ISRAEL JOURNAL OF MATHEMATICS, Vol. 30, No. 4, 1978

BIFURCATION IN THE NEIGHBOURHOOD OF A NON-ISOLATED SINGULAR POINT

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ABSTRACT

The Lyapunov–Schmidt method for bifurcation problems has, until recently, been applied only to operator equations whose singular points are isolated in the solution set of the equation. For bifurcation at a multiple eigenvalue involving several parameters, however, singular points are often non-isolated. In this paper, the case of intersecting curves of singular points is considered. Under natural hypotheses on these curves, and assuming suitable transversality conditions on the first order nonlinearity of the operator, it is shown that the solution set of the equation may be completely determined locally in terms of the solutions of associated finite dimensional polynomial equations.

1. Introduction

Let E and Y be real Banach spaces. In this paper, we discuss the nature of the set of small solutions of a class of operator equations of the form

$$G(u)=0, \qquad u\in E \tag{1.1}$$

where $G\colon E\to Y$ is a mapping of class C^n for some $n \geqq 3$, $G(0)=0$ and $DG(0)\colon E\to Y$ is a Fredholm operator with index $m\geqq 2$.

Since $DG(u)$ is assumed continuous in u, there exists a neighbourhood U of zero in E such that $DG(u)$ is a Fredholm operator of index m for all $u\in U$ [6]. Suppose $u\in U$ is a solution of (1.1) and $DG(u)$ is onto Y. Then, by the implicit function theorem, the solution set of (1.1) is locally (i.e. near u) C^n homeomorphic to an open ball in $\mathbf{R}^m$.

From the point of view of bifurcation theory, it is important to investigate the nature of the solution set of equation (1.1) in the neighbourhood of a point $u\in U$ such that $G(u)=0$ and $DG(u)$ is *not* onto Y. Such points are called *singular*. Henceforth we shall suppose $u=0$ is a singular point. Note that E must now be at least three-dimensional.

Received July 8, 1977

The set of small solutions of (1.1) may be visualised as consisting of an m-dimensional C^n manifold, on which $DG(u)$ has range Y, together with a set S of singular points. The simplest possible case, when $S \cap W = \{0\}$ for some neighbourhood W of zero in E, has been discussed thoroughly by Magnus [9–11]. In this case, the singular point $u = 0$ is called *isolated.*

Magnus' analysis is of particular interest in bifurcation problems involving one parameter. Such problems may be formulated as an equation of the form (1.1) with $m = 1$ ([9], [14]).

Now, the range of $DG(0)$ is a subspace of Y with codimension $\beta \geqq 1$. If $\beta = 1$ and $m \geqq 2$ then the singular point $u = 0$ cannot be expected to be isolated in general. If $\beta \geqq 2$ and $m \geqq 2$ then the case when $u = 0$ is an isolated singular point must be regarded as exceptional. Since bifurcation problems involving several parameters correspond to the case $m \geqq 2$, it is important to consider classes of equation (1.1) such that $u = 0$ is a non-isolated singular point.

To further motivate this last statement, consider the usual formulation of bifurcation problems involving several parameters. Let X be a real Banach space and let $E = \mathbf{R}^{p+1} \times X$ where $p \geqq 1$. Write $G(u)$ as $G(\lambda, \mu, x)$ for $u = (\lambda, \mu, x) \in \mathbf{R} \times \mathbf{R}^p \times X$. Suppose $G(\lambda, 0, \theta) = 0$ for each $\lambda \in \mathbf{R}$, where $0 \in \mathbf{R}^p$, $\theta \in X$ are the zeroes of $\mathbf{R}^p$, X respectively. $G_x(0,0,\theta)$: $X \to Y$ is assumed to be a Fredholm operator with index zero, null space $N \neq \{0\}$ and range R.

It is common to assume, for $p = 1$, that $G_\mu(0,0,\theta) \not\in R$ (e.g. [3, 4, 7, 8, 13]). In this case, $DG(0)$: $E \to Y$ has index two (i.e. $m = 2$) so that it is reasonable to assume that the singular point $u = 0$ $((\lambda, \mu, x) = (0,0,\theta))$ is isolated, provided that N has dimension one or two. In fact, the knowledge of a line $\{(\lambda, 0, \theta)$: $\lambda \in \mathbf{R}\}$ of trivial solutions considerably helps the analysis, and Magnus' transversality conditions may be modified in order to retain and exploit the significance of the parameter space, as in [4]. However, when $\dim N \geqq 3$ the assumption of an isolated singular point at zero is no longer acceptable in a theory attempting any degree of generality. It is worth noting that if N has dimension one, then $DG(0,0,\theta)$ is onto Y.

