepsilon > 0$, there exists an uncountable family $\{l_\alpha\}_{\alpha \in \Gamma}$ contained in L, and a family of pairwise-singular Borel measures $\{\mu_\alpha\}_{\alpha \in \Gamma}$ such that for all α,*

$$\|\mu_\alpha - l_\alpha\| \leqq \varepsilon \text{ and } \|\mu_\alpha\| \geqq \delta.$$

Moreover in the case where L equals the unit ball of a non-separable subspace of $C(K)^*$, δ may be chosen equal to 1.

PROOF. Let Γ denote the set of all countable ordinals in the usual ordering. Suppose first that L is the unit ball of a non-separable subspace Y of $C(K)^*$, and let $\varepsilon'/(1-\varepsilon') < \varepsilon$. Then given any $\mu \in C(K)^*$, there is an $l \in L$ with $\| l \| = 1$ and $\| dl/d\mu \cdot \mu \| < \varepsilon'$, for otherwise Y would be isometric to a subspace of the separable space $L^1(|\mu|)$. Now let l_0 be some element of L of norm one, and let $\nu_0 = 0$.

Having chosen l_α for all $\alpha < \beta$, let $\nu_\beta = \bigvee \{l_\alpha : \alpha < \beta\}$ and choose $l_\beta \in L$ with $\| l_\beta \| = 1$ and $\| dl_\beta / d\nu_\beta \cdot \nu_\beta \| < \varepsilon'$. This defines the l_α's and ν_α's by induction; for each α, put $\mu_\alpha = l_\alpha - dl_\alpha/d\nu_\alpha \cdot \nu_\alpha$; thus $\mu_\alpha \ll l_\alpha$. Now if $\alpha < \beta$, since then $l_\alpha \ll \nu_\beta$, $\mu_\alpha \ll \nu_\beta$; but $\mu_\beta \perp \nu_\beta$. Hence $\mu_\alpha \perp \mu_\beta$. Hence the μ_α's are pairwise singular; of course $\| \mu_\alpha - l_\alpha \| < \varepsilon'$. It then follows easily that setting $\bar{\mu}_\alpha = \mu_\alpha / \| \mu_\alpha \|$ and $\bar{l}_\alpha = l_\alpha / \| \mu_\alpha \|$, we have that the $\bar{\mu}_\alpha$'s and $\bar{l}_\alpha$'s satisfy the conclusion of the lemma. Indeed, $\| \mu_\alpha \| \geqq 1 - \varepsilon'$, hence $\| \, l_\alpha / \| u_\alpha \| - u_\alpha / \| u_\alpha \| \, \| \leqq \varepsilon'/(1-\varepsilon') < \varepsilon$ for all α.

We now pass to the general case. Observe that if $\mu \in C(K)^*$, then there is a $\lambda \in L$ with $\lambda - d\lambda/d\mu \cdot \mu \neq 0$, for otherwise $L \subset L^1(|\mu|)$. Let $\lambda_0 \in L$, $\lambda_0 \neq 0$, and $\nu_0 = 0$. By an argument similar to one already given, it follows that we may define $\{\lambda_\alpha\}_{\alpha \in \Gamma}$ and $\{\nu_\alpha\}_{\alpha \in \Gamma}$ by induction so that for all $\beta > 0$, $\nu_\beta = \bigvee \{\lambda_\alpha : \alpha < \beta\}$ and $\| \lambda_\beta - d\lambda_\beta / d\nu_\beta \cdot \nu_\beta \| > 0$, with $\lambda_\alpha \in L$ for all α. We have that

$$\text{if } \alpha < \beta, \text{ then } \lambda_\alpha \ll \nu_\beta \text{ and } \nu_\alpha \ll \nu_\beta. \tag{9}$$

