Quantum Correlations in Two-Photon Amplification (*).

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Some time ago, Glauber (1) showed that if the Heisenberg equations of motion of a system of coupled oscillators assume a certain general form, then states of the system which are initially coherent remain coherent at all times. Examples of physical systems exhibiting this behaviour (2) include a single free oscillator, a forced oscillator, a damped oscillator, the phase-diffusion model of a laser beam and the parametric frequency converter. The quantum correlations present in these systems are easily described in terms of the Glauber coherent states and associated diagonal $P(\alpha)$ representation (3). Photon-counting experiments that have been performed (4) give results in agreement with predictions. We note that in all of the above examples only one-photon emission and absorption processes occur.

It is known that quantum parametric amplification systems exhibit quite different behaviour. The reason for this is that the Heisenberg equations of motion in this case express the time derivatives of operators $a_i(t)$ in terms of the adjoint operators $a_j^\dagger(t)$ as well as $a_j(t)$. For such cases Glauber's theorem (1) fails to apply. As a result alternative modes of description such as dynamic characteristics functions (5) or Wigner distribution functions (6) have been used.

The purpose of the present paper is to show that the recently proposed new coherent states (7) which are generalizations of Glauber coherent states allow a generalization of Glauber's theorem (1) in the following sense: the quantum correlations arising from processes in which two photons are simultaneously absorbed or emitted, of which parametric amplification and the two-photon amplifier are examples, are described by the new coherent states in much the same way that Glauber coherent states describe the correlations present in systems where only one photon emission and absorption processes occur. Quantum statistics of two-photon transitions have previously been discus-

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(1) R. J. Glauber: *Phys. Lett.*, **21**, 650 (1966).

(2) See, for example, W. H. Louisell: *Radiation and Noise in Quantum Electronics* (New York, 1964).

(3) R. J. Glauber: *Phys. Rev. Lett.*, **10**, 84 (1963); *Phys. Rev.*, **131**, 2766 (1963); E. C. G. Sudarshan: *Phys. Rev. Lett*, **10**, 277 (1963).

(4) See, for example, the lectures of F. T. Arecchi and H. A. Haus: *Quantum optics*, in *Proceedings of the International School of Physics, Enrico Fermi*, Course XLII, edited by R. J. Glauber (New York, 1969).

(5) D. Holliday and A. E. Glassgold: *Phys. Rev.*, **139** A, 1717 (1965).

(6) B. R. Mollow and R. J. Glauber: *Phys. Rev.*, **160**, 1097 (1967).

(7) E. Y. C. Lu: *Lett. Nuovo Cimento*, **2**, 1241 (1971).

sed [5,6,8] using different approaches. Our results will be more explicit. We shall show that the quantum uncertainty of the radiation field arising from two-photon transitions fluctuates in time and may become less than $\hbar/2$, the uncertainty value of the usual coherent states, and cannot, therefore, be described in terms of the usual diagonal coherent state representation with positive definite weight functions [9]. They are, however, very naturally described by the new coherent states $|\alpha\rangle_\theta$ which are minimum uncertainty states with $\Delta p\,\Delta q = \hbar/2$ and the ratio $\xi \equiv \Delta p/\Delta q = \exp[2\theta]$ any positive number. In fact, the radiation field, if initially coherent, is a pure new coherent state $|\alpha\rangle_\theta$ for all t in the interaction picture, θ being determined by the parametric-coupling constant of transition matrix element and is a linear function of time.

Consider first the case of two photons of equal frequency: the Hamiltonian may be written in the Heisenberg picture as [2]

$$H = \hbar\omega a^\dagger(t)\,a(t) - \frac{i\hbar\varkappa}{2}\left(a^\dagger(t)\,a^\dagger(t)\exp[-2i\omega t] - a(t)\,a(t)\exp[2i\omega t]\right), \tag{1}$$

The Hamiltonian (1) may represent a parametric amplifier being pumped at twice its output frequency ω or atomic transitions with emission or absorptions of two photons of equal frequency, the pump field or the atomic medium being treated classically. It can be shown that the number of photons in the mode $N(t) \equiv a^\dagger(t)\,a(t)$ increases, as a result of the external pumping, exponenitally in time as $\exp[2\varkappa t]$ for large t and in this sense it behaves as a linear amplifier. In the interaction picture, the interaction Hamiltonian becomes time independent

$$H_I^{ip}(t) = -\frac{i\hbar\varkappa}{2}\left(a^\dagger(0)\,a^\dagger(0) - a(0)\,a(0)\right). \tag{2}$$