Another situation of interest arises when $p = 1$ and $G(\lambda, \mu, \theta) = 0$ for all $(\lambda, \mu) \in \mathbf{R}^2$. In this case, $G_\mu(0,0,\theta) = 0$ so that $DG(0)$: $E \to Y$ again has index two ($m = 2$). If N has dimension one ($\beta = 1$), it is conceivable that the singular point $\lambda = \mu = 0$, $x = \theta$ is isolated. However, this will only be the case if $\lambda G_{\lambda x}(0,0,\theta) + \mu G_{\mu x}(0,0,\theta)$ maps N into R for all $(\lambda, \mu) \in \mathbf{R}^2$; otherwise there exists a unique curve of singular points in $\mathbf{R}^2 \times \{\theta\}$ through zero in E. This fact follows easily from the analysis of Crandall and Rabinowitz [5] for bifurcation from simple eigenvalues (see [15] for details).

If N has dimension two, it is reasonable to expect two curves of singular points in $\mathbf{R}^2 \times \{\theta\}$ passing through zero. These curves may be coincident. This situation is discussed for two examples by Mallet-Paret [12]. Examples of non-isolated singular points occurring as secondary bifurcation points are discussed by Bauer, Keller and Riess [1].

In this paper we suppose the set S of singular points near $u = 0$ to consist of distinct curves (each of class C^n) each passing through zero. Let C be one of these curves and suppose C is parameterised by $w: (-1, 1) \to E$ such that $w(0) = 0$ and $w'(0) \neq 0$ (see (H1) of section two). Let X be a subspace of E such that $E = X \oplus \operatorname{span}\{w'(0)\}$. We might attempt Magnus' analysis on the equation

$$G(w(\alpha) + X) = 0, \qquad x \in X \tag{1.2}$$

separately for each $\alpha \neq 0$, and then try to piece the "slices" of solutions together to obtain the full picture in E. Clearly, we first need to characterise the null-space of $DG(w(\alpha))$ for each $\alpha \neq 0$. Hypothesis (H2) enables us to do this (see Lemma 2.4). The next step is to generalise Magnus' non-degeneracy condition to apply along C. This generalisation reduces to a single non-degeneracy condition at zero and the piecing together of the slices of solutions follows naturally, giving cone-shaped sets of solutions of (1.1) near C. Finally, we show how the two definitions of non-degeneracy can completely describe the set of small solutions of (1.1).

Having obtained preliminary results in section two, the analysis proceeds by analogy with that of [11]. The difference is that each result is one step removed from the corresponding result in [11] since we shall be considering the structure of solutions of (1.1) near a given curve of singular points, whereas the results in [11] are for the whole set of small solutions of (1.1).

2. Preliminary results

Let $N(A)$, $R(A)$ denote respectively the null set and range of a linear operator A. The notation we shall use for Frechet derivatives is that of [11]. Set $V = N(G'(0))$, $R = R(G'(0))$. Let $Z \subset E$, $Y_0 \subset Y$ be complementary subspaces of V and R respectively, and let P: $Y \to Y_0$ be the projection given by $P(f + y) = y$ for $f \in R$, $y \in Y_0$. Let k be the largest integer such that $2 \leqq k \leqq n$ and

$$G^{(j)}(0)v^j = 0, \qquad j = 1, \cdots, k-1 \quad \text{for all } v \in V. \tag{2.1}$$

Let C be a curve in E satisfying

(H1) *$C = \{w(\alpha): |\alpha| < 1\}$ where $w: (-1,1) \to E$ is of class C^p for some $p \geqq k$ such that $w(0) = 0$, $w'(0) \neq 0$, $G(C) = \{0\}$ and $R(G'(x)) \neq Y$ for each $x \in C$. We shall always suppose $p \geqq n$.*

LEMMA 2.1. *Let C satisfy* (H1). *Then*

(a) $w^{(j)}(0) \in V \quad j = 1, \cdots, k-1$,

(b) $G'(0)w^{(k)}(0) + G^{(k)}(0)(w'(0))^k = 0$,

(c) $PG^{(k)}(0)(w'(0))^k = 0$.

PROOF. By (H1)

$$G(w(\alpha)) = 0 \qquad \text{for } |\alpha| < 1. \tag{2.2}$$

Differentiate (2.2) with respect to α and set $\alpha = 0$ to get $G'(0)w'(0) = 0$. This proves (a) for $j = 1$.

