ISRAEL JOURNAL OF MATHEMATICS, Vol. 38, No. 4, 1981

ON THE CONVEX APPROXIMATION PROPERTY AND THE ASYMPTOTIC BEHAVIOR OF NONLINEAR CONTRACTIONS IN BANACH SPACES

BY

RONALD E. BRUCK[†]

ABSTRACT

We prove that if C is a bounded closed convex subset of a uniformly convex Banach space, $T:C\to C$ is a nonlinear contraction, and $S_n = (I+T+\cdots+T^{n-1})/n$, then $\lim_n \|S_n(x)-TS_n(x)\|=0$ uniformly in x in C. T also satisfies an inequality analogous to Zarantonello's Hilbert space inequality, which permits the study of the structure of the weak ω-limit set of an orbit. These results are valid for B-convex spaces if some additional condition is imposed on the mapping.

Introduction

Throughout this paper E denotes a real or complex Banach space and C is a nonempty bounded closed convex subset of E. A (nonlinear) contraction on C is a mapping $T:C\to E$ for which $\|Tx-Ty\|\leqq\|x-y\|$ for all x, y in C. We denote the set of all contractions $T:C\to C$, i.e. contractive self-mappings of C, by $\mathrm{Cont}(C)$, and the set of fixed-points of T by $F(T)$. The weak ω-limit set $\omega_w(x)$ of x in C is the set of weak subsequential limits of $\{T^n(x)\}$; since we do not assume C is weakly compact, $F(T)$ and $\omega_w(x)$ may very well be empty.

In this paper we introduce the convex approximation property on E, prove that it is equivalent to the B-convexity of E, and use it to study the asymptotic behavior of T (especially of the Cèsaro means $S_n=(I+T+\cdots+T^{n-1})/n$) and the structure of $\omega_w(x)$. The results may be regarded as the extension to Banach spaces of results of [5], and are related to [6].

We do not directly consider the mean ergodic theorem because our results are obtained under more stringent assumptions (in one sense) than needed for the

[†]Partially supported by NSF Grant MCS–7802305A01.
Received December 27, 1979 and in revised form August 10, 1980

MET. This narrowing (to B-convex spaces) is justified by three facts: first, since we do not assume C is weakly compact, our results may apply to mappings which do not have a fixed point. Second, we show that

$$\lim_n \| S_n(x) - TS_n(x) \| = 0,$$

even if T does not have a fixed-point; heretofore, this has been known only in Hilbert space. Third, we give an analogue of Zarantonello's inequality [12, p. 248]

$$\left\| T\left(\sum t_i x_i \right) - \sum t_i T x_i \right\|^2 \leqq \sum_{i<j} t_i t_j [\| x_i - x_j \|^2 - \| T x_i - T x_j \|^2].$$

Zarantonello's inequality is valid in Hilbert space for *any* n and $t \in \Delta^{n-1}$, $x_1, \cdots, x_n \in C$; it was rediscovered by Baillon [1] who used it as the basis of the first proof of the mean ergodic theorem in Hilbert space.

While we state our results for discrete semigroups $I, T, T^2, \cdots$, they are equally valid for continuous contraction semigroups $S(t)$. The analogues may be proved by step-by-step imitation, or in a more principled fashion by a discretization device of Reich [10].

In the first version of this paper we proved that superreflexive spaces have the convex approximation property. We have adopted J. Baillon's elegant proof (devised with the aid of W. B. Johnson and G. Pisier) that this property is actually equivalent to B-convexity. (W. Davis has independently, and simultaneously, communicated to us another proof of this equivalence.)

Notation. The convex hull of a set M is denoted by $\operatorname{co} M$, the closed convex hull by $\operatorname{clco} M$. We put

$$\Delta^{n-1} = \{\lambda = (\lambda_1, \cdots, \lambda_n) \text{: each } \lambda_i \geqq 0 \text{ and } \Sigma \lambda_i = 1\}.$$

The open ball of radius r centered at 0 is denoted by B_r. Weak convergence is denoted by $\rightharpoonup$.

