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RATIONAL ERGODICITY, BOUNDED RATIONAL ERGODICITY AND SOME CONTINUOUS MEASURES ON THE CIRCLE

BY

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To the memory of Shlomo Horowitz

ABSTRACT

Two ratio limit concepts for transformations preserving infinite measures, rational ergodicity and bounded rational ergodicity, are discussed and compared. The concept of rational ergodicity is used to construct some continuous measures on the circle, which show that the exceptional set in the weak mixing theorem may be rather large.

§0. Introduction

We study two ratio limit properties of conservative ergodic, measure preserving transformations (c.e.m.p.t.s) of infinite measure spaces. The weaker property is rational ergodicity. Let $(X, \mathcal{B}, \mu, T)$ be a c.e.m.p.t. Recall from [1] that T is said to be *rationally ergodic* (r.e.) if there is a set $A \in \mathcal{B}$, $0 < \mu(A) < \infty$ such that

$$\sup_{n \geq 1} \int_A \left(\sum_{k=0}^{n-1} 1_A \circ T^k / a_n(A) \right)^2 d\mu < \infty \tag{0.1}$$

where here and throughout $a_n(A) = \sum_{k=0}^{n-1} \mu(A \cap T^{-k}A)$. The collection of sets satisfying (0.1) is denoted by $B(T)$. It was shown in [1] that if T is r.e. then $\exists a_n(T) \uparrow \infty$ such that

$$\frac{1}{a_n(T)} \sum_{k=0}^{n-1} \mu(B \cap T^{-k}C) \to \mu(B)\mu(C) \qquad \text{as } n \to \infty \text{ if } B, C \in \mathcal{B}, \tag{0.2}$$

$$B \cup C \in B(T)$$

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and that

$$\varliminf_{n\to\infty} \frac{1}{a_n(T)} \sum_{k=0}^{n-1} \mu(B \cap T^{-k}C) \geqq \mu(B)\mu(C), \qquad \forall B, C \in \mathcal{B}. \tag{0.3}$$

The sequence $\{a_n(T)\}_n$ is clearly defined uniquely up to asymptotic equality, and is known as a *return sequence for T*. The collection of all sequences asymptotically proportional to $a_n(T)$ (i.e. $a_n/a_n(T) \to c \in (0,\infty)$) is known as the *asymptotic type of T* and denoted by $\mathcal{A}(T)$.

It was also shown in [1] that ergodicity is not sufficient for rational ergodicity.

In §1, we show that dyadic towers over the adding machine are rationally ergodic.

It turns out that the proof of this result actually establishes the stronger ratio limit property, *bounded rational ergodicity*, which we study in §2.

By a result in [5], bounded rational ergodicity characterises the occurrence of *mixed ratio limit theorems*, which were introduced in [5, §4] for Markov operators. A consideration of transformations admitting recurrent events (shown to be rationally ergodic in [1]) yields that some are boundedly rationally ergodic, and that some are not. Thus rational ergodicity is not sufficient for bounded rational ergodicity.

In the last section, we use the result of §1 to construct some continuous measures on the circle, which show that the exceptional set in the weak mixing theorem may be rather large.

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§1. Dyadic towers over the adding machine

We first recall the well known definition and basic properties of the (dyadic) adding machine.

Let $\Omega = \{0,1\}^N$ and let

$$l(x) = \inf\{n \geqq 1 : \varepsilon_n(x) = 0\} \quad (\leqq \infty)$$

where $x = (\varepsilon_1(x), \varepsilon_2(x), \cdots) = (1, 1, \cdots, 1, 0, \varepsilon_{l(x)+1}(x), \cdots) \in \Omega$.

The (dyadic) *adding machine* $\tau : \Omega \to \Omega$ is defined by

$$\tau x = (0, \cdots, 0, 1, \varepsilon_{l(x)+1}(x), \cdots).$$

The transformation τ is called the "adding machine" because

$$\sum_{k=1}^{n} 2^{k-1}\varepsilon_k(\tau x)=\sum_{k=1}^{n} 2^{k-1}\varepsilon_k(x)+1 \bmod 2^n, \qquad \forall n \geqq 1, \quad x\in\Omega \tag{1.1}$$

and this has the consequence

$$\{(\varepsilon_1(\tau^k x),\cdots,\varepsilon_n(\tau^k x)):0\leqq k<2^n\}=\{0,1\}^n, \qquad \forall n\geqq 1,\quad x\in\Omega. \tag{1.2}$$

Now let $\mathscr{A}$ be the σ-algebra of subsets of Ω generated by cylinders, and let P be the product measure $(\frac{1}{2},\frac{1}{2})^N$ defined on $(\Omega,\mathscr{A})$. It follows from (1.2) that if $f:\Omega\to\mathbf{R}$ is a function depending on only finitely many coordinates,

$$\frac{1}{n}\sum_{k=0}^{n-1} f(\tau^k x)\to\int_\Omega f dP. \tag{1.3}$$

In particular, $(\Omega,\mathscr{A},P,\tau)$ is an e.m.p.t.