If $k = 2$ we have proved (a), so suppose $k \geqq 3$ and that we have proved (a) for $j = 1, \cdots, m-1$ for some $m \leqq k-1$. Set $f(\alpha) = G(w(\alpha))$. Then $f^{(m)}(0)$ is given by

$$f^{(m)}(0) = m! \sum_{j=1}^{m} \sum_{\substack{|r|=m \\ 1 \leq r_i \leq m}} a_{rj} G^{(j)}(0) w^{(r_1)}(0) \cdots w^{(r_j)}(0), \tag{2.3}$$

where $r = (r_1, \cdots, r_j)$, $|r| = r_1 + \cdots + r_j$ and $a_{rj} = (r_1! \cdots r_j! j!)^{-1}$. Since $m \leqq k-1$ and $w^{(j)}(0) \in V$ for $j = 1, \cdots, m-1$, (2.1) and (2.3) imply $f^{(m)}(0) = G'(0)w^{(m)}(0)$. But $f^{(m)}(0) = 0$ by (2.2). This proves (a) for $j = 1, \cdots, m$, and so for $j = 1, \cdots, k-1$.

To prove (b), note that $f^{(k)}(0)$ is given by (2.3) with $m = k$. Thus by (a) and (2.2)

$$0 = f^{(k)}(0) = G'(0)w^{(k)}(0) + G^{(k)}(0)(w'(0))^k,$$

which proves (b) and so (c).

Set $v_0 = w'(0)$ and define $B_k: V \to Y$ by $B_k(v) = (1/(k-1)!)G^{(k)}(0)v_0^{k-1}v$. Set $V_0 = N(PB_k)$ and let X_0 be a subspace of V such that $V_0 \oplus X_0 = V$. Set $R_0 = B_k(X_0)$. Then $R_0 \cap R = \{0\}$ so let $Y_1 \subset Y$ be a complementary subspace of $R_0 \oplus R$. Since $Y_1 \oplus R_0$ is complementary to R, we may suppose $Y_0 = Y_1 \oplus R_0$. Define $P_1: Y \to Y_1$ by $P_1(f + y) = y$ if $f \in R_0 \oplus R$ and $y \in Y_1$.

Clearly B_k is a linear homeomorphism between X_0 and R_0. Now set $f(\alpha) = G'(w(\alpha))$ for $|\alpha| < 1$. From (2.1) and Lemma 2.1(a) it is easy to show that $f^{(j)}(0)v = 0$, $j = 0, 1, \cdots, k-2$ for all $v \in V$, and $f^{(k-1)}(0)v = B_k v$ for $v \in V$.

The results 2.2 to 2.4 below are perturbation results for the family $\{f(\alpha): |\alpha| < 1\}$ of linear operators from E to Y. These and similar results appear in [15].

LEMMA 2.2. *Suppose* (H1) *is satisfied. Then there exists* $\delta > 0$ *such that* $\dim N(G'(w(\alpha)) \leqq \dim V_0$ *for all* α, $0 < |\alpha| < \delta$.

To extend the method of [11], we require the following hypothesis on C:

(H2) *There exists* $\xi > 0$ *such that, for* $|\alpha| < \xi$, $\dim N(G'(w(\alpha))) \geqq \dim V_0$.

A subset C of E satisfying (H1), (H2) is called an *arc of singularities* of (1.1).

If C is an arc of singularities of (1.1) and v_0 is defined as above, we shall say v_0 *corresponds to* the arc of singularities C.

COROLLARY 2.3. *If* C *is an arc of singularities of* (1.1) *then there exists* $\varepsilon > 0$ *such that for* $0 < |\alpha| < \varepsilon$, $\dim N(G'(w(\alpha))) = \dim V_0$ *and* $\operatorname{codim} R(G'(w(\alpha))) = \dim Y_1$.

For $r > 0$ let I_r denote the open interval $(-r, r)$ and let $\tilde{I}_r$ denote $I_r - \{0\}$.