1. The convex approximation property and means

We say that E has the convex approximation property (C.A.P.) if for each $\varepsilon > 0$ there exists a positive integer p such that for every $M \subset B_1$,

$$(1.1) \qquad \operatorname{co} M \subset \operatorname{co}_p M + B_\varepsilon,$$

where $\operatorname{co}_p M$ denotes the set of sums $\lambda_1 x_1 + \cdots + \lambda_p x_p$ with $\lambda \in \Delta^{p-1}$, $x_1, \cdots, x_p \in M$ variable but p fixed; or in other words, such that each convex combination of elements of M can be approximated by a convex combination of no more than p elements of M.

The condition $M \subset B_1$ is a scaling normalization and we shall frequently use (1.1) for sets M whose diameters are uniformly bounded.

THEOREM 1.1. *E has the C.A.P. iff E is B-convex.*

PROOF. First, it is well known that E is B-convex iff l_1 is not finitely representable in E. Another characterization, due to Pisier [9] (see also [3, lemma 3]), is that E is B-convex iff there exist constants $c > 0$ and $q > 1$ such that for all independent random variables $X_1, X_2, \cdots, X_n$ with values in E,

$$E\left(\left\|\sum_{i=1}^{n} X_i\right\|^q\right) \leqq c^q \sum_{i=1}^{n} E(\|X_i\|^q).$$

Now suppose E is B-convex. If $\varepsilon > 0$ we can choose p so that

$$2cp^{1/q} < \varepsilon. \tag{1.2}$$

Let $x \in \operatorname{co} M$; then we can find an M-valued random variable X which is centered at x. Let $X_1, X_2, \cdots, X_p$ be independent M-valued random variables centered at x. Since $\|X_i(\omega) - x\| \leqq 2$ for any ω, by virtue of (1.2) we have

$$\left\|\frac{1}{p}\sum_{i=1}^{p}(X_i - x)\right\|_{L^q(E)} < 2cp^{1/q} \leqq \varepsilon. \tag{1.3}$$

Hence there exists ω such that

$$\left\|\frac{1}{p}\sum_{i=1}^{p} X_i(\omega) - x\right\| \leqq \varepsilon. \tag{1.4}$$

This shows $x \in \operatorname{co}_p(M) + B_\varepsilon$.

Conversely, if l_1 *is* finitely representable in E then for any n there exist unit vectors $x_1, x_2, \cdots, x_n$ of E such that

$$\frac{1}{2}\sum_{i=1}^{n} |a_i| \leqq \left\|\sum_{i=1}^{n} a_i x_i\right\| \qquad \text{for all } a_1, \cdots, a_n \in \mathbf{R}.$$

Thus if $y^* = (x_1 + x_2 \cdots + x_n)/n$ we have

$$\operatorname{dis}(y^*, \operatorname{co}_p(\{x_i\})) \geqq \tfrac{1}{2}(n-p)/n. \tag{1.5}$$

No matter what value of p we choose, we can therefore find a set M in the unit ball and a point y^* in $\operatorname{co} M$ such that $\operatorname{dis}(y^*, \operatorname{co}_p M) > \frac{1}{4}$, so that E cannot have the C.A.P. Q.E.D.

As in [6] we denote by Γ the set of continuous, strictly increasing, convex functions $\gamma : R \to R$ with $\gamma(0) = 0$. Recall that $T : C \to E$ is said to be of type (γ) if $\gamma \in \Gamma$ and

$$\gamma(\| cTx + (1-c)Ty - T(cx + (1-c)y)\|) \leqq \| x - y \| - \| Tx - Ty \|$$

for all x, y in C and c in $[0,1]$. If E is uniformly convex then every contraction $T : C \to E$ is of type (γ); moreover, γ can be chosen to depend only on $\operatorname{diam} C$ and not on T.

For $\varepsilon > 0$ we let $F_\varepsilon(T)$ be the set of ε-approximate fixed points of T, i.e., we put $F_\varepsilon(T) = \{x \in C : \| x - Tx \| \leqq \varepsilon\}$. We have:

THEOREM 1.2. *Suppose E is B-convex and $\gamma \in \Gamma$. Then for each $\varepsilon > 0$ there exists $\delta > 0$, depending only on ε, γ, and* $\operatorname{diam} C$, *such that for each contraction $T : C \to E$ of type (γ),*

$$\operatorname{clco} F_\delta(T) \subset F_\varepsilon(T). \tag{1.6}$$

There is an interesting historical note to Theorem 1.2. In [4] F. Browder proved that if E is a uniformly convex Banach space and $T : C \to E$ is a contraction then $I - T$ is demiclosed, i.e. if $\lim_n (x_n - Tx_n) = 0$ and $w - \lim_n x_n = x$ then $x \in F(T)$. The crux of his proof was that for each $\varepsilon > 0$ there exists $\delta > 0$ with $\operatorname{co}_2 F_\delta(T) \subset F_\varepsilon(T)$; by a clever passage to a subsequence of $\{x_n\}$ this is sufficient to prove demiclosedness.