Now let $\{\gamma(n)\}_{n=1}^{\infty}\subseteq \mathbf{N}$. We define the *dyadic height function with heights* $\{\gamma(n)\}$ by

$$\phi(x)=\gamma(l(x))$$

and the *dyadic tower over the adding machine with* (*dyadic*) *height function* $\phi(x)$ as follows:

$$X=\{(x,n):\phi(x)\geqq n\geqq 1\},\qquad \mathscr{B}=\bigvee_{n=1}^{\infty}(\mathscr{A}\cap[\phi\geqq n],n),$$

$$\mu=\sum_{n=1}^{\infty} P|_{(\mathscr{A}\cap[\phi\geqq n],n)},$$

$$T(x,n)=\begin{cases}(x,n+1) & \text{if } (x,n+1)\in X,\\ (\tau x,1) & \text{else.}\end{cases}$$

It follows ([8]) that $(X,\mathscr{B},\mu,T)$ is a c.e.m.p.t. and that $\mu(X)=\int_\Omega\phi dP=\sum_{n=1}^{\infty}\gamma(n)/2^n$.

The purpose of this section is to show that $(X,\mathscr{B},\mu,T)$ is rationally ergodic.

First, we introduce some more notation which will help identify the asymptotic type of T.

Let $\Gamma(n)=\sum_{k=1}^{n}2^{n-k}\gamma(k)$ and $\beta(n)=\Gamma(n)+\gamma(n+1)$ for $n\geqq 1$ and $\beta(0)=\gamma(1)$. It follows that

(1.4) $$\gamma(n)=\beta(n-1)-\sum_{k=0}^{n-2}\beta(k) \qquad \text{for } n\geqq 2 \text{ (and } \gamma(1)=\beta(0)).$$

We will refer to $\{\beta(n)\}$ as the *growth sequence of the dyadic height function* $\gamma(l(x))$. It is clear that if $\{\beta(n)\}\subseteq N$ is the growth sequence of some dyadic height function, then

(1.5) $$\beta(n)>\sum_{k=0}^{n-1}\beta(k) \qquad \text{for } n\geqq 1.$$

Conversely, any sequence $\{\beta(n)\}_{n=0}^{\infty}$ satisfying (1.5) is the growth sequence of the dyadic height function with heights defined by (1.4), so we term such a sequence a *growth sequence.* Note that any growth sequence must satisfy $\beta(n)\geqq 2^n$.

The reason for the name "growth sequence" is

LEMMA 1. *Let $\phi:\Omega\to\mathbf{N}$ be the dyadic height function with heights $\{\gamma(n)\}$ and growth sequence $\{\beta(n)\}$, then*

(1.6) $$\phi_{2^n}(x)=\Gamma(n)+\gamma(n+l(\sigma^n x)) \quad \text{where } \sigma(\varepsilon_1\cdots)=(\varepsilon_2\cdots)\quad \forall x\in\Omega,$$

(1.7) $$\phi_{2^n}(x)\geqq\beta(n-1),$$

(1.8) $$\mathrm{P}(\phi_{2^n}=\beta(n))\geqq\tfrac{1}{2},$$

where $\phi_n(x)=\sum_{k=0}^{n-1}\phi(\tau^k x)$.

PROOF. By (1.2), $\forall x\in\Omega$, $n\geqq 1$, $\exists!\, k=k_n(x)<2^n$ such that $\varepsilon_1(\tau^{k_n}x)=\cdots=\varepsilon_n(\tau^{k_n}x)=1$ and $\varepsilon_m(\tau^{k_n}x)=\varepsilon_m(x)$, $\forall m\geqq n+1$. Hence $l(\tau^{k_n}x)=n+l(\sigma^n x)$. Now, also by (1.2)

$$\begin{aligned}\phi_{2^n}(x)&=\sum_{k=1}^{n}\gamma(k)\times\#\{\varepsilon\in\{0,1\}^n:\varepsilon_j=1 \\ &\qquad\qquad \text{for } j<k \text{ and } \varepsilon_k=0\}+\phi(\tau^{k_n}x)\\ &=\sum_{k=1}^{n}\gamma(k)2^{n-k}+\gamma(l(\tau^{k_n}x))\\ &=\Gamma(n)+\gamma(n+l(\sigma^n x)).\end{aligned}$$

This is (1.6). It implies (1.7) as $\Gamma(n)\geqq\beta(n-1)$; and (1.8) as $\mathrm{P}(\phi_{2^n}=\beta(n))\geqq \mathrm{P}(l\circ\sigma^n=1)=\tfrac{1}{2}$. □

THEOREM 1. *Let* $(X, \mathcal{B}, \mu, T)$ *be the dyadic tower over the adding machine with height function* $\phi(x)$ *with growth sequence* $\{\beta(n)\}$. *Let* $c(n) = \inf\{k \geqq 1 : \beta(k) \geqq n\}$. *Then* T *is rationally ergodic and* $a_n(T) \overset{\cup}{\cap} 2^{c(n)}$ (*i.e.* $\underline{\lim}\, a_n(T)/2^{c(n)} > 0$ *and* $\overline{\lim}\, a_n(T)/2^{c(n)} < \infty$).