Because of this the time development operator of the system in the interaction picture is easily written down

$$U^{ip}(\varkappa t) = \exp\left[\frac{-iH_I^{ip}t}{\hbar}\right] = \exp\left[-\frac{\varkappa t}{2}\,(a^\dagger a^\dagger - aa)\right]. \tag{3}$$

The unitary transformation generated by the time development operator in the interaction picture can be calculated by explicit power series expansion to be

$$a^{ip}(t) = U^{ip}(\varkappa t)^{-1}a(0)\,U^{ip}(\varkappa t) = a(0)\cosh\varkappa t - a^\dagger(0)\sinh\varkappa t\,. \tag{4}$$

It is immediately evident that the unitary operator $U^{ip}(\varkappa t)$ generates the canonical transformation definition in ref. [7]. Thus if the radiation field is initially coherent, *i.e.*

$$a(0)|t=0\rangle = a(0)|\alpha\rangle = \alpha|t=0\rangle\,, \tag{5}$$

then the state at time t in the interaction picture is given by

$$|t\rangle^{ip} = U^{ip}(t)|t=0\rangle = U^{ip}(t)|\alpha\rangle\,, \tag{6}$$

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[8] P. LAMBROPOULOS, C. KIKUCHI and R. K. OSBORN: *Phys. Rev.*, **144**, 1081 (1966); P. LAMBROPOULOS: *Phys. Rev.*, **156**, 286 (1967); Y. R. SHEN: *Phys. Rev.*, **155**, 921 (1967).

[9] B. R. MOLLOW and R. J. GLAUBER: *Phys. Rev.*, **160**, 1076 (1967).

and satisfies

$$(7)\qquad a_\theta|t\rangle^{ip} \equiv \big(a(0)\cosh\theta + a^\dagger(0)\sinh\theta\big)|t\rangle^{ip} = U(\theta)\,a(0)\,U^{-1}(\theta)\,U^{ip}(\varkappa t)|\alpha\rangle = \alpha|t\rangle^{ip}$$

if $\theta = \varkappa t$.

Thus we have shown that in the interaction picture the radiation field arising from parametric second-subharmonic generation is a pure new coherent state $|\alpha\rangle_\theta$ with $\theta = \varkappa t$ if the state is initially coherent. If the initial state has a positive definite $P(\alpha)$ representation, then the radiation field in the interaction picture has the same $P(\alpha)$ representation in terms of the new coherent states for all time:

$$(8)\qquad \varrho^{ip}(t) = U^{ip}(t)\,\varrho(0)\,U^{ip-1}(t) = \int d^2\alpha P(\alpha)\,U^{ip}(t)|\alpha\rangle\langle\alpha|U^{ip}(t)^{-1} = \int d^2\alpha P(\alpha)|\alpha\rangle_{\theta\theta}\langle\alpha|$$

with $\theta = \varkappa t$.

In general, if the initial radiation field possesses a generalized diagonal coherent state representation ([7]) in the form

$$(9)\qquad \varrho(0) = \int d^2\alpha P(\alpha)|\alpha\rangle_{\theta_0\,\theta_0}\langle\alpha|\,,$$

then

$$(10)\qquad \varrho^{ip}(t) = \int d^2\alpha P(\alpha)|\alpha\rangle_{\theta\,\theta}\langle\alpha|$$

with $\theta = \theta_0 + \varkappa t$.

