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On the distribution and expectation of success runs in nonhomogeneous Markov dependent trials

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The number of success runs for nonhomogeneous markov dependent trials are represented as the sum of Bernoulli trials and the expected value of runs are obtained by using this representation. The distribution and bounds for the distribution of the longest run are derived for markov dependent trials.

Key words: Success run, Markov chain, Bernoulli trials, expected value.

1. Introduction

Runs are important in applied probability and statistical inference. They are used in many areas, such as hypothesis testing, system reliability, quality control. There have been various publications dealing with the distribution theory of runs. Early discussions for runs appeared in the works of Mood (1940), Levene and Wolfowitz (1944), Wolfowitz (1944), Dobrushin (1953). New results on runs have been derived by many authors including Philippou and Makri (1986), Fu and Koutras (1994), Koutras and Alexandrou (1995), Han and Aki (1999). Recent investigations are due to Stefanov (2000), Chadjiconstantinidis and Koutras (2001) . In a sequence of markov dependent trials success runs are discussed in the works of Schwager (1983), Hirano and Aki (1993), Mohanty (1994), Antzoulakos (1999), Antzoulakos and Chadjiconstantinidis (2001), Vaggelatou (2003). For the longest success run and its applications, we refer to Philippou and Makri (1985), Philippou (1986), Makri and Philippou (1994), Makri and Philippou (1996), Lou (1996).

There are various definitions of success runs. In the present paper, we consider the number of success runs of size exactly "k", the number of success runs of size greater than or equal to "k" and the longest success run for markov dependent trials. Markov dependent trials appear in a large number of natural, physical, biological and economic phenomena. Denote by $S_n(k)$ and $G_n(k)$ the number of success runs of size exactly "k" and the number of success runs of size greater than or equal to "k" respectively, in n trials. Let L_n be the length of the longest success run in n trials. For illustration, consider the sequence 1110010011. Then $S_{10}(1) = 1, S_{10}(3) = 1, G_{10}(1) = 3, G_{10}(2) = 2, L_{10} = 3$.

Fu and Koutras (1994) investigated the distribution of runs in a sequence of Bernoulli trials by using Markov chain imbedding technique. Lou (1996) has used to same technique to find the exact joint and conditional distributions of the success runs and the longest success run given the number of successes. Antzoulakos and Chadjiconstantinidis (2001) considered the number of success runs for homogeneous markov dependent trials. Eryılmaz and Tütüncü (2002) studied the success run model based on record indicators of independent and identically distributed random variables. Recently, Vaggelatou (2003) presents asymptotic results for the longest run distribution in a multi state markov chain.

The present paper is organized as follows. In the second section we define a useful Markov chain for our investigations which denotes the length of the success run at the end of the nth step and obtain its one dimensional distribution. In the third section we represent the number of success runs for nonhomogeneous markov dependent trials as the sum of dependent Bernoulli trials having different success probabilities. By using this representation we obtain the expected values of $S_n(k)$ and $G_n(k)$ for nonhomogeneous markov dependent trials. In the fourth section we study the distribution of the longest success run and in the last section we give a numerical example to illustrate the findings of the paper. Our approach of finding expectation of success runs and the distribution of the longest run departs from the Markov chain imbedding technique. In the present paper, success runs are represented as the sum of dependent Bernoulli indicators and the longest run is expressed as the maximum of sample whose members are subject to a markov chain condition.

2. Preliminary results

Let $\{\xi_n, n \geq 1\}$ be a nonhomogeneous two-state Markov chain with transition probabilities:

$$p_{ij}^{(n)} = P\{\xi_n = j \mid \xi_{n-1} = i\} \quad , \quad n \geq 2, \quad i,j = 0,1$$

and initial probabilities $p_j = P\{\xi_1 = j\}, j = 0, 1.$

Denote by η_n the length of the success run at the end of the nth step. It can be easily verified that η_n is nonhomogeneous Markov chain with transition probabilities:

$$P\{\eta_n = i \mid \eta_{n-1} = i-1\} = P\{\xi_n = 1 \mid \xi_{n-1} = 1\} = p_{11}^{(n)} \quad , \quad 2 \leq i \leq n \; ; \; n \geq 2$$

$$P\{\eta_n = 0 \mid \eta_{n-1} = i\} = P\{\xi_n = 0 \mid \xi_{n-1} = 1\} = p_{10}^{(n)} \quad , \; 1 \leq i \leq n \; ; \; n \geq 2$$

$$P\left\{\eta_n = 0 \mid \eta_{n-1} = 0\right\} = P\left\{\xi_n = 0 \mid \xi_{n-1} = 0\right\} = p_{00}^{(n)} \quad , \quad n \geq 2$$

$$P\left\{\eta_n = 1 \mid \eta_{n-1} = 0\right\} = P\left\{\xi_n = 1 \mid \xi_{n-1} = 0\right\} = p_{01}^{(n)} \quad , \quad n \geq 2.$$

In the following lemma the one dimensional distribution of η_n is obtained.