LEMMA 2.4. *If* C *is an arc of singularities of* (1.1) *then there exists* $r > 0$ *and a mapping* T: $I_r \to B(E, E)$ *such that*

(1) T *is of class* C^n *on* $\tilde{I}_r$ *and of class* C^{n-k} *on* I_r,

(2) $T(\alpha)$ *is an isomorphism between* V_0 *and* $N(G'(w(\alpha)))$ *for each* $\alpha \in \tilde{I}_r$,

(3) $T(0)x = x$ *for each* $x \in E$,

(4) $T(\alpha)$ *has the form, for* $v \in V_0$, $x \in X_0 \oplus Z$,

$$T(\alpha)(v + x) = v + x + L(\alpha)v + \alpha^{k-1}M(\alpha)v,$$

where L: $I_r \to B(V_0, X_0)$, M: $I_r \to B(V_0, Z)$ *are of class* C^n *on* $\tilde{I}_r$ *and of class* C^{n-k} *on* I_r, $L(0)$ *is the zero operator and* $M(0)v$ *is given by*

$$G'(0)M(0)v + (1/(k-1)!)G^{(k)}(0)v_0^{k-1}v = 0.$$

The degree of degeneracy of $G(x)$ along C needs to be assessed, as (2.1) assesses the degree of degeneracy of $G(x)$ at $x = 0$. Let s be the largest integer with the following properties: (a) $s \leqq k$, (b) there exists $\eta > 0$ such that for $\alpha \in I_\eta$

$$G^{(j)}(w(\alpha))v^j = 0, \qquad j = 1, \cdots, s-1 \quad \text{for all } v \in N(G'(w(\alpha))). \tag{2.4}$$

Note that $2 \leqq s \leqq k$ and that in general s will be two. However, there is no simplification in assuming $s = 2$.

LEMMA 2.5. *Suppose* C *is an arc of singularities of* (1.1) *and that* $s \geqq 3$. *Then*

$$G^{(k)}(0)v_0^{k-j}v^j = 0, \qquad j = 2, \cdots, s-1 \quad \textit{for all } v \in V_0. \tag{2.5}$$

PROOF. For $j = 2, \cdots, s-1$ consider the Taylor expansion around $\alpha = 0$ of the expression

$$G^{(j)}(w(\alpha))(v + L(\alpha)v + \alpha^{k-1}M(\alpha)v)^j.$$

The first $k - j$ terms are included in the form

$$\sum_{q \geqq j} \sum_{p=q-j}^{k} \sum_{\substack{|r|=p \\ r_i \geqq 1}} a_{rq-j}\alpha^p G^{(q)}(0)w^{(r_1)}(0)\cdots w^{(r_{q-j})}(0)(v + L(\alpha)v)^j,$$

where $r = (r_1, \cdots, r_{q-j})$, $|r| = r_1 + \cdots + r_{q-j}$ and a_{rq-j} is the real number given in the proof of Lemma 2.1.

Now consider the coefficient of α^p for $p \leqq k - j - 1$. Since $r_i \geqq 1$ we have $q - j \leqq p \leqq k - j - 1$ so that $q \leqq k - 1$ and $r_i \leqq k - 1$ $(i = 1, \cdots, q - j)$. Since $L(\alpha)$ maps V_0 into V, Lemma 2.1 and (2.1) imply that this coefficient is zero. For $p = k - j$ the only contribution to the coefficient of α^p in the summation is, by the same argument, $G^{(k)}(0)v_0^{k-j}(v + L(\alpha)v)^j$. Thus, for $2 \leqq j \leqq s - 1$, and for all $v \in V_0$,

$$0 = \alpha^{j-k}G^{(j)}(w(\alpha))(T(\alpha)v)^j \to (1/(k-j)!)G^{(k)}(0)v_0^{k-j}v^j \quad \text{as } \alpha \to 0.$$

This completes the proof.

It is well known (see, for instance, [3, 4, 7–13]) that the investigation of small solutions of equation (1.1) may be reduced (using the Liapunov–Schmidt method) to a discussion of the small zeroes of an operator $\tilde{G}: V \to Y_0$ of the form $\tilde{G}(v) = PG^{(k)}(0)v^k + o(\|v\|^k)$. Accordingly, various forms of the following definition are frequently used in order to establish the existence of small zeroes of $\tilde{G}$ (and so of G) using the implicit function theorem.

A solution $w \in V$ of the equation

$$(2.6) \qquad PG^{(k)}(0)w^k = 0 \qquad w \in V \quad w \neq 0$$

is called *non-degenerate* if the map $\phi(w)$: $V \to Y_0$: $v \mapsto PG^{(k)}(0)w^{k-1}v$ is onto Y_0. If $\phi(w)$ is not onto Y_0, then w is called a *degenerate* solution of (2.6).

Note that if v_0 corresponds to an arc of singularities then v_0 is a degenerate solution of (2.6) and $\phi(v_0) = (k-1)!PB_k$.