PROOF OF THEOREM 1.2. It was shown in [6, lemma 1.2] that the inverse function σ of $t \to \gamma^{-1}(2t) + t$ satisfies

$$\operatorname{co}_2 F_{\sigma(t)}(T) \subset F_t(T).$$

Hence by induction

$$\operatorname{co}_p F_{\sigma^p(t)}(T) \subset F_t(T). \tag{1.7}$$

By Theorem 1.1 E has the C.A.P., and since C is bounded, given $\varepsilon > 0$ we can choose p so

$$\operatorname{co} M \subset \operatorname{co}_p M + B_{\varepsilon/3}$$

for all $M \subset C$. From (1.7) we get

$$\operatorname{co} F_\delta(T) \subset F_{\varepsilon/3}(T) + B_{\varepsilon/3} \tag{1.8}$$

with $\delta = \sigma^p(\varepsilon/3)$. But

$$F_{\varepsilon/3}(T) + B_{\varepsilon/3} \subset F_\varepsilon(T) \tag{1.9}$$

because

$$\|x - Tx\| \leqq \|x - y\| + \|y - Ty\| + \|Ty - Tx\|$$
$$\leqq 2\|x - y\| + \|y - Ty\|;$$

finally, putting (1.8) and (1.9) together and noting that $F_\varepsilon(T)$ is closed we get (1.6). Q.E.D.

Since C is bounded it is a standard result that $F_\varepsilon(T) \neq \emptyset$ for every $T \in \mathrm{Cont}(C)$. If in addition C is weakly compact and T is of type (γ) then in fact $F(T) \neq \emptyset$ (see the remark on p. 110 of [6]), but it is not clear T must have a fixed-point if C is not weakly compact.

THEOREM 1.3. *Suppose E is B-convex and $\gamma \in \Gamma$. Then for each $\eta > 0$ there exist $\delta > 0$ and $N > 0$, depending only on n, γ, and* $\operatorname{diam} C$, *such that for every contraction $T: C \to E$ of type (γ) and each sequence $\{x_n\}$ in C satisfying $\|x_{n+1} - Tx_n\| \leqq \delta$ for all n, there holds*

$$\frac{1}{n}\sum_{i=1}^{n} x_i \in F_\eta(T) \qquad \textit{for all } n \geqq N. \tag{1.10}$$

For an application where it is important that $\{x_n\}$ is not an actual orbit, see [7].

PROOF. The proof is essentially the same as theorem 1.1 of [6]. Since the details are rather complicated we sketch only the differences.

First, choose $\varepsilon > 0$ using Theorem 1.2 so

$$\operatorname{clo} F_\varepsilon(T) \subset F_{\eta/3}(T);$$

thus

$$\operatorname{clco} F_\varepsilon(T) + B_{\eta/3} \subset F_\eta(T).$$

Next, choose p in Z so $\operatorname{diam} C \leqq p\varepsilon^2/2$. Next, put $q_n(t) = \gamma^{-1}(\operatorname{diam} C/n + t) + t$, $q(t) = \gamma^{-1}(t) + t$ and choose $0 < \delta < \eta/3$ so small that

$$q^{p-1}(\delta) < \varepsilon^2/2.$$

Finally, choose N so large that

$$q_n^{p-1}(\delta) < \varepsilon^2/2 \qquad \text{for all } n \geqq N.$$

As promised, N and δ depend only on γ, η, and $\operatorname{diam} C$.

Put $w_i = 1/p\,\Sigma_{j=0}^{p-1} x_{j+i}$. Paralleling the proof of lemma 1.5 of [6] we find

$$\frac{1}{n}\sum_{i=0}^{n-1}\|w_{i+1} - Tw_i\| \leqq q_n^{p-1}(\delta)$$

provided $\|x_{i+1} - Tx_i\| \leqq \delta$ for all i. Following the rest of the proof in [6] with $\operatorname{clco} F_\varepsilon(T)$ playing the role of W' there, we end with (1.10). Q.E.D.