LEMMA 2.1. *It is true that for $n \geq 1$,*

$$P\left\{\eta_n = k\right\} = \begin{cases} a(n) & , \quad k = 0 \\ p_{01}^{(n-k+1)} a(n-k) \prod_{i=1}^{k-1} p_{11}^{(n-i+1)} & , \quad k = 1, 2, ..., n-1 \\ p_1 \prod_{i=1}^{n-1} p_{11}^{(n-i+1)} & , \quad k = n \end{cases} \tag{2.1}$$

where $\Pi_\phi = 1, a(j) = P\left\{\xi_j = 0\right\}$ and

$$a(j) = (p_{00}^{(j)} - p_{10}^{(j)})a(j-1) + p_{10}^{(j)} \quad j \geq 2, \quad a(1) = p_0. \tag{2.2}$$

PROOF. For $k = 0$, $P\left\{\eta_n = 0\right\} = P\left\{\xi_n = 0\right\}$. For $k = 1, 2, ..., n-1$ and $k = n$ the probability of the event $\left\{\eta_n = k\right\}$ is represented respectively by,

$$P\left\{\eta_n = k\right\} = P\left\{\xi_n = 1, \xi_{n-1} = 1, ..., \xi_{n-k+1} = 1, \xi_{n-k} = 0\right\}$$

and

$$P\left\{\eta_n = n\right\} = P\left\{\xi_n = 1, \xi_{n-1} = 1, ..., \xi_1 = 1\right\}.$$

The proof follows by the markovian property of $\left\{\xi_n, n \geq 1\right\}$. The recurrence relation given in (2.2) is obtained by applying the total probability law□

If $\left\{\xi_n, n \geq 1\right\}$ is a homogeneous two-state Markov chain, i.e. $p_{ij}^{(n)} = p_{ij}$ the one dimensional distribution of the length of the success run is

$$P\left\{\eta_n^h = k\right\} = \begin{cases} b(n) & , \quad k = 0 \\ p_{01} p_{11}^{k-1} b(n-k) & , \quad k = 1, 2, ..., n-1 \\ p_1 p_{11}^{n-1} & , \quad k = n \end{cases} \tag{2.3}$$

where $b(j) = P\left\{\xi_j = 0\right\}$ and

$$b(j) - (p_{00} - p_{10})b(j-1) = p_{10} \quad j \geq 2, \quad b(1) = p_0. \tag{2.4}$$

Solving (2.4) and using it in (2.3) we obtain the following corollary.

COROLLARY 2.1. *If* $\{\xi_n, n \geq 1\}$ *is a homogeneous two-state Markov chain, i.e.* $p_{ij}^{(n)} = p_{ij}$ *then*

$$P\left\{\eta_n^h = k\right\}$$

$$= \begin{cases} \left[p_0(p_{00}-p_{10})^{n-1} + \frac{p_{10}(1-(p_{00}-p_{10})^{n-1})}{1-p_{00}+p_{10}}\right] & ,k=0 \\ p_{01}p_{11}^{k-1}\left[p_0(p_{00}-p_{10})^{n-k-1} + \frac{p_{10}(1-(p_{00}-p_{10})^{n-k-1})}{1-p_{00}+p_{10}}\right] & ,k=1,2,...,n-1 \\ p_1 p_{11}^{n-1} & ,k=n. \end{cases}$$

3. Expected values of $S_n(k)$ and $G_n(k)$

It is possible to establish the number of success runs for nonhomogeneous markov trials in the following way.

Define the following random variables:

$$X_{jk} = \begin{cases} 1 & , \quad \eta_j = k \text{ and } \eta_{j+1} = 0 \\ 0 & , \quad otherwise \end{cases}$$

$$k \geq 1;\ j = k, ..., n\ ;\ P\left\{\eta_{n+1} = 0\right\} = 1$$

$$S_n(k) = \sum_{j=k}^{n} X_{jk}$$

and

$$Y_{jk} = \begin{cases} 1 & , \quad \eta_j \geq k \text{ and } \eta_{j+1} = 0 \\ 0 & , \quad otherwise \end{cases}$$

$$k \geq 1;\ j = k, ..., n\ ;\ P\left\{\eta_{n+1} = 0\right\} = 1$$

$$G_n(k) = \sum_{j=k}^{n} Y_{jk}.$$

It is evident that $S_n(k)$ is the number of success runs of size exactly "k" and $G_n(k)$ is the number of success runs of size greater than or equal to "k".