In [11] it is shown that each non-degenerate solution of (2.6) directs a manifold M of solutions of (1.1) with the property that $G'(x)$ is onto Y for each $x \in M - \{0\}$. The statement "v directs M" means that M has the shape of a double cone with vertex at zero and axis span $\{v\}$.

Suppose $v_0 \in V$ is a degenerate solution of (2.6) and define V_0, Y_1 and P_1: $Y \to Y_1$ as though v_0 corresponded to an arc of singularities. Suppose in addition that (2.5) holds for some $s \geqq 2$ $(s \leqq k)$. Let V_1 be a subspace of V such that $V_1 \oplus \operatorname{span}\{v_0\} = V_0$ and set $\tilde{V} = V_1 \oplus X_0$, $X = \tilde{V} \oplus Z$.

A solution $v_1 \in V_1$ of the equation

$$P_1 G^{(k)}(0) v_0^{k-s} v^s = 0, \qquad v \in V_1, \quad v \neq 0 \tag{2.7}$$

is called *non-degenerate* if the map $\psi(v_1)$: $V_1 \to Y_1$: $v \mapsto P_1 G^{(k)}(0) v_0^{k-s} v_1^{s-1} v$ is onto Y_1. If $\psi(v_1)$ is not onto Y_1, then v_1 is called a *degenerate* solution of (2.7). If every solution of (2.7) is non-degenerate, then $v_0 \in V$ is called a *quasi-degenerate* solution of (2.6).

Note that if $m \leqq 1$ then $\dim V_1 \leqq \dim Y_1$, so that every solution of (2.7) is degenerate. Since we shall only be concerned with non-degenerate solutions of (2.7), we suppose $m \geqq 2$, as in section one.

Let $F(v) = PG^{(k)}(0)(v_0 + v)^k$ for $v \in \tilde{V}$. Using Magnus' argument ([11]) for F, we see that each non-degenerate solution v_1 of (2.7) directs a manifold $M(v_1) \subset \tilde{V}$ of small zeroes of F with the property that $F'(v)$ is onto Y_0 for each $v \in M(v_1) - \{0\}$. But then $v_0 + v \in V$ is a non-degenerate solution of (2.6) and so directs a manifold of small solutions of (1.1). However, with this analysis, we find little more information about the structure of small solutions of (1.1) than if we had studied only non-degenerate solutions of (2.6) from the beginning. Moreover, even if v_0 is a quasi-degenerate solution of (2.6), this is not enough alone to guarantee that the above analysis will catch all small solutions of (1.1) near the ray span $\{v_0\}$.

Now suppose that v_0 corresponds to an arc of singularities. We define the following sets of solutions of (1.1). Let $v \in E$ and $\rho > 0$. Set

$$S(v,\rho) = \{\alpha(v+x) \colon (\alpha, x) \in \mathbf{R} \times E, |\alpha| < \rho, \|x\| < \rho\} \cap G^{-1}(0),$$

$$\tilde{S}(v_0,\rho) = \{w(\alpha) + \alpha T(\alpha)x \colon (\alpha, x) \in \mathbf{R} \times X, |\alpha| < \rho, \|x\| < \rho\} \cap G^{-1}(0),$$

$$\tilde{D}(v_0,\rho) = \{w(\alpha) + \alpha\beta T(\alpha)(v+x) \colon$$
$$(\alpha,\beta,x) \in \mathbf{R}^2 \times X, |\alpha| < \rho, |\beta| < \rho, \|x\| < \rho\} \cap G^{-1}(0)$$

($\tilde{S}$ and $\tilde{D}$ are defined only for $\rho \leqq r$).

We shall obtain solutions of (1.1) from non-degenerate solutions of (2.7) as follows. We first establish the form of small solutions of (1.1) near an arc of singularities C (see Lemmas 2.7, 2.8) and the differentiability properties of a mapping to which we shall apply the implicit function theorem (Lemma 2.6). It is then an easy matter to show the existence of a manifold of solutions of (1.1) directed by v_0 for each non-degenerate solution of (2.7) (Theorem 3.2). The analysis is similar to the discussion above, except that stronger differentiability properties of the solutions are obtained than is possible for the more general case, and we require only one step. The difficult part is to show how completely the two definitions of non-degeneracy describe the set of small solutions of (1.1). This analysis occupies most of section three.

For the rest of this section, we suppose that $v_0 \in E$ corresponds to an arc of singularities, with $w:(-1,1)\to E$ given by (H1).

