THE EXTENSION PROPERTY FOR COMPACT CONVEX SETS

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ABSTRACT

A closed convex subset Q of a compact convex set K is said to have the extension property if every continuous affine function on Q can be extended to a continuous affine function on K. It is proved that the extension property is equivalent to the existence of a number N such that is any direction in which Q has positive width, the ratio of the width of K to the width of Q is less than N.

K will always denote a non-empty, compact, convex set in a real locally convex topological vector space E. $A(K)$ will be the Banach space of continuous affine functions on K with sup norm, and Q will be a closed convex subset of K. We will denote by $\| \cdot \|_Q$ and $\| \cdot \|_K$ the norms in $A(Q)$ and $A(K)$. As a subset of K, Q is said to have the *extension property* if for every $f \in A(Q)$ there is $\bar{f} \in A(K)$ such that $\bar{f}|_Q = f$. Q is said to have the *bounded extension property* if there is a number N such that for every $f \in A(Q)$ there is $\bar{f} \in A(K)$ with $\bar{f}|_Q = f$ and $\|\bar{f}\|_K \leqq N \|f\|_Q$. A simple application of the open mapping theorem for Banach spaces (for example see Alfsen [2, theorem II.5.9]) shows that the bounded extension property is equivalent to the extension property. Alfsen [1, prop. 10] has given an example of a compact convex set K with a closed face which does not have the extension property. In Theorem 1 we give a simple geometric condition equivalent to the bounded extension property.

If $d \in E$, $d \neq 0$ we define the *d-width of* K to be

$$|K|_d = \sup\{t : \exists x \in K \text{ and } x + td \in K\}.$$

THEOREM 1. *If Q is a non-empty closed convex subset of K then the following are equivalent.*

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1) *Q has the bounded extension property.*

2) *there exists N such that for every $d \in E$ with $|Q|_d \neq 0$ we have $|K|_d \leqq N|Q|_d$.*

We remark that the condition $|Q|_d \neq 0$ is an essential part of 2). For example, if Q contains only a single point, then certainly Q has the extension property, but for any d with $|K|_d > 0$ we have $|K|_d > N|Q|_d = 0$ for every N.

We prove first a few lemmas. If $g \in A(K)$ then the *K-variation* of g is

$$V_K(g) = \sup\{g(x) - g(y) : x, y \in K\}.$$

Since members of E^* are by restriction in $A(K)$, this definition will apply in particular to them.

LEMMA 1. *Suppose $x, y \in K$, $f \in A(K)$ and $d = x - y \neq 0$. Then*

$$|K|_d |f(x) - f(y)| \leqq V_K(f).$$

PROOF. Since $x, y \in K$, $|K|_d \neq 0$. Choose $\lambda > 0$ so $|K|_d > \lambda$. Choose $z \in K$ so $z + \lambda d \in K$. Applying the affine function f to both sides of the equality

$$((z + \lambda d) + \lambda y)/(1 + \lambda) = (z + \lambda x)/(1 + \lambda)$$

we obtain

$$f(z + \lambda d)/(1 + \lambda) + \lambda f(y)/(1 + \lambda) = f(z)/(1 + \lambda) + \lambda f(x)/(1 + \lambda)$$

from which $f(z + \lambda d) - f(z) = \lambda(f(x) - f(y))$. So

$$V_K(f) \geqq \lambda |f(x) - f(y)|.$$

LEMMA 2. *Suppose $0 \in K$ and let L be the convex symmetric hull of K. Then for $d \in E$, $d \neq 0$*

$$|L|_d \leqq 2|K|_d.$$

PROOF. If $|L|_d = 0$ there is nothing to prove. Otherwise suppose $|L|_d > \lambda > 0$. Choose $z \in L$ so that $z + \lambda d \in L$. Since L is the convex hull of K and $-K$ we can find $u, v, x, y \in K$ so that $z = \alpha u + (1 - \alpha)(-v)$ and $z + \lambda d = \beta x + (1 - \beta)(-y)$ with $0 \leqq \alpha, \beta \leqq 1$. Subtracting,

$$\lambda d = (z + \lambda d) - z = [\beta x + (1 - \alpha)v] - [\alpha u + (1 - \beta)y]$$

since each square-bracketed term is in $2K$ (recall $0 \in K$), we deduce $|2K|_d \geqq \lambda$. Hence

$$2|K|_d \geqq |2K|_d \geqq \lambda,$$

the first inequality following from the observation $x \in 2K \Rightarrow x/2 \in K$. This completes the proof of the lemma.

We remark that the proofs we have given of these two lemmas use no topology and that the lemmas are in fact true for general convex sets, with f in Lemma 1 any affine function.

LEMMA 3. *Suppose* $|K|_d > 0$ *for some* $d \neq 0$ *in* E. *Then* $\exists g \in E^*$ *such that* $V_K(g) \leqq 4|K|_d$ *and* $g(d) = 1$.

PROOF. Let $|K|_d = \lambda > 0$. We may suppose $0 \in K$ (translate K if necessary). Let L be the convex symmetric hull of K. Then L is closed and $|L|_d < 2\lambda$ by Lemma 2. So $2\lambda d \notin L$ (since L symmetric) and by a standard separation theorem [3, 14.4], we can choose $g \in E^*$ such that $g(L) \leqq 2\lambda$ and $g(2\lambda d) = 2\lambda$. Hence $g(d) = 1$ and

$$V_K(g) \leqq V_L(g) \leqq 4\lambda \qquad \text{(by symmetry of } L\text{)}.$$

PROOF OF THEOREM 1.

