ON SETS OF HAAR MEASURE ZERO IN ABELIAN POLISH GROUPS

BY

JENS PETER REUS CHRISTENSEN

ABSTRACT

It is shown that the concept of zero set for the Haar measure can be generalized to abeli ın Polish groups which are not necessarily locally compact. It turns out that these groups, in many respects,behave like locally compact groups. Suitably modified, many theorems from harmonic analysis carry over to this case. A few applications are given and some open problems are mentioned.

This paper deals with a curious analogy between Polish and locally compact groups.

It contains at most the beginning of a theory. We hope that the questions presented here will stimulate further research on the problem.

We always tacitly assume that the topologies under consideration are separated. Universally measurable means measurable with respect to the completion of the Borel field with respect to the family of all measures. Since our nonabelian results are still rather unsatisfactory, we restrict ourselves to abelian groups. First we give some motivating remarks of an entirely heuristic nature.

The finite dimensional Banach spaces are much easier to deal with because of the existence of a Haar measure which provides us with more tools. But although infinite dimensional Banach spaces do exist, it is well known that an invariant measure with reasonable properties does not exist unless the space is locally compact and therefore finite dimensional. So if one agrees that the σ-ideal of zero sets is almost as good as the Haar measure itself, the zero sets may suffice.

And why should there not be zero sets for the Haar measure even if this itself has disappeared? We shall characterize the sets of Haar measure zero without using the Haar measure in order that this concept can be generalised immediately to nonlocally compact groups. Then, of course, the next thing to do is to find as many analogies as possible with the locally compact case.

Let $(G, +)$ be an abelian Polish group and $A \subseteq G$ a universally measurable set. By definition, A is a Haar zero set (or simply zero set if no confusion is likely to arise) if there exists a probability measure u on G (not unique) such that $\mathscr{X}_A * u = 0$. This means that every translate of A has u measure zero (every measure is a countable additive Borel measure extended to the universally measurable sets). Let G be locally compact and let h denote the Haar measure. Note that h is σ-finite and therefore we may use the Fubini theorem. From the equality

$$\iint \mathscr{X}_A(x + y)u(dx)h(dy) = \iint \mathscr{X}_A(y + x)h(dy)u(dx),$$

it follows easily that our definition coincides with the usual one, in the locally compact case.

THEOREM 1. *In an abelian Polish group every countable union of Haar zero sets is a Haar zero set.*

PROOF. It is well known that the space $\mathscr{P}$ of probability measures on G is a Polish space equipped with the weak topology (induced from the bounded continuous functions). Let d be a complete metric on $\mathscr{P}$ generating this topology. For every $u \in \mathscr{P}$, the mapping $L_u: \mathscr{P} \to \mathscr{P}$ defined by $L_u(v) = u * v$ is continuous. Now let A_n be a sequence of universally measurable sets which are zero sets and let u_n be corresponding probabilities. If u_n' is a probability with density with respect to a translation of u_n, we also have $\mathscr{X}_{A_n} * u_n' = 0$. Then it is easily seen that such a u_n' can be found in every neighbourhood of e where e is the neutral element of the semigroup $\mathscr{P}$ (unit mass in O_G).

By induction we now choose a sequence of such u_n' satisfying $d(x, x * u_n') \leq 1/2^n$ where x is any convolution of different u_i' $i = 1, \dots, n-1$.

Then $u = u_1' * u_2' \dots$ is well defined. Moreover for any n, we have $u = x_n * u_n * y_n$; hence $\mathscr{X}_{A_n} * u = 0$.

Therefore $\mathscr{X}_A * u = 0$ where $A = \bigcup_n A_n$. Hence A is a zero set. The next result shows a more striking analogy between Polish and locally compact groups.

THEOREM 2. *Let A and B be arbitrary universally measurable sets in the abelian Polish group G. We put*

$$F(A,B) = \{g \in G \mid (g + A) \cap B \text{ is not a zero set}\}.$$

Then $F(A, B)$ is an open subset of G (possibly empty).