PROOF. We first show that $\bar{\Omega} = (\Omega, 1) \in B(T)$.
Note that

$$\sum_{k=1}^{\phi_n(x)} 1_{\bar{\Omega}} \circ T^k(x, 1) = n, \qquad \forall x \in \Omega, \quad n \geqq 1.$$

Consequently, if $x \in \Omega$ then

$$\begin{aligned}\sum_{k=1}^{n} 1_{\bar{\Omega}} \circ T^k(x,1) &\leqq \sum_{k=1}^{\beta(c(n))} 1_{\bar{\Omega}} \circ T^k(x,1) \quad \text{since } \beta(c(n)) \geqq n \\ &< \sum_{k=1}^{\phi_{2^{c(n)+1}}(x)} 1_{\bar{\Omega}} \circ T^k(x,1) \quad \text{by (1.7)} \\ &= 2^{c(n)+1}.\end{aligned}$$

This implies

$$\sum_{k=1}^{n} 1_{\bar{\Omega}} \circ T^k \leqq 2^{c(n)+1} \qquad \text{on } X. \tag{1.9}$$

It will now follow from

$$\underline{\lim}_{n\to\infty}\, a_n(\bar{\Omega})/2^{c(n)} \geqq 1/4 \tag{1.10}$$

that $\bar{\Omega} \in B(T)$, since if this is true, then

$$\int_{\bar{\Omega}} \left(\sum_{k=1}^{n} 1_{\bar{\Omega}} \circ T^k \right)^2 d\mu \leqq 4(2^{c(n)})^2 \leqq 100 a_n(\Omega)^2 \qquad \text{for } n \text{ large.}$$

We now establish (1.10).

By the mean ergodic theorem, if $\mu(X) < \infty$ then T is rationally ergodic. We therefore consider the case

$$\mu(X) = \sum_{n=1}^{\infty} \gamma(n)/2^n = \infty.$$

It is not hard to see that

$$a_n(\bar{\Omega}) = \sum_{k=1}^{n} \mu(\bar{\Omega} \cap T^{-k}\bar{\Omega}) = \sum_{k=1}^{n} \mathrm{P}(\phi_k \leqq n)$$

and hence that, if $l = [\log_2 n]$, then

$$a_n(\bar{\Omega}) \geqq \sum_{k=1}^{2^l} \mathrm{P}(\phi_k \leqq n) \geqq \frac{1}{2} \sum_{k=1}^{l} \mathrm{P}(\phi_{2^k} \leqq n) 2^k.$$

Now, since we assume $\mu(X) = \infty$, we have that

$$\beta([\log_2 n]) \geqq \Gamma([\log_2 n]) > n \qquad \text{for } n \text{ large enough.}$$

Thus we obtain, for n large enough,

$$\begin{aligned}
\sum_{k=1}^{l} \mathrm{P}(\phi_{2^k} \leqq n) 2^k &\geqq \sum_{\substack{k=1 \\ \beta(k)<n}}^{l} \mathrm{P}(\phi_{2^k} = \beta(k)) 2^k \\
&= \sum_{k=1}^{c(n)-1} \mathrm{P}(\phi_{2^k} = \beta(k)) 2^k \\
&\geqq \frac{1}{2} \sum_{k=1}^{c(n)-1} 2^k \qquad \text{by (1.8)} \\
&= \tfrac{1}{2}(2^{c(n)} - 2).
\end{aligned}$$

This establishes (1.10), and the rational ergodicity of T. We also have since $\Omega \in \bar{B}(T)$, $a_n(T) \sim a_n(\Omega)$, and it follows from (1.9) and (1.10) that $a_n(\Omega) \overset{\cup}{\underset{\cap}{}} 2^{c(n)}$.

□

Corollary. *Let $(X, \mathcal{B}, \mu, T)$ be a dyadic tower over the adding machine. Then any measurable, μ-non-singular transformation of $(X, \mathcal{B}, \mu)$ which commutes with T preserves μ.*

Proof. The result follows immediately from the rational ergodicity of T.

This corollary was established, using different methods, in [7] for the dyadic tower over the adding machine with heights $\gamma(n) = (1 + 2^{2n-1})/3$.

§2. Bounded rational ergodicity

Let $(X, \mathcal{B}, \mu, T)$ be a c.e.m.p.t. We will say that T is *boundedly rationally ergodic* (b.r.e.) if $\exists A \in \mathcal{B}$, $0 < \mu(A) < \infty$ such that

$$\sup_{n \geqq 1} \operatorname*{ess\text{-}sup}_{x \in X} \left| \frac{1}{a_n(A)} \sum_{k=0}^{n-1} 1_A(T^k x) \right| < \infty. \tag{2.1}$$

We denote the collection of sets satisfying (2.1) by $S(T)$. It is clear that

$S(T) \subseteq B(T)$, and it follows from theorem 4.1 of [5] that $A \in S(T)$ iff T satisfies a *mixed ratio limit theorem on* A, i.e.

$$\frac{1}{a_n(A)} \sum_{k=0}^{n-1} \mathrm{P}(T^{-k}B) \to \frac{\mu(B)}{\mu(A)^2}, \tag{2.2}$$
$$\forall B \in \mathcal{B} \cap A; \qquad \mathrm{P} \ll \mu, \quad \mathrm{P}(X) = 1.$$

It follows immediately from (1.9) and (1.10) that *if T is a dyadic tower over the adding machine, then* $\bar{\Omega} \in S(T)$.

The rest of this section is devoted to identifying some other b.r.e.m.p.t.s, and to showing that rational ergodicity does not imply bounded rational ergodicity.