The random variables $S_n(k)$ and $G_n(k)$ take values $0, 1, ..., \left[\frac{n-k}{k+1}\right] + 1$, where $[x]$ denotes the integer part of x.

X_{jk} and Y_{jk}'s correspond to the dependent Bernoulli trials with success probabilities:

$$P\{X_{jk} = 1\} = \begin{cases} p_{10}^{(j+1)} P\{\eta_j = k\} & , \quad j = k, ..., n-1 \\ P\{\eta_n = k\} & , \quad j = n \end{cases}$$

and

$$P\{Y_{jk} = 1\} = \begin{cases} p_{10}^{(j+1)} P\{\eta_j \geq k\} & , \quad j = k, ..., n-1 \\ P\{\eta_n \geq k\} & , \quad j = n. \end{cases}$$

Hence $S_n(k)$ and $G_n(k)$ are the sum of dependent Bernoulli trials having different success probabilities. By using this representation the expected values of $S_n(k)$ and $G_n(k)$ are

$$E(S_n(k)) = \begin{cases} \sum_{j=k}^{n-1} p_{10}^{(j+1)} P\{\eta_j = k\} + P\{\eta_n = k\} & , \quad 1 \leq k \leq n-1 \\ p_1 \prod_{i=1}^{n-1} p_{11}^{(n-i+1)} & , \quad k = n \end{cases}$$

and

$$E(G_n(k)) = \begin{cases} \sum_{j=k}^{n-1} p_{10}^{(j+1)} P\{\eta_j \geq k\} + P\{\eta_n \geq k\} & , \quad 1 \leq k \leq n-1 \\ p_1 \prod_{i=1}^{n-1} p_{11}^{(n-i+1)} & , \quad k = n \end{cases}$$

where $P\{\eta_j = k\}$ is given by 2.1. It is easy to write the expected values for homogeneous markov trials taking $p_{ij}^{(n)} = p_{ij}$ and using η_n^h instead of η_n.

4. The distribution of the longest success run

In this section we derive the distribution of the longest success run L_n. Bounds for the distribution of the longest run are also given. For this purpose, we firstly establish the following lemma since it is helpful for our further investigations.

LEMMA 4.1. *It is true that for $n \geq 2$,*

$$P\left\{\eta_{n-1} < k, \eta_n < k\right\}$$

$$= \begin{cases} p_{00}^{(n)} P\left\{\eta_{n-1} = 0\right\} & , \quad k = 1 \\ \sum_{j=0}^{k-2} P\left\{\eta_{n-1} = j\right\} + p_{10}^{(n)} P\left\{\eta_{n-1} = k-1\right\} & , \quad k = 2, 3, ..., n-1 \\ 1 - p_1 \prod_{j=1}^{n-1} p_{11}^{(n-j+1)} & , \quad k = n \end{cases}$$

PROOF. For $k = 1$

$$P\left\{\eta_{n-1} < k, \eta_n < k\right\} = P\left\{\eta_{n-1} = 0, \eta_n = 0\right\}$$

$$= p_{00}^{(n)} P\left\{\eta_{n-1} = 0\right\}$$

observe next that for $k = 2, 3, ..., n-1$

$$P\left\{\eta_{n-1} < k, \eta_n < k\right\}$$

$$= P\left\{\eta_{n-1} = 0, \eta_n = 0\right\} + \sum_{j=1}^{k-1} P\left\{\eta_{n-1} = j, \eta_n = 0\right\}$$

$$+ P\left\{\eta_{n-1} = 0, \eta_n = 1\right\} + \sum_{j=2}^{k-1} P\left\{\eta_{n-1} = j-1, \eta_n = j\right\}$$

$$= P\left\{\eta_{n-1} = 0\right\} + \sum_{j=1}^{k-2} P\left\{\eta_{n-1} = j\right\} + p_{10}^{(n)} P\left\{\eta_{n-1} = k-1\right\}$$

and for $k = n$

$$P\left\{\eta_{n-1} < k, \eta_n < k\right\} = P\left\{\eta_n < n\right\}$$

$$= 1 - p_1 \prod_{j=1}^{n-1} p_{11}^{(n-j+1)}.$$

THEOREM 4.1. *It is true that, for* $k = 1, 2, ..., n-1$

$$P\{L_n < k\}$$

$$= \sum_{i_j(j=k,...,n)=0}^{k-1} \left\{ P\{\eta_k = i_k\} \prod_{m=k+1}^{n} P\left\{\eta_m = i_m \mid \eta_{m-1} = i_{m-1}\right\} \right\}$$