PROOF. If $g \in F(A,B)$ and $C = (g + A) \cap B$, then we have $g + F(C,C) \subseteq F(A,B)$. It is therefore enough to consider the case $A = B$ and to show that if A is universally measurable and nonzero, then $F(A,A)$ is a neighbourhood of zero.

Suppose this is not the case. Then we may choose a sequence g_n in G not belonging to $F(A,A)$ and such that $d(x,x+g_n) \leqq 1/2^n$ where x is any sum of different g_i $i = 1, n-1$ (d is a complete metric on G compatible with the topology).

We put

$$A' = A \setminus (\textstyle\bigcup_n (g_n + A) \cap A).$$

Because we have only removed countable many zero sets, A' is not a Haar zero set.

Let $K = \{0,1\}^N$ be the Cantor group. With the usual product topology and group structure, K is a compact metrizable abelian group. We define the mapping $\theta: K \to G$ by

$$\theta(x) = \sum_{n=1}^{\infty} x(n) g_n.$$

Because A' is not a zero set, there exists $g \in G$ such that $\theta^{-1}(g + A')$ has nonzero Haar measure in K.

Hence the set $\theta^{-1}(g + A) - \theta^{-1}(g + A) = U$ is a neighbourhood in K (the subtraction is performed with respect to the group structure of K). Let $e_v = (0, 0, \ldots, 1, 0, 0, \ldots)$ (1 is in the vth place). Because U is a neighbourhood, $e_v \in U$ for a suitable v. But this implies $g_v \in (A' - A')$. Hence $(g_v + A') \cap A' \neq \emptyset$. Since this contradicts the definition of A', the proof of Theorem 2 is finished.

It is easy to prove that every analytic hyperplane in a separable Fréchet space is closed. It does not seem to be known whether or not every universally measurable hyperplane is closed. Of course, it follows from Theorem 2 that a universally measurable hyperplane is a Haar zero set. Therefore the problem is the measure-theoretic analogue to the (also open) problem of whether or not every first category hyperplane is closed. The problems seems to be of the same degree of difficulty. The proof of the following theorem is an adaptation of a similar reasoning shown to the author for the category case by W. Roelcke (oral communication). Of course it is not at all trivial that there exist nonuniversally measurable hyperplanes.

THEOREM 3. *Let E be a separable Fréchet space and let a_i, $i \in I$, be an algebraic basis of E and b_i the coefficient functionals. Then the hyperplanes $b_i^{-1}(0)$ are universally measurable for at most finitely many $i \in I$.*

PROOF. Suppose i_n is a sequence of different elements of I such that each $b_{i_n}^{-1}(0)$ is universally measurable. We put $L_n = \bigcap_{\nu \geq n} b_{i\nu}^{-1}(0)$. Each L_n is a universally measurable (proper) linear subspace and therefore a Haar zero set. Clearly the union of the L_n's is the whole of E. Since E is not a zero set, we have proved the theorem.

Let G be an abelian Polish group. Consider the space $L^\infty(G)$ of bounded universally measurable functions where two functions are identified if they coincide on the complement of an Haar zero set. With the norm

$$\|f\|_\infty = \text{ess.sup}\,\{|f(x)|\},$$

$L^\infty(G)$ is a C^*-algebra. We now have

THEOREM 4. *$L^\infty(G)$ is a W^*-algebra (norm isomorphic with a von Neumann algebra) if and only if G is locally compact.*

PROOF. Only the "only if" part need be proved. Suppose $L^\infty(G)$ is a W^*-algebra. Let u be a positive normal functional on $L^\infty(G)$ (we can suppose $u(1) = 1$). From u we get a Borel probability measure v such that $v(A) = 0$ for every Haar zero set A. Because G is Polish, there is a compact set with $v(K) \neq 0$. Then $(K - K) \supseteq F(K, K)$ is a neighbourhood and Theorem 4 is proved.

Consider the Banach algebra A of bounded complex measures on G (with convolution as multiplication and total variation as norm). We call an ideal $I \subseteq A$ an H-ideal if I is norm closed and if for every bounded universally measurable function f and every $u \in I$, also $f \cdot u \in I$. If G is locally compact, it is very easy to prove that $L_1(G)$ is an H-ideal contained in every nontrivial H-ideal. Therefore in the general case, we define $L_1(G)$ to be the intersection of all nontrivial H-ideals.