We now outline the generalization to two-photon transitions of different frequencies. The Hamiltonian may be written as

$$(11)\qquad H = \hbar\omega_1 a^\dagger(t)\,a(t) + \hbar\omega_2 b^\dagger(t)\,b(t) - i\hbar\varkappa\big(a^\dagger(t)\,b^\dagger(t)\exp[-i(\omega_1+\omega_2)t] - a(t)\,b(t)\exp[i(\omega_1+\omega_2)t]\big)\,.$$

The interaction Hamiltonian is again time-independent in the interaction picture:

$$(12)\qquad H_I^{ip}(t) = -i\hbar\varkappa\big(a^\dagger(0)\,b^\dagger(0) - a(0)\,b(0)\big)\,.$$

It is convenient to define $C_\pm$-operators

$$(13)\qquad C_\pm = \frac{1}{\sqrt{2}}(a \pm b)\,,$$

which satisfy canonical commutation relations

$$(14)\qquad [C_\pm, C_\pm^\dagger] = 1\,,$$

while all other commutators vanish. In terms of the $C_\pm$-operators, the two modes decouple, namely

$$(15)\qquad H_I^{ip}(t) = \frac{i\hbar\varkappa}{2}\,(C_+C_+ - C_+^\dagger C^\dagger - C_-C_- + C_-^\dagger C_-^\dagger)\,.$$

The decoupling allows the previous analysis to go through in the present case with the result that in obvious notations

$$\varrho^{ip}(t) = \int d^2\gamma_+ d^2\gamma_- P(\gamma_+, \gamma_-) |\gamma_+, \gamma_-\rangle_{\theta_+\theta_-\;\theta_+\theta_-}\langle\gamma_+, \gamma_-| \,, \tag{16}$$

where $|\gamma_+, \gamma_-\rangle$ are right eigenstates of $C_\pm$-operators and

$$\theta_\pm = \theta_0 \pm \varkappa t \,. \tag{17}$$

From the known properties [7,10] of new coherent states we can draw the following conclusions on radiation fields arising from two-photon transitions:

1) Since the new coherent states do not lead to factorization of usual correlation functions, one may expect nonzero coincidence photon-counting effects. Our calculations [10] on $G^2(x_1, x_2, x_2, x_1)$ show that one may expect positive as well as negative correlation counting rates. It would be extremely interesting to have direct experimental verification of this effect since neither thermal nor laser radiation give negative correlation counting.

2) The photon statistics is given by

$$\langle N(t)\rangle = |\alpha|^2(\cosh^2\varkappa t + \sinh^2\varkappa t) - (\alpha^2 + \alpha^{*2}) \sinh\varkappa t \cosh\varkappa t + \sinh^2\varkappa t \,, \tag{18}$$

$$\Delta N^2(\varkappa t) = |\alpha|^2 \cosh 4\varkappa t + |\alpha|^2(\alpha^2 + \alpha^{*2}) \sinh 4\varkappa t + \tfrac{1}{2} \sinh^2 2\varkappa t \,. \tag{19}$$

Thus the photon distribution is neither Bose-Einstein nor Poisson.

3) The quantum uncertainties in canonial momenta and co-ordinate which may be looked upon as the H and E fields of a two-photon oscillator are given by

$$\begin{cases} \Delta p^2(t) = \dfrac{\hbar}{2} |\cos\omega t \exp[\varkappa t] + i \sin\omega t \exp[-\varkappa t]|^2 \,, \\[2ex] \Delta q^2(t) = \dfrac{\hbar}{2} |\cos\omega t \exp[-\varkappa t] + i \sin\omega t \exp[\varkappa t]|^2 \,, \end{cases} \tag{20}$$

while the fields themselves oscillate as

$$\begin{cases} p(t) = p(0) \cos\omega t \exp[-\varkappa t] - \omega q(0) \sin\omega t \exp[\varkappa t] \,, \\[2ex] q(t) = q(0) \cos\omega t \exp[\varkappa t] + \dfrac{p(0)}{\omega} \sin\omega t \exp[-\varkappa t] \,. \end{cases} \tag{21}$$

4) From the above equations it is clear that parametric or two-photon amplification fundamentally alters the quantum correlations and statistics of the initial-

(10) E. Y. C. Lu and M. E. Smithers: in preparation.

radiation field. It appears to offer the practical means by which new coherent states can be generated and controlled and allow possible new tests of predictions of quantum mechanics of the electromagnetic field.

Finally, we remark that this is but one example of the use of generalized coherent state representation to represent a wider class of quantum states as suggested in ref. ([7]). It is known ([5,6]) that in the present case the radiation fields possess Gaussian characteristic functions. In general, generalized coherent state representation allows the use of positive definite weight functions to represent all states with Gaussian characteristic functions.