LEMMA 2.6. *Let F: $I_r \times \mathbf{R} \times V_1 \times X_0 \times Z \to Y$ be the mapping defined by*

$$F(\alpha,\beta,v,x,z)=\alpha^{-k}\beta^{-s}G(w(\alpha)+\alpha\beta T(\alpha)v+\alpha\beta^s x+\alpha^k\beta^s z) \quad \text{if} \quad \alpha\beta\neq 0,$$

$$F(0,\beta,v,x,z)=\beta^{-s}\left\{\frac{1}{k!}G^{(k)}(0)(v_0+\beta v+\beta^s x)^k+G'(0)\left(\beta M(0)v+\frac{1}{k!}w^{(k)}(0)\right)\right\}$$
$$+G'(0)z \quad \text{if} \quad \beta\neq 0,$$

$$F(\alpha,0,v,x,z)=\frac{\alpha^{s-k}}{s!}G^{(s)}(w(\alpha))(T(\alpha)v)^s+G'(w(\alpha))z$$
$$+\alpha^{1-k}G'(w(\alpha))x \quad \text{if} \quad \alpha\neq 0,$$

$$F(0,0,v,x,z)=\frac{1}{(k-s)!s!}G^{(k)}(0)v_0^{k-s}v^s+\frac{1}{(k-1)!}G^{(k)}(0)v_0^{k-1}x+G'(0)z.$$

Then F is of class C^n in the region defined by $\alpha\beta\neq 0$, of class C^{n-s} in the region defined by $\alpha\neq 0$, of class C^{n-k} everywhere, and of class C^n with respect to v,x,z everywhere.

PROOF. From the definition of k and s, we may observe that F has the desired properties except possibly at $\alpha=\beta=0$. (See [9] for details.) The Taylor expansion of $G(x+y)$ around x may be written in the form

$$G(x+y)=\sum_{j=0}^{k}(1/j!)G^{(j)}(x)y^j+g(x;y)y^k,$$

where $g(x;y)$ is of class C^{n-k} in y for each x and $g(x;0)=0$. Since G is of class C^n, $G^{(j)}(x)$ is of class C^{n-j} with respect to x for each $j=1,\cdots,k$. Thus $g(x;y)$ is of class C^{n-k} in (x,y).

Set $r(\alpha,\beta,v,x,z)=T(\alpha)v+\beta^{s-1}x+\alpha^{k-1}\beta^{s-1}z$ and substitute $x=w(\alpha)$, $y=\alpha\beta r$ into the above expansion:

$$G(w(\alpha)+\alpha\beta r(\alpha,\beta,v,x,z))=\sum_{j=1}^{k}(1/j!)G^{(j)}(w(\alpha))(\alpha\beta r(\alpha,\beta,v,x,z))^j$$
$$+\alpha^k\beta^k g(w(\alpha);\alpha\beta r)(r(\alpha,\beta,v,x,z))^k.$$

Let $f(\alpha,\beta,v,x,z)$, $h(\alpha,\beta,v,x,z)$ denote respectively the first term, and $\alpha^{-k}\beta^{-s}$

times the second term in the expansion above. Then h is of class C^{n-k} at $\alpha=\beta=0$. By (2.4), we may observe that $f(\alpha,\beta,v,x,z)$ is a polynomial in β divisible by β^s. Furthermore, f is of class C^n in α, and by (2.1) $D_\alpha^{(j)}f(0,\beta,v,x,z)=0$ for $j=1,\cdots,k-1$. Thus, the function $H\colon \mathbf{R}^2\times V_1\times X_0\times Z\to Y$ given by

$$H(\alpha,\beta,v,x,z)=\alpha^{-k}\beta^{-s}f(\alpha,\beta,v,x,z)\qquad(\alpha\neq 0),$$

$$H(0,\beta,v,x,z)=\beta^{-s}D_\alpha^{(k)}f(0,\beta,v,x,z)$$

is of class C^{n-k} at $\alpha=\beta=0$. Since $F=h+H$ and $F(\alpha,\beta,v,x,z)$ is clearly of class C^n with respect to (v,x,z) at $\alpha=\beta=0$, the proof is complete.

Let $\tilde{E}$ denote the space $\mathbf{R}\times V_1\times X_0\times Z$ with zero θ.