THEOREM 5. *Let G be an abelian Polish group. $L_1(G) \neq \{0\}$ if and only if the group G is locally compact.*

PROOF. Suppose $L_1(G) \neq \{0\}$. Then we may choose a probability measure $u \in L_1(G)$. Also we may choose a compact set K with $u(K) > 0$. Suppose K is a zero set. Then $L_K = \{v \in A \mid \chi_K * |v| = 0\}$ is a nontrivial H-ideal. Therefore $L_1(G) \subseteq L_K$ and $u \in L_1(G)$ but $u \notin L_K$. This contradiction shows that K is nonzero. But if K is nonzero, then $(K - K) \supseteq F(K, K)$ is a neighbourhood according to Theorem 2. This concludes the proof.

One may conclude that there is no L_1 theory in the nonlocally compact case. But perhaps it is better to consider the filtering family of H-ideals I_A where A is a universally measurable zero set as a substitute for L_1.

As an application of the preceding results, it can easily be shown that universally measurable homomorphisms between Polish groups are continuous. This is already known (see [1]).

The next result seems to banish any hope for a reasonable analogue of the Fubini theorem. However a very weak version of the Fubini theorem is valid as we shall point out in the sequel.

THEOREM 6. *Let H be a separable infinite dimensional Hilbert space and let T be the unit circle in the complex plane. There exists in the product group $H \times T$ a Borel measurable set A such that*

i) *For every $h \in H$, the section $A(h) = \{t \in T \mid (h, t) \in A\}$ has Haar measure one in T.*

ii) *For every $t \in T$, the section $A(t)$ is a Haar zero set in H.*

iii) *The complement of A is a Haar zero set in the product group $H \times T$ (which of course is Polish).*

PROOF. Let H be the space of measurable functions on T which are square integrable with respect to the Haar measure (functions equal almost everywhere are identified). H is equipped with the usual Hilbert space structure. Let A be the the set of all (f, t) such that the Fourier series associated with f is convergent at the point t. From the Carleson theorem (the Fourier series for any L_2 function is almost everywhere convergent), it follows that i) is fulfilled. The section $A(t)$ is a Borel measurable linear subspace; if it were not a Haar zero set, it would be all of H (use Theorem 2). But is it well known that for each particular $t \in T$ there exists a $f \in H$ whose Fourier series is not convergent at the point t. This shows that ii) is satisfied. Let u be the probability measure on $H \times T$ defined as the product of the one point measure in H with mass 1 at zero and the Haar measure in T. From the Carleson theorem and the very definition of zero sets, it follows that iii) is fulfilled.

If H were an arbitrary abelian Polish group and T a locally compact abelian Polish group, it could easily be shown (with a slight modification of the above reasoning) that, for a universally measurable set $A \subseteq H \times T$, the following conditions are equivalent

i) The section $A(h)$ is a zero set for the Haar measure in T for almost every $h \in H$.

ii) The set A is a Haar zero set in the product group.

Hence we have a Fubini theorem in one direction if one of the groups is locally compact.

Some problems regarding the preceding results remain open:

Problem 1. If the sets A and B in Theorem 2 are not Haar zero sets, then is $F(A,B) \neq \emptyset$? This seems to be obvious only if $A = B$.

Problem 2. Is any family of mutually disjoint universally measurable nonzero sets at most countable?

Problem 3. (Very important for applications.) Does any bijective in (both directions) Lipschitz mapping between separable Banach spaces preserve the zero sets?

Problem 4. If F is any ultrafilter on the Polish group G (abelian), does there exist a filter set $A \in F$ such that $(A - A)$ is not dense in G?

The last problem may be unrelated to the preceding results but we have good reason to believe that it is fundamental to the theory of abelian topological groups.

Reference

1. J. P. R. Christensen, *Borel structures in groups and semigroups*, Math. Scand. **28** (1971), 124–128.

University of Copenhagen