One interpretation of Theorem 1.3 is that Cèsaro averaging is stable with respect to rounding (or other) error in the evaluation of T. Probably the most important consequence, however, is the following, previously known only in Hilbert space (cf. Reich [11]):

Corollary 1.1. *Suppose E is B-convex and $\gamma \in \Gamma$. Then*

$$\lim_n \|S_n(y) - TS_n(y)\| = 0$$

uniformly in y in C and in T in $\operatorname{Cont}(C)$ *of type* (γ), *where* $S_n = (I + T + \cdots + T^{n-1})/n$.

The proof is such a trivial application of Theorem 1.3 that we omit it. Equally trivial is:

Corollary 1.2. *If E is B-convex, $T: C \to E$ is of type (γ), and $\{x_n\} \subset C$ satisfies* $\lim_n \|x_{n+1} - Tx_n\| = 0$, *then*

$$\lim_n \left\| T\left(\frac{1}{n}\sum_{i=1}^n x_{i+k}\right) - \frac{1}{n}\sum_{i=1}^n x_{i+k} \right\| = 0$$

uniformly in $k > 0$.

It is even possible to reach the conclusion of Corollary 1.2 from the assumption that $\|x_{n+1} - Tx_n\|$ is only $(C, 1)$ convergent to 0.

Corollary 1.3. *Suppose E is B-convex and $T, T_0 \in \operatorname{Cont}(C)$ are both of type (γ) and commute. Then for each $\varepsilon > 0$, $F_\varepsilon(T) \cap F_\varepsilon(T_0) \neq \varnothing$.*

Proof. Given $\varepsilon > 0$ choose $\delta > 0$ so $\operatorname{clco} F_\delta(T_0) \subset F_\varepsilon(T_0)$. Since T commutes with T_0, $F_\delta(T_0)$ is seen to be T-invariant, and we have already remarked it is nonempty. Thus for any x in $F_\delta(T_0)$,

$$S_n(x) = (x + Tx + \cdots + T^{n-1}x)/n \in \operatorname{clco} F_\delta(T_0) \subset F_\varepsilon(T_0).$$

But by Corollary 1.1 we have $S_n(x) \in F_\varepsilon(T)$ for sufficiently large n. Q.E.D.

It would be interesting to know whether Corollary 1.3 is true without the hypotheses that E is B-convex and the mappings are type (γ).

2. The weak ω-limit set

We return to condition (γ).

LEMMA 2.1. *Suppose $\gamma \in \Gamma$. Then for each positive integer p there exists $\gamma_p \in \Gamma$ such that for any $T : C \to E$ of type (γ) and any $\lambda \in \Delta^{p-1}$ and $x_1, \cdots, x_p$ in C,*

$$\gamma_p(\| T(\Sigma \lambda_i x_i) - \Sigma \lambda_i T x_i \|) \leqq \max_{1 \leqq i,j \leqq p} (\| x_i - x_j \| - \| T x_i - T x_j \|). \tag{2.1}$$

PROOF. By induction on p. We begin by setting $\gamma_2 = \gamma$. Once γ_p has been defined we define γ_{p+1} to be any function in Γ satisfying

$$\gamma_{p+1}^{-1}(t) \geqq \gamma_2^{-1}(t) + \gamma_p^{-1}(t + 2_2^{-1}(t));$$

for example, if $*$ denotes the inf-convolution, I the identity function on R, and the functions in Γ are extended to be $+\infty$ on $(-\infty, 0)$, then we may take

$$\gamma_{p+1} = \tfrac{1}{2}(I * \gamma_2) \circ (\tfrac{1}{2}(\gamma \,{}^{*}_{2}\, \gamma_p)).$$