and for $k = n$

$$P\{L_n < k\} = 1 - p_1 \prod_{i=1}^{n-1} p_{11}^{(n-i+1)}.$$

PROOF. One can write for the probability of the event $\{L_n < k\}$ for $k = 1, 2, ..., n-1$

$$P\{L_n < k\} = P\left\{ \max_{k \le i \le n} \eta_i < k \right\}$$

$$= P\left\{\eta_k < k, \eta_{k+1} < k, ..., \eta_n < k\right\}$$

$$= \sum_{i_k=0}^{k-1} \sum_{i_{k+1}=0}^{k-1} \cdots \sum_{i_n=0}^{k-1} P\left\{\eta_k = i_k, \eta_{k+1} = i_{k+1}, ..., \eta_n = i_n\right\}$$

$$= \sum_{i_k=0}^{k-1} \sum_{i_{k+1}=0}^{k-1} \cdots \sum_{i_n=0}^{k-1} P\left\{\eta_n = i_n \mid \eta_{n-1} = i_{n-1}\right\} P\left\{\eta_{n-1} = i_{n-1} \mid \eta_{n-2} = i_{n-2}\right\} \times$$

$$\times ... P\left\{\eta_{k+1} = i_{k+1} \mid \eta_k = i_k\right\} P\left\{\eta_k = i_k\right\}$$

and for $k = n$

$$P\{L_n < n\} = 1 - P\{L_n \ge n\}$$

$$= 1 - P\{L_n = n\}$$

$$= 1 - P\left\{\xi_n = 1, \xi_{n-1} = 1, ..., \xi_1 = 1\right\}$$

$$= 1 - p_1 \prod_{i=1}^{n-1} p_{11}^{(n-i+1)}.$$

According to the theorem given above the distribution of the longest run is characterized by the transition probabilities and the one dimensional distribution of the length of the success run. The complexity of the exact distribution of L_n may prevent its direct use for large values of n. Hence we investigate bounds for the probability $P\{L_n < k\}$. Bounds which are obtained in this paper are based on following probabilities:

$$\theta_1(i,k) := P\{\eta_i \geq k\}$$

$$\theta_2(i,k) := P\left\{\eta_i \geq k, \eta_{i+1} \geq k\right\}.$$

THEOREM 4.2. For $1 < k < n$, the following inequalities hold:

$$\max\left(0, 1 - \sum_{i=k}^{n} \theta_1(i,k) + \sum_{i=k}^{n-1} \theta_2(i,k)\right) \leq P\{L_n < k\} \leq \min_{k \leq i \leq n}(1 - \theta_1(i,k)).$$

PROOF. By using the same representation given in Theorem 4.1, one can write

$$P\{L_n < k\} = P\left\{\bigcap_{i=k}^{n} A_i\right\}$$

where $A_i \equiv \{\eta_i < k\}$. For the lower bound by using Worsley's variant of a Bonferroni type inequality we obtain

$$P\left\{\bigcap_{i=k}^{n} A_i\right\} \geq 1 - \sum_{i=k}^{n} P\{A_i^c\} + \sum_{i=k}^{n-1} P\left\{A_i^c A_{i+1}^c\right\}.$$

The upper bound is obtained by using the following inequality:

$$P\left\{\bigcap_{i=k}^{n} A_i\right\} \leq \min_{k \leq i \leq n}\left(P\{A_i\}\right).$$

□

An easier representation for the probability $\theta_2(i,k)$ is given as follows:

$$\theta_2(i,k) = 1 - P\{\eta_i < k\} - P\left\{\eta_{i+1} < k\right\} + P\left\{\eta_i < k, \eta_{i+1} < k\right\}$$

where $P\left\{\eta_i < k, \eta_{i+1} < k\right\}$ is given by Lemma 4.1. Denote by $l(n,k)$ and $u(n,k)$ the lower and upper bound for the probability $P\{L_n < k\}$ respectively. By considering Lemma 4.1 in Theorem 4.2 the explicit formulas for $l(n,k)$ and $u(n,k)$ are

$$l(n,k) = \max\left(0, 1 - \sum_{i=k}^{n}\left(1 - \sum_{j=0}^{k-1} P\{\eta_i = j\}\right)\right.$$

$$\left.+\sum_{i=k}^{n-1}\left(1 - (1 - p_{10}^{(i+1)})P\{\eta_i = k-1\} - \sum_{j=0}^{k-1} P\{\eta_{i+1} = j\}\right)\right)$$

and

$$u(n,k) = \min_{k \le i \le n}\left(\sum_{j=0}^{k-1} P\{\eta_i = j\}\right)$$

$$1 < k < n.$$

Since the exact computation is easy for $k = 1$ and $k = n$ it is needless to use inequalities in these cases. It is easy to rewrite the theorems given above for homogeneous markov trials taking $p_{ij}^{(n)} = p_{ij}$ and using η_n^h instead of η_n.