1) $\Rightarrow$ 2). Suppose Q has the bounded extension property. Choose N so every $g \in A(Q)$ has an extension $f \in A(K)$ with $\|f\|_K \leqq N\|g\|_Q$. Now suppose $|Q|_d \neq 0$ for some $d \in E$. We will show $|K|_d \leqq 8N|Q|_d$. By multiplying d by a constant we may suppose $|Q|_d > 1$. Then we can choose $x, y \in Q$ with $d = x - y$. Use Lemma 3 to choose $g \in E^*$ with $V_Q(g) \leqq 4|Q|_d$ and $g(d) = 1$.

Then $h = g - g(x)$ is a member of $A(Q)$ and since h is zero at $x \in Q$, $\|h\|_Q \leqq V_Q(h)$. Using 1), choose an extension $f \in A(K)$ of h with $\|f\|_K \leqq N\|h\|_Q$. Then $f(x) - f(y) = h(x) - h(y) = g(d) = 1$ and so by Lemma 1

$$|K|_d \leqq V_K(f) \leqq 2\|f\|_K \leqq 2N\|h\|_Q \leqq 2NV_Q(h) = 2NV_Q(g) \leqq 8N|Q|_d.$$

2) $\Rightarrow$ 1). Suppose N is such that for every $d \in E$ with $|Q|_d \neq 0$ we have $|K|_d \leqq N|Q|_d$. We will show that the restriction map from $A(K)$ to $A(Q)$ has the property that the closure of the image of the unit ball of $A(K)$ contains the δ-ball of $A(Q)$ where $\delta = 1/(8N+1)$. It will follow from [4, 7G lemma 1] that the image of the unit ball of $A(K)$ contains the δ-ball of $A(Q)$, hence that Q has the bounded extension property with constant $8N + 1$.

So suppose $f \in A(Q)$ and $\|f\|_Q < \delta$. Choose any $\varepsilon > 0$. We will find $h \in A(K)$, $\|h\|_K \leqq 1$ and $\|h|_Q - f\|_Q \leqq \varepsilon$. Let

$$K_1 = \{(k, 1) \in E \times R : k \in K\},$$

and

$$K_2 = \{(k, -1) \in E \times R : k \in K\}.$$

Let L_1 be the convex hull of K_1 and the graph of $f + \varepsilon$.
Let L_2 be the convex hull of K_2 and the graph of $f - \varepsilon$.
We will show that $L_1 \cap L_2 = \emptyset$.

Since the graphs of $f + \varepsilon$ and $f - \varepsilon$ are convex, it will be enough to show that if q_1 and q_2 are in Q then the convex hull of $(q_1, f(q_1) + \varepsilon)$ and K_1 is disjoint from the convex hull of $(q_2, f(q_2) - \varepsilon)$ and K_2. If $q_1 = q_2$ this is obvious; otherwise let $d = q_2 - q_1$. Since $|K|_d \neq 0$ apply Lemma 3 to find $g \in E^*$ such that

$$V_K(g) \leqq 4|K|_d \text{ and } g(d) = 1.$$

If $k \in K$ let $F(k) = f(q_1) + (g(k) - g(q_1))(f(q_2) - f(q_1))$. Then F is affine on K and

$$\begin{aligned} |F(k)| &\leqq |f(q_1)| + |g(k) - g(q_1)| \cdot |f(q_2) - f(q_1)| \\ &\leqq \delta + V_K(g) \cdot V_Q(f)/|Q|_d \qquad \text{(by Lemma 1)} \\ &\leqq \delta + 4|K|_d \cdot 2\|f\|_Q/|Q|_d \\ &< \delta + 8N\delta = \delta(1 + 8N) = 1. \qquad (\text{since } \|f\|_Q < \delta) \end{aligned}$$

Also $F(q_1) = f(q_1)$ and $F(q_2) = f(q_2)$ since $g(d) = 1$. So K_1 and $(q_1, f(q_1) + \varepsilon)$ lie strictly above graph(F) and K_2 and $(q_2, f(q_2) - \varepsilon)$ lie strictly below graph(F). Since F is affine, graph(F) strictly separates the convex hulls, which must then be disjoint.

So L_1 and L_2 are disjoint convex sets in $E \times R$. They are also compact since K_1, K_2, graph($f + \varepsilon$) and graph($f - \varepsilon$) are all compact. So L_1 and L_2 can be separated by a closed hyperplane H [3, 14.4]. If H' is the translate of H which passes through the origin, then H' is the graph of a linear functional on E. Since H' is closed this functional has closed nullspace $H' \cap E$, and so is continuous [3, 5.4]. Hence H is the graph of a continuous affine function on E and so the set of points

$$\{(k, \gamma) \in H : k \in K\}$$

is the graph of a continuous affine function h on K. Since H separates K_1 and K_2, $|h(k)| \leqq 1$ for $k \in K$. Since H separates graph $(f + \varepsilon)$ and graph $(f - \varepsilon)$, $\| h|_Q - f \|_Q \leqq \varepsilon$.

Remarks. 1) The possibility of characterizing the extension property with a notion of "relative width" seems to have been first considered by Alfsen. In [1, theorem 5] he showed that for an Archimedean face of K the extension property and the bounded extension property are both equivalent to a condition of bounded "relative width" somewhat different from ours. Indeed it was this theorem together with his example [1, prop. 10] which motivated our Theorem 1.

2) I am grateful to David Gregory and George Elliott for several enlightening discussions about the extension property. In particular, Elliott pointed out that the use of [4, 7G, Lemma 1] greatly simplifies my original proof of 2) $\Rightarrow$ 1).

3) I am grateful to the referee for providing the neat proof of Lemma 1.

4) L. Asimow has pointed out to me that in [5, theorem 3.1] he has a condition for the extension property easily seen to be equivalent to condition 2) of Theorem 1.

References

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