It follows that there exists an uncountable set $\Gamma_1 \subset \Gamma$ and a $\delta > 0$ such that

$$\left\| \lambda_\beta - \frac{d\lambda_\beta}{d\nu_\beta} \cdot \nu_\beta \right\| > 2\delta \text{ for all } \beta \in \Gamma_1. \tag{10}$$

Now let $\varepsilon > 0$. We shall choose the l_α's by induction, to be one-half the difference of suitable pairs of λ_β's. The observation which allows our induction to proceed, is that

$$\begin{gathered} \text{for all } \nu \in C(K)^* \text{ and } \alpha \in \Gamma, \text{ there exist} \\ \beta_1 \text{ and } \beta_2 \text{ in } \Gamma_1, \text{ with } \alpha < \beta_1 < \beta_2 \text{ such that} \\ \left\| \frac{d\lambda_{\beta_1}}{d\nu} \cdot \nu - \frac{d\lambda_{\beta_2}}{d\nu} \cdot \nu \right\| < 2\varepsilon. \end{gathered} \tag{11}$$

Indeed, this is simply because $\{d\lambda_\beta / d\nu : \beta > \alpha, \beta \in \Gamma_1\}$ is an uncountable subset of the separable metric space $L^1(|\nu|)$, and hence has a cluster point. We now construct by induction, elements l_γ of L and measures $\mu_\gamma \in C(K)^*$ satisfying the following conditions for all $\gamma \in \Gamma$:

(12) $\| u_\gamma \| \geqq \delta$; $\| u_\gamma - l_\gamma \| \leqq \varepsilon$; for all $\beta \neq \gamma$, $\mu_\gamma \perp \mu_\beta$; there exists a $\tau(\gamma) \in \Gamma$ with $\mu_\gamma \ll l_\gamma \ll \nu_{\tau(\gamma)}$.

Let β_0 be the first element of Γ_1, and put $l_0 = \lambda_{\beta_0} = \mu_0$. We then let $\tau(0)$ be the successor to β_0. Now fix $\eta > 0$ a countable ordinal, and suppose that l_γ and μ_γ have been constructed satisfying (12) for all $\gamma < \eta$ and $\beta \neq \gamma$, $\beta < \eta$. Choose α a countable ordinal so that $\tau(\gamma) < \alpha$ for all $\gamma < \eta$. Choose β_1 and β_2 satisfying (11) for $\nu = \nu_\alpha$. Now put $l_\eta = (\lambda_{\beta_1} - \lambda_{\beta_2})/2$ and $\mu_\eta = l_\eta - dl_\eta/d\nu_\alpha \cdot \nu_\alpha$; then let $\tau(\eta)$ be the successor to β_2. By (9) and the definition of l_η and μ_η, $\mu_\eta \ll l_\eta \ll \nu_{\tau(\eta)}$.

$$\begin{aligned} \| \mu_\eta \| &\geqq \left\| l_\eta - \frac{dl_\eta}{d\nu_{\beta_2}} \cdot \nu_{\beta_2} \right\| \quad \text{since } \nu_\alpha \ll \nu_{\beta_2} \text{ by (9)} \\ &= \tfrac{1}{2} \left\| \lambda_{\beta_2} - \frac{d\lambda_{\beta_2}}{d\nu_{\beta_2}} \cdot \nu_{\beta_2} \right\| \quad \text{by the definition of } l_\eta \\ &> \delta \quad \text{by (10).} \end{aligned}$$

By definition $\mu_\eta \perp \nu_\alpha$ and by (9), (12), and the definition of α, if $\gamma < \eta$, then $\mu_\gamma \ll \nu_\alpha$, from which $\mu_\gamma \perp \mu_\eta$. This completes the induction and hence the proof of Lemma 4.

Q.E.D.