We must verify (2.1) for $p+1$. Let $T : C \to E$ be of type (γ) and fix $\lambda \in \Delta^p$ and $x_1, \cdots, x_{p+1} \in C$. The case $\lambda_{p+1} = 1$ is trivial (and omitted). For the rest of the proof we let subscript i range through $1 \leqq i \leqq p+1$, while j ranges through $1 \leqq i \leqq p$. We put

$$u_j = (1 - \lambda_{p+1}) x_j + \lambda_{p+1} x_{p+1},$$

$$x'_i = T x_i,$$

$$u'_j = (1 - \lambda_{p+1}) x'_j + \lambda_{p+1} x'_{p+1},$$

$$\mu_j = \lambda_j / (1 - \lambda_{p+1}),$$

so $\mu \in \Delta^{p-1}$. We lay out the computations (most of which are trivial) as follows:

$$\Sigma \lambda_i x_i = \Sigma \mu_j u_j, \qquad \Sigma \lambda_i x'_i = \Sigma \mu_j u'_j;$$

$$\begin{aligned} \| T(\Sigma \lambda_i x_i) - \Sigma \lambda_i x'_i \| &= \| T(\Sigma \mu_j u_j) - \Sigma \mu_j u'_j \| \\ &\leqq \| T(\Sigma \mu_j u_j) - \Sigma \mu_j T u_j \| + \Sigma \mu_j \| T u_j - u'_j \|; \end{aligned} \tag{2.2}$$

$$\gamma_p(\| T(\Sigma \mu_j u_j) - \Sigma \mu_j T u_j \|) \leqq \max_{i \leqq j,k \leqq p} (\| u_j - u_k \| - \| T u_j - T u_k \|); \tag{2.3}$$

$$\| u_j - u_k \| - \| T u_j - T u_k \| \leqq \| u_j - u_k \| - \| u'_j - u'_k \| + \| u'_k - T u_k \| + \| u'_j - T u_j \|; \tag{2.4}$$

$$\gamma_2(\| T u_j - u'_j \|) \leqq \| x_j - x_{p+1} \| - \| x'_j - x'_{p+1} \|; \tag{2.5}$$

$$\begin{aligned} \| u_j - u_k \| - \| u'_j - u'_k \| &= (1 - \lambda_{p+1})(\| x_j - x_k \| - \| x'_j - x'_k \|) \\ &\leqq \| x_j - x_k \| - \| x'_j - x'_k \|. \end{aligned} \tag{2.6}$$

Of these, (2.3) represents the induction hypothesis and (2.5) the case $p = 2$.

Put $t = \max\{\|x_i - x_k\| - \|x'_i - x'_k\| : 1 \leqq i, k \leqq p+1\}$. Then by (2.5)

$$\|Tu_j - u'_j\| \leqq \gamma_2^{-1}(t),$$

which combined with (2.6) and used in (2.4) yields

$$\|u_j - u_k\| - \|Tu_j - Tu_k\| \leqq t + 2\gamma_2^{-1}(t). \tag{2.7}$$

When used in (2.3) this yields

$$\|T(\Sigma\, \mu_j u_j) - \Sigma\, \mu_j Tu_j\| \leqq \gamma_p^{-1}(t + \gamma_2^{-1}(t)). \tag{2.8}$$

Finally, when (2.7) and (2.8) are used with (2.2) we get

$$\|T(\Sigma\, \lambda_i x_i) - \Sigma\, \lambda_i Tx_i\| \leqq \gamma_2^{-1}(t + 2\gamma_p^{-1}(t + 2\gamma_2^{-1}(t))).$$

By the definition of γ_{p+1}, therefore

$$\gamma_{p+1}(\|T(\Sigma\, \lambda_i x_i) - \Sigma\, \lambda_i Tx_i\|) \leqq t = \max_{1 \leqq i,k \leqq p+1} (\|x_i - x_j\| - \|Tx_i - Tx_j\|).$$

Q.E.D.

It would seem from the lemma that as p increases, γ_p decreases (quite rapidly, in fact) and the estimates are useless for convex combinations of infinite point sets. This is not the case in B-convex spaces, however.

THEOREM 2.1. *Suppose E is B-convex and $\gamma_2 \in \Gamma$. Then there exists $\gamma \in \Gamma$, depending only on γ_2 and* diam C, *such that*

$$\gamma(\|T(\Sigma\, \lambda_i x_i) - \Sigma\, \lambda_i Tx_i\|) \leqq \max_{1 \leqq i,j \leqq n} (\|x_i - x_j\| - \|Tx_i - Tx_j\|) \tag{2.9}$$

for any contraction $T : C \to E$ of type (γ_2), and $\lambda \in \Delta^{n-1}$, any $x_1, \cdots, x_n \in C$, and, most importantly, for any n.