LEMMA 2. *Let $(X, \mathcal{B}, \mu, T)$ be a c.e.m.p.t., then T is b.r.e. iff T is r.e. and* $\exists \alpha = \alpha(T) \in [1, \infty)$ *such that*

$$\overline{\lim_{n\to\infty}} \frac{1}{a_n(T)} \sum_{k=0}^{n-1} f \circ T^k = \alpha \int_X f d\mu \quad \text{a.e.}, \tag{2.3}$$
$$\forall f \in L^1(X), \quad f \geqq 0.$$

PROOF. First, suppose that T is r.e., and let, for $f \in L^1$, $f \geqq 0$

$$\alpha(f, x) = \overline{\lim_{n\to\infty}} \frac{1}{a_n(T)} \sum_{k=0}^{n-1} f(T^k x).$$

Clearly $\alpha(f, Tx) \geqq \alpha(f, x)$ and hence $\alpha(f, x) = \alpha(f)$ for μ-a.e. x. Furthermore, the Hopf ergodic theorem yields that

$$\alpha(f) \Big/ \int_X f d\mu = \alpha(g) \Big/ \int_X g d\mu, \qquad \forall f, g \in L^1(X), \quad f, g \geqq 0.$$

Hence, $\exists \alpha \in [0, \infty]$ satisfying (2.3). Now, if $\alpha < 1$ then we can choose $A \in B(T)$ such that $\mu(A) > 0$, $\alpha' \in (\alpha, 1)$, $n_0 \in \mathbf{N}$, and $B \in \mathcal{B} \cap A$ such that

$$\frac{1}{a_n(T)} \sum_{k=0}^{n-1} 1_A(T^k x) \leqq \alpha' \mu(A), \qquad \forall n \geqq n_0, \quad x \in B.$$

Integrating this inequality on B violates (0.2) and therefore the assumption that $A \in B(T)$. Hence $\alpha \geqq 1$.

We have shown that if T is r.e. then (2.3) is satisfied with $1 \leqq \alpha \leqq \infty$.

Suppose that $\alpha < \infty$ and choose $A \in B(T)$. We can find $B \subseteq A$, $\mu(B) > 0$ and $M < \infty$ such that

$$\frac{1}{a_n(T)}\sum_{k=0}^{n-1} 1_A(T^k x) \leqq M, \qquad \forall n \geqq 1, \quad x \in B$$

whence

$$\frac{1}{a_n(T)}\sum_{k=0}^{n-1} 1_B(T^k x) \leqq M, \qquad \forall n \geqq 1, \quad x \in B$$

and hence $\forall x \in X$. Thus $B \in S(T)$, since $a_n(B) \sim \mu(B)^2 a_n(T)$.

Conversely, suppose that T is b.r.e., then T is r.e., and if $A \in S(T)$ then $a_n(A) \sim \mu(A)^2 a_n(T)$ and hence

$$\alpha\mu(A) = \overline{\lim_{n\to\infty}}\frac{1}{a_n(T)}\sum_{k=0}^{n-1} 1_A \circ T^k < \infty \quad \text{a.e.}$$

by (2.1). □

Note that (1.9) and (1.10) show that $\alpha(T) \leqq 8$ for T a dyadic tower over the adding machine.

We now turn our attention to c.e.m.p.t.s admitting recurrent events, whose definition we now recall from [1].

Let $(X, \mathcal{B}, \mu, T)$ be a c.e.m.p.t. A set $A \in \mathcal{B}$, $0 < \mu(A) < \infty$ is called a *recurrent event* for T if, for $0 \leqq n_1 \leqq \cdots \leqq n_k$,

$$\mu(A \cap T^{-n_1}A \cap \cdots \cap T^{-n_k}A) = \mu(A)\prod_{j=2}^{k}\mu(A \cap T^{-(n_j - n_{j-1})}A)/\mu(A).$$

The collection of recurrent events for T is denoted by $M(T)$, and T is said to *admit recurrent events* if $M(T) \neq \varnothing$ (all Markov shifts admit recurrent events).

It was shown in [1] that transformations admitting recurrent events are rationally ergodic and that $M(T) \subseteq B(T)$.

As the next results show, the bounded rational ergodicity of a transformation admitting recurrent events is dependent on its asymptotic type. This is in contrast to the situation with dyadic towers over the adding machine.

Theorem 2. *Let $(X, \mathcal{B}, \mu, T)$ be a c.e.m.p.t. admitting recurrent events, and assume that $a_n(T) \sim n/L(n)$, $L(n) = \exp(\int_0^n \varepsilon(t)dt)$ and $t \operatorname{Log}\operatorname{Log} t \cdot \varepsilon(t) \to 0$ then T is b.r.e. and $\alpha(T) \leqq e$.*

Proof. Let $A \in M(T)$ and let, for $x \in A$,

$$\phi(x) = \inf\{n \geqq 1 : T^n x \in A\},$$

$T_A x = T^{\phi(x)}x$ — the induced transformation on A ([8]) and $\phi_n(x) = \sum_{k=0}^{n-1}\phi(T_A^k x)$; then

$$1_A(T^n x) = \sum_{k=1}^{n} 1_{[\phi_k = n]}(x), \qquad \forall n \geqq 1, \quad x \in A.$$

Now, it is not hard to show that, since $A \in M(T)$: $\{\phi \circ T_A^n\}_{n=0}^{\infty}$ are i.i.d.r.v.s defined on the probability space $(A, B \cap A, \mu_A)$ where $\mu_A(B) = \mu(A)^{-1}\mu(A \cap B)$. Consequently, the proof of formula (i) in [3] applies and we have

$$\overline{\lim_{n\to\infty}} \frac{1}{a(n)} \sum_{k=0}^{n-1} 1_A \circ T^k \leqq e \quad \text{a.e. on } A$$

$$\text{where } a(n) = \sum_{k=0}^{n-1} \mu_A(T^{-k}A).$$

Since the above $\overline{\lim}$ is constant a.e. on X, and $a(n) \sim \mu(A)a_n(T)$, we have shown that $\alpha(T) \leqq e$. Bounded rational ergodicity follows from Lemma 2. □