We are finally prepared for the

Proof of Theorem 1. Let $L = T^* S_{X^*}$. Now choose $\delta > 0$ according to Lemma 4; and let $\varepsilon > 0$; then choose an uncountable family $\{l_\alpha\}_{\alpha \in \Gamma}$ contained in L, and a family of pairwise singular Borel-measures $\{\mu_\alpha\}_{\alpha \in \Gamma}$, so that for all α

(13) $$\| u_\alpha - l_\alpha \| \ll \varepsilon \text{ and } \| \mu_\alpha \| \geqq \delta.$$

Since $\{\mu_\alpha / \| \mu_\alpha \| : \alpha \in \Gamma\}$ is an uncountable set, there is a countable subset $\alpha_1, \alpha_2, \cdots$ of Γ, so that putting $f_n = \mu_{\alpha_n} / \| \mu_{\alpha_n} \|$ for all n, (f_n) is dense-in-itself in the weak* topology. Moreover (f_n) is isometric to the usual basis of l^1; hence by Proposition 3, there exists a subspace U of X^* isometric to $C(\Delta)^*$, satisfying (7) for all $x \in C(K)$. Now choose Z a subspace of U isometric to L^1; by Proposition 2, there is a Borel probability measure μ on K and a σ-algebra $\mathcal{S}$ of the Borel subsets of K such that $Z = L^1(\mu \mid \mathcal{S})$ and of course $\mu \mid \mathcal{S}$ is purely non-atomic. Now choose $F_i^n \in \mathcal{S}$ and K_i^n compact subsets of K satisfying (i)–(iv) of Lemma 1 for all $1 \leqq i \leqq 2^n$ and $n = 0, 1, 2, \cdots$. Let $\tilde{K} = K_1^0$, let A denote the closed linear span of $\{\chi_{K_i^n}: 1 \leqq i \leqq 2^n,\ n = 0, 1, 2, \cdots\}$ in $C(\tilde{K})$, and choose a norm-one extension operator $E: C(\tilde{K}) \to C(K)$; i.e., E is a norm-one linear operator such that $(Ef) \mid \tilde{K} = f$ for all $f \in C(\tilde{K})$. (The existence of such an operator follows from the

Borsuk-Dugundji extension theorem (see [1] and [7]); the proof is an easy and well-known exercise in the case where $K = [0,1]$). Finally, let $Y = E(A)$. Since A is isometric to $C(\Delta)$ and E is an isometry, Y is isometric to $C(\Delta)$. By our remarks following the statement of Lemma 1, it follows from (1) that

$$\|\phi\| \leqq (1-2\varepsilon)^{-1} \sup_{f \in S_Z} |f(\phi)| \text{ for all } \phi \in Y.$$

Since $S_Z \subset S_U$, we have by the definition of the f_n's and (7) that for all $\phi \in Y$,

$$\|\phi\| \leqq (1-2\varepsilon)^{-1} \sup_{\alpha \in \Gamma} \left| \int \phi \frac{d\mu_\alpha}{\|\mu_\alpha\|} \right|. \tag{14}$$

Fixing $\phi \in Y$ with $\|\phi\| = 1$ and $\alpha \in \Gamma$, we have by (13) that $|\int \phi d\mu_\alpha| \leqq |\int \phi dl_\alpha| + \varepsilon$. Combining this with (14) and the second inequality of (13),

$$\delta \leqq (1-2\varepsilon)^{-1} \sup_{\alpha \in \Gamma} \left| \int \phi d\mu_\alpha \right| \leqq (1-2\varepsilon)^{-1} \left(\sup_{\alpha \in \Gamma} \left| \int \phi dl_\alpha \right| + \varepsilon \right).$$

So long as $\varepsilon(1-2\varepsilon)^{-1} < \delta$, we thus obtain that for all $\phi \in Y$,

$$\begin{aligned} \|\phi\| &\leqq (\delta - 2\varepsilon\delta - \varepsilon)^{-1} \sup_{\alpha \in \Gamma} \left| \int \phi dl_\alpha \right| \\ &\leqq (\delta - 2\varepsilon\delta - \varepsilon)^{-1} \| T\phi \|. \end{aligned} \tag{15}$$

Hence $T|\,Y$ is an isomorphism. Also in the case where T is a quotient map, δ may be chosen equal to 1 by Lemma 4, and hence by (15), $\|(T|\,Y)^{-1}\| \leqq (1-3\varepsilon)^{-1}$. The proof of Theorem 1 is now complete.