Theorem 4.1 may be fruitful for small values of n. However, bounds given in Theorem 4.2 give good approximation for some values of n.

5. Numerical example

Denote by ξ_n the quality of the nth item produced by a production system with $\xi_n = 0$ meaning "Defective" and $\xi_n = 1$ meaning "Good". Suppose that ξ_n evolves as a Markov chain with transition probabilities:

$$P\{\xi_n = 1 \mid \xi_{n-1} = 1\} = p_{11}^{(n)} = \frac{1}{n} \quad , \quad P\{\xi_n = 0 \mid \xi_{n-1} = 1\} = p_{10}^{(n)} = 1 - \frac{1}{n}$$

$$P\{\xi_n = 0 \mid \xi_{n-1} = 0\} = p_{00}^{(n)} = \frac{1}{n^2} \quad , \quad P\{\xi_n = 1 \mid \xi_{n-1} = 0\} = p_{01}^{(n)} = 1 - \frac{1}{n^2}$$

$n \geq 2$,with $p_0 = P\{\xi_1 = 0\} = \frac{1}{2}, p_1 = P\{\xi_1 = 1\} = \frac{1}{2}$.

i) What is the expected value of producing "Good" items of size exactly "2" at the end of the tenth stage?

$$ES_{10}(2) = \sum_{j=2}^{9} p_{10}^{(j+1)} P\{\eta_j = 2\} + P\{\eta_{10} = 2\} = 0.658\,.$$

ii) What can be said about the probability of the event $\{S_{10}(2) \geq 2\}$?

By using Markov inequality,

$$P\{S_{10}(2) \geq 2\} \leq \frac{ES_{10}(2)}{2} = 0.329\,.$$

iii) What is the expected value of producing "Good" items of size greater than or equal to "2" at the end of the tenth stage?

$$EG_{10}(2) = \sum_{j=2}^{9} p_{10}^{(j+1)} P\{\eta_j \geq 2\} + P\{\eta_{10} \geq 2\} = 0.833\,.$$

iv) What is the probability that the length of the longest success run will be equal to one at the end of third stage?

$$P\{L_3 = 1\} = P\{L_3 < 2\} - P\{L_3 < 1\} = 0.625 - 0.014 = 0.611.$$

In addition to the foregoing numerical example, in Table 1 some numerics for the expected values of $S_n(k), G_n(k)$, in Table 2 and Table 3 exact probabilities and bounds for $P\{L_n < k\}$ are given respectively.

n	k	$ES_n(k)$	$EG_n(k)$
3	2	0.29167	0.37500
10	4	0.02859	0.03467
10	5	0.00515	0.00608
15	5	0.00529	0.00624
20	3	0.16212	0.19880
30	3	0.16989	0.20693
30	4	0.03074	0.03704
50	3	0.17627	0.21349
50	8	1.38246×10^{-5}	1.54964×10^{-5}
60	3	0.17789	0.21514
60	6	8.28845×10^{-4}	9.57917×10^{-4}

Table 1. Some numerics for $ES_n(k)$ and $EG_n(k)$.

n	k	$P\{L_n < k\}$
5	1	3.472×10^{-5}
5	2	0.48698
5	3	0.86875
5	4	0.97292
5	5	0.99583
10	1	3.797×10^{-14}
10	2	0.34472
10	3	0.82882
10	4	0.96540
10	5	0.99391
10	6	0.99906
10	7	1.00000
10	8	1.00000
10	9	1.00000
10	10	1.00000

Table 2. Exact probabilities for $P\{L_n < k\}$.

n	k	$l(n,k)$	$u(n,k)$
3	2	0.62500	0.75000
10	4	0.96533	0.97917
10	5	0.99391	0.99583
15	5	0.99375	0.99583
20	3	0.80120	0.91667
30	3	0.79307	0.91667
30	4	0.96296	0.97917
50	3	0.78651	0.91667
50	8	0.99998	0.99999
60	3	0.78486	0.91667
60	6	0.99904	0.99931

Table 3. Bounds for $P\{L_n < k\}$.

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