THEOREM 3. *Let $(X, \mathcal{B}, \mu, T)$ be a c.e.m.p.t. admitting recurrent events, and assume that $a_n(T) = n^{\delta}L(n)$ where $0 \leqq \delta < 1$ and $L(n)$ is slowly varying as $n \to \infty$; then T is not b.r.e., moreover, $\forall n_k \to \infty$*

$$\overline{\lim_{k\to\infty}} \frac{1}{a_{n_k}(T)} \sum_{j=0}^{n_k-1} f \circ T^j = \infty \quad \text{a.e.,}$$

(2.4)

$$\forall f \in L^1(X), \quad f \geqq 0, \quad \int f > 0.$$

PROOF. Let $A \in M(T)$, $u_n = \mu(A \cap T^{-n}A)/\mu(A)$, $a(n) = \sum_{k=1}^{n} u_k$ and $\psi_n = \sum_{k=1}^{n} 1_A \circ T^k$. We will establish that

$$\overline{\lim_{k\to\infty}} \frac{1}{a(n_k)} \psi_{n_k} = \infty \quad \text{a.e.,} \qquad \forall n_k \to \infty \tag{2.5}$$

which, by the Hopf ergodic theorem, implies (2.4) (and hence the result) since $a(n) \sim \mu(A)a_n(T)$.

If (2.5) is not satisfied for $n_k \to \infty$, then, owing to the T-super-invariance of the $\overline{\lim}$ in (2.5), $\exists M < \infty$ such that

$$\overline{\lim_{k\to\infty}} \frac{1}{a(n_k)} \psi_{n_k} < M \quad \text{a.e.}$$

Now let, for $x \in A$,

$$\varphi(x) = \inf\{n \geqq 1 : T^n x \in A\},$$

$$T_A x = T^{\varphi(x)}x \quad \text{and} \quad \varphi_n(x) = \sum_{k=0}^{n-1} \varphi(T_A^k x).$$

Then, $\psi_{\varphi_n(x)}(x) \equiv n$ and $\varphi_m \leqq n \Rightarrow \psi_n \geqq m$. Moreover, since $A \in M(T)$, $\{\varphi \circ T_A^n\}_{n=0}^{\infty}$ are independent identically distributed random variables on the probability space $(A, \mathcal{B} \cap A, \mu_A)$, and

$$\sum_{k=0}^{n} u_{n-k} \cdot \mu_A(\varphi > k) \equiv n \qquad (n \geqq 1).$$

It follows from Karanata's Tauberian theory (see [9] and [11]) that $\mu_A(\varphi \geqq n) \sim c_\alpha / a(n)$ as $n \to \infty$, where $0 < c_\alpha < \infty$ depends only on α. In this situation, it is known (see [11] pp. 448–449) that

$$\mu_A(\varphi_n \leqq xb(n)) \to \int_0^x f_\alpha(y)dy \qquad \text{as } n \to \infty \quad \text{for all } x \geqq 0,$$

where $b(a(n)) = n$ and $f(y) \geqq 0$ is characterised by its Laplace transform:

$$\int_0^\infty e^{-yx} f_\alpha(y)dy = \exp(-d_\alpha x^\alpha) \qquad \text{for } x \geqq 0.$$

We shall need the fact that $\int_0^\varepsilon f_\alpha(y)dy > 0$ for all $\varepsilon > 0$ (which follows from the form of its Laplace transform).

In the light of this (2.6) is impossible, as

$$\begin{aligned} \mu_A(\psi_n \geqq x/2a(n)) &\geqq \mu_A(\psi_n \geqq a(x^{1/\alpha}n)) \quad \text{for } n \text{ large} \\ &\geqq \mu_A(\varphi_{a(x^{1/\alpha}n)} \leqq n) \\ &= \mu_A(\varphi_{a(x^{1/\alpha}n)} \leqq x^{-1/\alpha} b(a(x^{1/\alpha}n))) \\ &\to \int_0^{x^{-1/\alpha}} f_\alpha(y)dy > 0 \qquad \text{for all } x > 0. \end{aligned}$$

□

§3. Rational ergodicity and continuous measures on the circle

Let p be a continuous probability measure on the circle $\Gamma = \{\lambda \in C : |\lambda| = 1\}$. It is well known that there is an "*exceptional set*" $K \subseteq \mathbf{N}$ of density zero (i.e. $|K \cap [1, n]|/n \to 0$) such that

$$\hat{p}(n) = \int_\Gamma \lambda^n dp(\lambda) \to 0 \qquad \text{as } n \to \infty, \quad n \notin K. \tag{3.1}$$

The purpose of this section is to show, using the result of §1, that the exceptional set in (3.1) may in general be arbitrarily "thick" within the limitation of having density zero. We prove

THEOREM 4. *Let $a(n) > 0$, $a(n)/n \to 0$. Then $\exists$ a continuous probability measure p on Γ, and $L \subseteq \mathbf{N}$ such that*

$$\hat{p}(n) \to 1 \quad \textit{as} \quad n \to \infty, \quad n \in L \quad \textit{and}$$

$$|L \cap [1, n]|/a(n) \to \infty \quad \textit{as} \quad n \to \infty.$$

Theorem 4 has the following

COROLLARY. *Let $a(n) > 0$, $a(n)/n \to 0$. Then $\exists$ a weakly mixing m.p.t. (Ω, a, P, θ) and $L \subseteq \mathbf{N}$ such that $f \circ \theta^n \to f$ in $L^2(\Omega)$ as $n \to \infty$, $n \in L$, $\forall f \in L^2(\Omega)$, and $|L \cap [1, n]|/a(n) \to \infty$.*