LEMMA 2.7. *There exists $\xi>0$ and positive constants A_1,A_2 such that if $u\in\tilde{S}(v_0,\delta)$ for $\delta\leqq\xi$ then $u=w(\alpha)+\alpha(T(\alpha)v+x+z)$ with $(\alpha,v,x,z)\in\tilde{E}$ satisfying*

(a) $\|z\|\leqq A_1|\alpha|^{k-1}(\|x\|+\|v\|^s)$,

(b) $\|x\|\leqq A_2\|v\|^s$.

Furthermore, given $\varepsilon>0$, there exists $\delta>0$ ($\delta\leqq\xi$) such that if $u\in\tilde{S}(v_0,\delta)$ as above, then

(c) $\left\|\dfrac{\alpha^k}{(k-1)!}G^{(k)}(0)v_0^{k-1}x+\alpha G'(0)z+\dfrac{\alpha^k}{(k-s)!s!}G^{(k)}(0)v_0^{k-s}v^s\right\|\leqq\varepsilon|\alpha|^k\|v\|^s$.

PROOF. Let $F_{ks}\colon V\to Y$ be the symmetric s-linear operator defined by $F_{ks}v^s=(1/((k-s)!s!))G^{(k)}(0)v_0^{k-s}v^s$.

From the proofs of Lemmas 2.5, 2.6 it is straightforward to use the Taylor expansion of $G(w(\alpha)+y)$ $(y\in E)$ around $y=0$ followed by the Taylor expansions of $G^{(j)}(w(\alpha))$ around $\alpha=0$ $(j=1,\cdots,k)$ to obtain an expansion (for $\alpha\in I_r$) of the following form:

$$\begin{aligned}G(w(\alpha)+\alpha(T(\alpha)v+x+z))&=\alpha G'(0)z+\alpha f_1(\alpha,v,x,z)z\\&\quad+\alpha^kB_kx+\alpha^kf_2(\alpha,v,x,z)x+\alpha^kF_{ks}v^s\\&\quad+\alpha^kf_3(\alpha,v,x,z)v^s.\end{aligned}\tag{2.8}$$

Here, $f_i\colon\tilde{E}\to B(E,Y)$, $i=1,2$ and $f_3\colon\tilde{E}\to B^s(E,Y)$ (the bounded s-linear operators) are continuous, and each is the zero operator at θ.

Since $G'(0)\colon Z\to R$ and $B_k\colon X_0\to R_0$ have bounded inverses, there exist $K_i>0$ $(i=1,2)$ such that $\|G'(0)z\|\geqq K_1\|z\|$ for $z\in Z$ and $\|B_kx\|\geqq K_2\|x\|$ for $x\in X_0$. Choose $\xi>0$ so that $\|f_i(\alpha,v,x,z)\|<K_i/2$ $(i=1,2)$ for $|\alpha|<\xi$,

$\| v + x + z \| < \xi$. From (2.8), A_1 may be chosen to satisfy (a), and substituting (a) into (2.8), we see that A_2 may be chosen to satisfy (b).

Now choose $\delta > 0$ ($\delta \leqq \xi$) so that $|\alpha| < \delta$, $\| v + x + z \| < \delta$ imply

$$A_1(A_2 + 1)\| f_1(\alpha, v, x, z)\| + A_2 \| f_2(\alpha, v, x, z)\| + \| f_3(\alpha, v, x, z)\| < \varepsilon.$$

Then, from (a), (b) and (2.8), it is immediate that (c) holds for this choice of δ, and the proof is complete.

LEMMA 2.8. *Let $v_1 \in V_1$ be a solution of (2.7) and let U be a subspace of V_1 such that $V_1 = U \oplus \operatorname{span}\{v_1\}$. Then given $\varepsilon > 0$, there exists $\delta > 0$ such that $u \in \tilde{D}(v_1, \delta)$ implies $u = w(\alpha) + \alpha\beta T(\alpha)(v_1 + v) + \alpha\beta^s(x_0 + x) + \alpha^k\beta^s(z_0 + z)$ where $(\alpha, \beta, v, x, z) \in \mathbf{R} \times \mathbf{R} \times U \times X_0 \times Z$, $|\alpha|$, $|\beta|$, $\|v\|$, $\|x\|$, $\|z\|$ are each less than ε; $x_0 \in X_0$ and $z_0 \in Z$ are defined uniquely by*

$$G'(0)z_0 + \frac{1}{(k-1)!} G^{(k)}(0)v_0^{k-1}x_0 + \frac{1}{(k-s)!s!} G^{(k)}(0)v_0^{k-s}v_1^s = 0. \tag{2.9}$$

PROOF. The proof is in two parts. First we show that, given $\varepsilon_0 > 0$, there exists $\delta > 0$ such that $u \in \tilde{D}(v_1, \delta)$ implies $u = w(\alpha) + \alpha\beta(T(\alpha)(v_1 + v) + x + z)$ with $(\alpha, \beta, v, x, z) \in \mathbf{R}^2 \times U \times X_0 \times Z$, $|\alpha|$, $|\beta|$, $\|v\|$, $\|x\|$, $\|z\|$ each less than ε_0.