(2.9) is the analogue of Zarantonello's inequality we mentioned in the introduction.

PROOF. First, determine $\gamma_p \in \Gamma$ for $p = 2, 3, \cdots$ from Lemma 2.1. The product of B-convex spaces being B-convex (Giesy [8]), $E \times E$ has the C.A.P.; hence, given $\varepsilon > 0$ we can determine p so that

$$\operatorname{co} M \subset \operatorname{co}_p M + B_{\varepsilon/3} \times B_{\varepsilon/3}$$

for every $M \subset C \times C$. Finally, put $\delta = \gamma_p(\varepsilon/3)$.

Suppose $x_1, \cdots, x_n \in C$ satisfy

$$\|x_i - x_j\| - \|Tx_i - Tx_j\| \leqq \delta \qquad \text{for all } i, j.$$

Consider $M = \{[x_i, Tx_i] \in C \times C : i = 1, 2, \cdots, n\}$. Thus for each $\lambda \in \Delta^{n-1}$ there exist $\mu \in \Delta^{p-1}$ and $i_1, \cdots, i_p \in \{1, 2, \cdots, n\}$ such that

$$\|\Sigma \lambda_i x_i - \Sigma \mu_j x_{i_j}\| < \varepsilon/3,$$

$$\|\Sigma \lambda_i Tx_i - \Sigma \mu_j Tx_{i_j}\| < \varepsilon/3.$$

In other words, the C.A.P. on $E \times E$ guarantees simultaneous approximability in E. Now

$$\begin{aligned}\|T(\Sigma \lambda_i x_i) - \Sigma \lambda_i Tx_i\| &\leqq \|T(\Sigma \lambda_i x_i) - T(\Sigma \mu_j x_{i_j})\| + \|T(\Sigma \mu_j x_{i_j}) - \Sigma \mu_j Tx_{i_j}\| \\ &\quad + \|\Sigma \mu_j Tx_{i_j} - \Sigma \lambda_i Tx_i\| \\ &\leqq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.\end{aligned}$$

Thus whenever $\|x_i - x_j\| - \|Tx_i - Tx_j\| \leqq \delta$ for all i, j we have

$$\|T(\Sigma \lambda_i x_i) - \Sigma \lambda_i Tx_i\| \leqq \varepsilon. \tag{2.10}$$

The construction of $\gamma \in \Gamma$ such that $\gamma(\varepsilon) \leqq \delta$ for this $\varepsilon - \delta$ prescription is an elementary exercise. Q.E.D.

We recall two definitions from [5]: $T \in \mathrm{Cont}(C)$ is said to be asymptotically isometric on a subset S of C provided for all x, y in S the $\lim_n \|T^n x - T^{n+i} y\|$ exists uniformly in i; and T is said to be ε-approximately affine on a convex subset K of C provided (2.10) holds for all n, all $\lambda \in \Delta^{n-1}$, and all $x_1, \cdots, x_n$ in K.

THEOREM 2.2. *Suppose E is B-convex and $T \in \mathrm{Cont}(C)$ is of type (γ). If T is asymptotically isometric on a subset S of C, then T maps $\omega_w(x)$ into itself for each x in S and T maps* clco $\bigcup\{\omega_w(x) : x \in S\}$ *into itself affinely. If, in addition, C is weakly compact, "into" can be replaced by "onto" in these statements.*

PROOF. As in [5], but using Theorem 2.1 instead of Zarantonello's inequality, we see that for each $\varepsilon > 0$ and $x_1, \cdots, x_n$ in S there exists N such that T is ε-approximately affine on clco $\bigcup\{T^k x_i : 1 \leqq i \leqq m;\ k \geqq N\}$. It also follows, but not quite by the proof in [5], that if as $j \to \infty$ we have $n(i) \to \infty$ and $T^{n(i)}x \rightharpoonup y$, then $T^{1+n(i)}x \rightharpoonup Ty$. (Fix g in E^* and use the approximate affineness of T to show that all subsequential limits of $\{(T^{1+n(i)}x - Ty, q)\}$ are 0.) This implies that T maps $\omega_w(x)$ into itself.