Since, if p is a symmetric continuous probability measure on Γ, and $L \subseteq \mathbf{N}$ satisfying the conclusion of Theorem 4 for $a(n)$ (no generality is lost in assuming p symmetric) and (Ω, a, P, θ) is the shift of the real Gaussian process $\{X_n\}$ with correlation function $E(X_m X_n) = \hat{p}(m - n)$, then by the theorems of Girsanov and Maruyama (see [10]), θ is weakly mixing and has the maximal spectral type

$$\pi = \sum_{n=0}^{\infty} p^{n*}/2^{n+1}.$$

It follows that $\hat{\pi}(n) \to 1$ as $n \to \infty$, $n \in L$, and hence that $\hat{\eta}(n) \to \hat{\eta}(0)$ as $n \to \infty$, $n \in L$, $\forall \eta \ll \pi$. But if $f \in L^2$ then

$$\int_\Omega f\bar{f} \circ \theta^n df = \hat{\eta}_f(n) \qquad \text{where } \eta_f \ll \pi$$

and so $f \circ \theta^n \to f$ in L^2 as $n \to \infty$, $n \in L$.

Continuous measures, and weakly mixing m.p.t.s satisfying the conclusions of Theorem 4 and its corollary for thin sequences L are mentioned in [6]. Here, we quantify those constructions, "thickening" L by means of

THEOREM 5. *Let $\{\beta(n)\}$ be a growth sequence* (in the sense of §1) *such that $\beta(n)/2^n \to \infty$, and let $c(n) = \inf\{k \geqq 1 : \beta(k) \geqq n\}$.*

If p is a probability measure on T such that

$$\sum_{n=1}^{\infty} |1-\lambda^{\beta(n)}| < \infty \qquad \textit{for } p\textit{-a.e.}, \quad \lambda \in \Gamma$$

then $\exists L \subseteq \mathbf{N}$ *such that* $\hat{p}(n) \to 1$ *as* $n \to \infty$, $n \in L$; *and* $|L \cap [1,n]|/2^{c(n)} \to \infty$ *as* $n \to \infty$.

REMARK. If $\{\beta(n)\}$ is a growth sequence, then $\beta(n) \geqq 2^n$ and either $\overline{\lim}\, \beta(n)/2^n < \infty$, in which case $2^{c(n)} \asymp n$, or $\beta(n)/2^n \to \infty$, in which case $2^{c(n)}/n \to 0$.

We will first prove Theorem 5, and then deduce Theorem 4 from it. The following lemma establishes the connection between Theorem 5 and dyadic towers over the adding machine.

LEMMA 3 (cf. [6]). *Let* $(X, \mathcal{B}, \mu, T)$ *be a dyadic tower over the adding machine with height function* ϕ *and let* $\{\beta(n)\}$ *be the growth sequence of* ϕ.

If $\lambda \in \Gamma$ *and* $\sum_{n=1}^{\infty} |1-\lambda^{\beta(n)}| < \infty$ *then* λ *is an eigenvalue of* T, *i.e.*

$$\exists g : X \to \mathbf{R} \textit{ such that } |g(x)| = 1, \quad g(Tx) = \lambda g(x) \quad \mu\textit{-a.e.} \tag{3.2}$$

PROOF. To establish (3.2), it is sufficient to find $f : \Omega \to \mathbf{R}$ such that

$$|f(x)| = 1 \quad \text{and} \quad f(\tau x) = \lambda^{\phi(x)} f(x) \quad P\text{-a.e.} \tag{3.3}$$

where $(\Omega, \mathcal{A}, P, \tau)$ is the adding machine, for then $g(x,n) = \lambda^n f(x)$ will satisfy (3.2).

Using the notations of §1, we let $A_n = [l \geqq n+1]$, the definition of τ, we have that, $\forall n \geqq 1$, $\Omega = \bigcup_{k=1}^{2^n} \tau^{-k} A_n$ (disjoint), $\tau^{-2^n} A_n = A_n$, $\tau^{-2^n} A_{n+1} = [l = n+1]$, and $A_n = A_{n+1} \cup \tau^{-2^n} A_{n+1}$.

Let $f_n = \sum_{k=1}^{2^n} \bar{\lambda}^{\phi_k(x)} 1_{A_n}(\tau^k x)$. It follows that $|f_n(x)| = 1$ and

$$|f_n(\tau x) - \lambda^{\phi(x)} f_n(x)| \leqq 1_{A_n}(\tau x) \to 0 \quad \text{a.e.}$$

Hence, if $f_n(x) \to f(x)$ a.e. then f satisfies (3.3).