Remarks. The following example shows the necessity of proving something like Lemma 4, in order to obtain the proof of our main result, even in the case of quotient maps. Let $f_1, f_2, \cdots$ be a countable dense subset of the unit ball of C; let $h_1, h_2, \cdots$ in C^* be supported on $[3/4, 1]$ such that (h_n) is isometrically equivalent to the usual basis of l^2, and let W be the subspace of C^* consisting of all elements ν of the form

$$\nu = \mu + \sum_{i=1}^{\infty} \frac{\int f_i \, d\mu}{i} h_i$$

where μ is an arbitrary element of C^* supported on $[0, \frac{1}{2}]$. It is then easily seen that W is a weak* closed strictly convex non-norm-separable subspace of C^*. It follows that W contains no subspace isometric to l^1. Also by letting $Z = C/W^{\perp}$, we have that Z contains no subspace isometric to C since Z is a smooth Banach space. (It is also easily seen that Z is isomorphic to C, while the smoothness of Z

implies that given one of its subspaces X, there exists an $\varepsilon > 0$ such that X is not $(1+\varepsilon)$-isomorphic to C. Thus the conclusion of Corollary 2 cannot be sharpened to the assertion resulting from moving the quantifier "for all $\varepsilon > 0$" to the end of its statement.)

Our final result applies the proof of Theorem 1 and some results of [4]. (For the definition of norming and ε-norming sets, see the beginning of Section 2.)

THEOREM 2. *Let K be a compact Hausdorff space and W a bounded subset of $C(K)^*$.*

(a) *If K is metrizable and W is non-separable, there is a subspace Y of $C(K)$, isometric to $C(\Delta)$, so that W norms Y.*

(b) *If W is the unit ball of a subspace Z of $C(K)^*$, then for all $\varepsilon > 0$, there is a subspace Y of $C(K)$, isometric to $C(\Delta)$, such that W $(1+\varepsilon)$-norms Y, under the following circumstances*: (i) *K is metrizable and Z is non-separable.* (ii) *K is arbitrary and Z is isomorphic to L^1.*

PROOF. To see (a), let $L = \{(w_1 - w_2)/2 \colon w_1, w_2 \in W)\}$. Our proof of Lemma 4 and Theorem 1 yields that there is a subspace Y of $C(K)$, isometric to $C(\Delta)$, so that L norms Y. But then trivially W norms Y also.

Part (i) of (b) follows immediately from our proof of Theorem 1. We pass now to (ii) of (b). We may actually assume that K is metrizable, for there exists a separable subalgebra B of $C(K)$, such that $\| w \| = \sup_{f \in S_B} |w(f)|$ for all $w \in W$. Of course then B is algebraically isometric to $C(K_1)$ for some compact metrizable space K_1.

Assuming now that K is metrizable, it follows, by the proofs of Theorems III.1 and IV.3 of [4], that there is a subspace M of $C(K)^*$ such that M is isomorphic to $C(\Delta)^*$ and S_M is contained in the weak* closure of W. Of course S_M is non-separable, so part (ii) of (b) follows from part (i) of (b). (We note in passing, that when Z is isometric to L^1, part (ii) of (b) follows immediately from our proof of Theorem 1.) *Q.E.D.*

As an immediate consequence of Theorem 2(b) (ii), we have the

COROLLARY. *Let K be a compact Hausdorff space, let X be isometric to a quotient space of $C(K)$, and assume that X^* contains an isomorph of L^1. Then for all $\varepsilon > 0$, X contains a subspace $(1+\varepsilon)$-isometric to C.*

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University of California, Berkeley