Moreover, since we clearly have

$$\mathrm{co} \bigcup_{i=1}^{n} \omega_w(x_i) \subset \mathrm{clco} \bigcup_{i=1}^{n} \{T^k x_i : k \geqq N\}$$

for any N, it follows that T is ε-approximately affine on $\mathrm{co}\bigcup_{i=1}^{n}\omega_w(x_i)$ for any $\varepsilon > 0$ — and hence affine there for any choice of $x_1, \cdots, x_n$ in S. Thus T is affine on

$$K_0 := \mathrm{co}\bigcup\{\omega_w(x) : x \in S\},$$

and since T maps $\omega_w(x)$ into itself, $T(K_0) \subset K_0$.

It only remains to show that if C is weakly compact then T maps $\omega_w(x)$ onto itself and $\mathrm{cl}\, K_0$ onto itself. But if $z \in \omega_w(x)$ we can choose $n(i) \to \infty$ so $T^{n(i)}x \rightharpoonup z$, and (by weak compactness) a subsequence (which we again denote by $n(i)$) such that $\{T^{n(i)-1}x\}$ converges weakly to some y in C. By our earlier remark, we have $z = Ty$, so that $\omega_w(x) \subset T(\omega_w(x))$ as claimed. It also follows that $T(K_0) = K_0$; by continuity T is also affine on $\mathrm{cl}\, K_0$, and therefore continuous in the weak topology on $\mathrm{cl}\, K_0$. Since $\mathrm{cl}\, K_0$ is weakly compact, so is $T(\mathrm{cl}\, K_0)$, but as $K_0 = T(K_0)$ is dense in $\mathrm{cl}\, K_0$ this implies $T(\mathrm{cl}\, K_0) = \mathrm{cl}\, K_0$. Q.E.D.

In Hilbert space it is known that T is isometric on $\mathrm{cl}\, K$, but we do not know whether under the present circumstances T is isometric on even $\omega_w(x)$. It is also known that an odd mapping is asymptotically isometric in Hilbert space, but this too is unknown in general spaces.

Finally, to reconcile the abstract with the paper, we remark that every uniformly convex Banach space is B-convex, and every nonexpansive mapping in such a space is of type (γ), so that all the results of this paper apply to uniformly convex spaces.

References

1. J. B. Baillon, *Un théorème de type ergodique pour les contractions nonlinéaires dans un espace de Hilbert*, C. R. Acad. Sci. Paris **280** (1975), 1511–1514.
2. J. B. Baillon, Thèse, Université de Paris VI, 1978.
3. B. Beauzamy et P. Enflo, *Théorèmes de point fixe et d'approximation*, preprint.
4. F. Browder, *Nonlinear Operators and Nonlinear Equations of Evolution in Banach Spaces*, Proc. Symposia Pure Math. XVIII, pt. 2, Amer. Math. Soc., Providence, R.I., 1976.
5. R. E. Bruck, *On the almost-convergence of iterates of a nonexpansive mapping in Hilbert space and the structure of the weak ω-limit set*, Israel J. Math. **29** (1978), 1–16.
6. R. E. Bruck, *A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces*, Israel J. Math. **32** (1979), 279–282.
7. R. E. Bruck and G. B. Passty, *Almost convergence of the infinite product of resolvents in Banach spaces*, Nonlinear Analysis **3** (1979), 279–282.
8. D. P. Giesy, *On a convexity condition in normed linear spaces*, Trans. Amer. Math. Soc. **125** (1966), 114–146.
9. G. Pisier, *Sur les espaces de Banch qui ne contiennent pas uniformement de l_n^1*, C.R. Acad. Sci. Paris **277** (1973), A991–A994.
10. S. Reich, *Nonlinear evolution equations and nonlinear ergodic theorems*, Nonlinear Analysis **1** (1977), 319–330.

11. S. Reich, *Almost convergence and nonlinear ergodic theorems*, J. Approximation Theory **24** (1978), 269–272.

12. E. H. Zarantonello, *Protections on convex sets in Hilbert space and spectral theory*, in *Contributions to Nonlinear Functional Analysis*, Academic Press, New York, 1971, pp. 237–244.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SOUTHERN CALIFORNIA
LOS ANGELES, CA 90007 USA