Now

$$f_n = \sum_{k=1}^{2^n} \bar{\lambda}^{\phi_k} 1_{A_n} \circ \tau^k = \sum_{k=1}^{2^n} \bar{\lambda}^{\phi_k} (1_{A_{n+1}} \circ \tau^k + 1_{A_{n+1}} \circ \tau^{2^n + k}).$$

Hence

$$f_n - f_{n+1} = \sum_{k=1}^{2^n} \bar{\lambda}^{\phi_k} (1 - \bar{\lambda}^{\phi_{2^n}} \circ \tau^k) 1_{[l=n+1]} \circ \tau^k.$$

It follows that if $l = n + 1$ then $l \circ \sigma^n = 1$, and hence by (1.6), that $\phi_{2^n} = \beta(n)$. Thus $|f_n - f_{n+1}| \leqq |1 - \bar{\lambda}^{\beta(n)}| = |1 - \lambda^{\beta(n)}|$ so that, under the assumptions of the lemma, $f_n(x)$ converges a.e. □

LEMMA 4. *Let (G, d) be a separable, isometric group with identity 1 (i.e. $d(g, h) = d(gh^{-1}, 1)$, $\forall g, h \in G$), and let $(X, \mathcal{B}, \mu, T)$ be an r.e.m.p.t. preserving an infinite measure.*

If $g_0 \in G$ has the property that

(3.4) $\exists \pi : X \to G$ *measurable such that* $\pi(Tx) = g_0 \pi(x)$ μ*-a.e.*

then $\exists L \subseteq \mathbf{N}$ such that $g_0^n \to 1$ as $n \to \infty$, $n \in L$; and $|L \cap [1, n]|/a_n(T) \to \infty$ as $n \to \infty$.

PROOF. We denote, for $g \in G$ and $\varepsilon > 0$,

$$N(g, \varepsilon) = \{h \in G : d(g, h) < \varepsilon\} \quad \text{and} \quad A(g, \varepsilon) = \pi^{-1} N(g, \varepsilon).$$

STEP 1 (cf. [4]). *$\exists g \in G$ such that $\mu(A(g, \varepsilon)) > 0$, $\forall \varepsilon > 0$.*

PROOF. We show (as in [4]) that $\mu(A(\pi(x), \varepsilon)) > 0$ a.e., $\forall \varepsilon > 0$, which implies step 1. Let $\{g_n\}$ be dense in G, and let $\varepsilon > 0$. Then

$$G = \bigcup_n N(g_n, \varepsilon/2) \Rightarrow X = \bigcup_n A(g_n, \varepsilon/2)$$

$$\Rightarrow X = \bigcup_k A(g_{n_k}, \varepsilon/2) \quad \text{mod } \mu$$

where $\mu(A(g_{n_k}, \varepsilon/2)) > 0$, $\forall k$.

Now, $x \in A(g_{n_k}, \varepsilon/2) \Rightarrow d(g_{n_k}, \pi(x)) < \varepsilon/2 \Rightarrow A(g_{n_k}, \varepsilon/2) \subseteq A(\pi(x), \varepsilon)$. Hence $\mu(A(\pi(x), \varepsilon)) > 0$ a.e. □

We now fix $h_0 \in G$ with $\mu(A(h_0, \varepsilon)) > 0$, $\forall \varepsilon > 0$, and let

$$K(\varepsilon) = \{n \geqq 1 : d(g_0^n, 1) < \varepsilon\} \quad \text{and} \quad a_n(\varepsilon) = |K(\varepsilon) \cap [1, n]|.$$

STEP 2. $\mu(A(h_0, \varepsilon)) = \infty$, $\forall \varepsilon > 0$.

PROOF. Firstly, note that

(3.5) $$n \in K(\varepsilon),\ x \in A(h_0, \delta) \Rightarrow T^n x \in A(h_0, \varepsilon + \delta)$$

and that

$$A(h_0, \varepsilon) \cap T^{-n} A(h_0, \delta) \neq \varnothing \Rightarrow n \in K(\varepsilon + \delta). \tag{3.6}$$

If step 2 is wrong, then $\exists \varepsilon_0 > 0$ such that

$$0 < \mu(A(h_0, \varepsilon)) < \infty, \qquad \forall 0 < \varepsilon \leqq \varepsilon_0.$$

Choose $\varepsilon, \delta > 0$ such that $0 < \varepsilon - \delta < \varepsilon < \varepsilon + \delta < \varepsilon_0$. Then, for μ-a.e. $x \in A(h_0, \delta)$,

$$a_n(\varepsilon) = \sum_{k=1}^{n} 1_{K(\varepsilon)}(k) \leqq \sum_{k=1}^{n} 1_{A(h_0, \varepsilon+\delta)}(T^k x) \qquad \text{by (3.5)}$$

$$\sim M \sum_{k=1}^{n} 1_{A(h_0, \varepsilon-\delta)}(T^k x) \quad \text{as } n \to \infty$$

by the Hopf ergodic theorem

where $M = \mu(A(h_0, \varepsilon + \delta))/\mu(A(h_0, \varepsilon - \delta))$

$$\leqq M a_n(\varepsilon) \qquad \text{by (3.6).}$$

Since we assumed that $\mu(X) = \infty$, this violates theorem 2 of [2] thus establishing step 2. □

Now let $\varepsilon > 0$, and $A \subseteq A(h_0, \varepsilon/2)$, $\mu(A) = 1$. It follows from (3.6) that

$$a_n(\varepsilon) \geqq \sum_{k=1}^{n} \mu(A \cap T^{-k} A(h_0, \varepsilon/2))$$

and from step 2, rational ergodicity and (0.3), that

$$\frac{1}{a_n(T)} \sum_{k=1}^{n} \mu(A \cap T^{-k} A(h_0, \varepsilon/2)) \to \infty.$$

Thus $a_n(\varepsilon)/a_n(T) \to \infty$, $\forall \varepsilon > 0$.

Choose $\varepsilon_k \downarrow 0$, and $n_k \uparrow \infty$ such that $a_n(\varepsilon_k) \geqq k a_n(T)$, $\forall n \geqq n_k$, $k \geqq 1$.

Let $L = \bigcup_{k=1}^{\infty} K(\varepsilon_k) \cap [n_k, n_{k+1}]$. Then clearly $g_0^n \to 1$ as $n \to \infty$, $n \in L$, and for $n_k \leqq n < n_{k+1}$

$$|L \cap [1, n]| \geqq |K(\varepsilon_k) \cap [1, n]| = a_n(\varepsilon_k) \geqq k a_n(T). \qquad \square$$

Proof of Theorem 5. Let $(X, \mathscr{B}, \mu, T)$ be the dyadic tower over the adding machine with the height function $\phi(x)$, which has the growth sequence $\{\beta(n)\}$. By Lemma 3, the assumptions of the theorem mean that the measure p is supported on the collection of eigenvalues of T. It is well known that in this case

$$\exists \eta : X \times \Gamma \to \Gamma,\ \mu \times p \textit{ measurable such that}$$

$$\eta(Tx, \lambda) = \lambda \eta(x, \lambda),\ \mu \times p\text{-a.e.}$$

Now let G be the collection of all p-measurable functions $g : \Gamma \to \Gamma$, and let $d(g, h) = (\int_\Gamma |g - h|^2 dp)^{1/2}$, then (G, d) is a separable, isometric group and if

$$\pi(x)(\lambda) = \eta(x, \lambda) \qquad \text{and} \qquad g_0(\lambda) = \lambda$$

we have that

$$\pi : X \to G \text{ is measurable} \quad \text{and} \quad \pi(Tx) = g_0 \pi(x) \quad \mu\text{-a.e.}$$

By Theorem 1, T is rationally ergodic and $a_n(T) \overset{\cup}{\cap} 2^{c(n)}$, so by Lemma 4 we have that $\exists L \subseteq \mathbf{N}$ such that $|L \cap [1, n]|/2^{c(n)} \to \infty$, and $g_0^n \to 1$ as $n \to \infty$, $n \in L$. This proves Theorem 5, since

$$d(g_0^n, 1)^2 = \int |1 - \lambda^n|^2 dp(\lambda) = 2(1 - \operatorname{Re} \hat{p}(n))$$

$$\text{and} \quad |\hat{p}(n)| \leqq 1, \quad \forall n. \qquad \square$$

Proof of Theorem 4. Let $a(n) > 0$, $a(n)/n \to 0$ be given.

Let $q(n) = \inf\{[\log_2(k/a(k))] - 1 : k \geqq n\}$, then $q(n) \to \infty$, and

$$q([\log_2 n]) \leqq [\log_2(n/a(n))] - 1, \quad \forall n \geqq 1. \tag{3.7}$$

Choose $n_k \uparrow \infty$ such that $n_{k+1} > n_k + k$ and

$$q(n_k) \geqq k^2. \tag{3.8}$$

Let $W = \{e^{2\pi i \alpha} : \alpha = 0.\varepsilon_1 \varepsilon_2 \dots$ in binary expansion where $\varepsilon_n = 0$ unless $n = n_k + k$ (some k)$\}$, $L = \bigcup_{k=1}^{\infty} [n_k + 1, n_k + k]$ and $L^c = \{k(n)\}_{n=0}^{\infty}$ where $k(n+1) > k(n)$ and $\beta(n) = 2^{k(n)}$. It follows immediately that $\{\beta(n)\}$ is a growth sequence.

Let $c(n) = \inf\{k \geqq 1 : \beta(k) \geqq n\}$, then

$$c(n) \geqq \sup\{k \geqq 1 : \beta(k) \leqq n\} = [\log_2 n] - |L \cap [1, [\log_2 n]]|.$$

Now, if $n_k \leqq m < n_{k+1}$, we have from (3.8) that

$$|L \cap [1, m]| \leqq \sum_{j=1}^{k} j < k^2 \leqq q(n_k) \leqq q(m).$$

So

$$c(n) \geqq [\log_2 n] - q([\log_2 n])$$

and hence by (3.7)

$$2^{c(n)} \geqq a(n). \tag{3.9}$$

Now W is a Cantor set on Γ, and therefore $\exists$ a continuous probability measure $p \in \mathcal{P}(\Gamma)$ such that $p(W) = 1$. Theorem 4 will follow from (3.9) and Theorem 5 if we show that

$$\sum_{n=0}^{\infty} |1 - \lambda^{\beta(n)}| < \infty, \qquad \forall \lambda \in W. \tag{3.10}$$

Let $\lambda = e^{2\pi i\alpha} \in W$ where $\alpha \in [0,1]$. We have

$$\begin{aligned}\sum_n ((\beta(n)\alpha)) &= \sum_{n \notin L} ((2^n\alpha)) \\ &= \sum_{k=1}^{\infty} \sum_{j=n_k+k+1}^{n_{k+1}} ((2^j\alpha)).\end{aligned}$$

If $e^{2\pi i\alpha} \in W$ and $n_k + k + 1 \leqq j \leqq n_{k+1}$ then $((2^j\alpha)) < 1/2^{n_{k+1}+k-j}$. Thus

$$\sum_n ((\beta(n)\alpha)) \leqq \sum_{k=1}^{\infty} \sum_{j=n_k+k+1}^{n_{k+1}} 1/2^{n_{k+1}+k-j} \leqq 2.$$

Since $|1 - e^{2\pi i\alpha}| \sim 2((\alpha))$ as $((\alpha)) \downarrow 0$, this establishes (3.10), and Theorem 4.

□

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