Suppose $u \in \tilde{D}(v_1, \delta)$. Then $u = w(a) + T(a)ab(v_1 + x')$ where $(a, b, x') \in \mathbf{R}^2 \times X$, $|a|$, $|b|$, $\|x'\|$ are each less than δ. Let P_U, P_0, P_Z be projections of X onto U, X_0, Z respectively, and let f_1 be a continuous linear functional on X, such that $y \in X$ implies $y = f_1(y)v_1 + P_Uy + P_0y + P_Zy$. Set $\alpha = a$, $\beta = b(1 + f_1(x'))$, $v = (1 + f_1(x'))^{-1}P_Ux'$, $x = (1 + f_1(x'))^{-1}P_0x'$, $z = (1 + f_1(x'))^{-1}P_Zx'$. Then $u = w(\alpha) + \alpha\beta(T(\alpha)(v_1 + v) + x + z)$. Now choose $\delta > 0$ small enough that $\|f_1(x')\| < \frac{1}{2}$ if $\|x'\| < \delta$ and so that

$$\delta \max\{3/2, 2\|P_U\|, 2\|P_0\|, 2\|P_Z\|\} < \varepsilon_0.$$

Then $u \in D(v_1, \delta)$ has the desired form.

We can now complete the proof. From Lemma 2.7 (c) we see that given $\varepsilon' > 0$, there exists $\delta' > 0$ such that if $u = w(\alpha) + \alpha\beta(T(\alpha)(v_1 + v) + x + z) \in D(v_1, \delta')$ then

$$\| \alpha^k\beta B_k x + \alpha\beta G'(0)z + \alpha^k\beta^s F_{ks}(v_1 + v)^s \| \leqq \varepsilon' |\alpha|^k |\beta|^s \|v_1 + v\|^s.$$

Thus

$$\begin{aligned} &\| \alpha^{k-1}B_k(x - \beta^{s-1}x_0) + G'(0)(z - \alpha^{k-1}\beta^{s-1}z_0) + \alpha^{k-1}\beta^{s-1}(F_{ks}(v_1 + v)^s - F_{ks}v_1^s)\| \\ &\qquad \leqq \varepsilon' |\alpha|^{k-1} |\beta|^{s-1} \|v_1 + v\|^s. \end{aligned} \tag{2.10}$$

Now

$$\|F_{ks}(v_1+v)^s - F_{ks}v_1^s\| \leqq \|F_{ks}\| Q(\|v_1+v\|, \|v_1\|)\|v\|,$$

where $Q(a,b)=\sum_{i=0}^{s-1} a^{k-i+1}b^i$ is a homogeneous polynomial of degree $s-1$. Let K_1, K_2 be the positive constants defined in the proof of Lemma 2.7. Then, from (2.10)

$$K_1\|z-\alpha^{k-1}\beta^{s-1}z_0\| \leqq |\alpha|^{k-1}|\beta|^{s-1}\|I-P\|(\varepsilon'\|v_1+v\|^s + \|F_{ks}\|Q(\|v_1+v\|,\|v_1\|)\|v\|),$$

since $(I-P)B_kx=0$ for all $x\in X_0$, and

$$K_2\|x-\beta^{s-1}x_0\| \leqq |\beta|^{s-1}\|P-P_1\|(\varepsilon'\|v_1+v\|^s + \|F_{ks}\|Q(\|v_1+v\|,\|v_1\|)\|v\|),$$

since $PG'(0)z = P_1G'(0)z = 0$ for all $z \in Z$.

Now choose $\varepsilon'>0$ so that $\|I-P\|\|v_1\|^s\varepsilon' < (\varepsilon/4)K_1$ and $\|P-P_1\|\|v_1\|^s\varepsilon' < (\varepsilon/4)K_2$, and let $\delta'>0$ be chosen as above. Choose $\varepsilon_0>0$ so that $\varepsilon_0<\varepsilon$ and if $\|v\|<\varepsilon_0$ then $\|F_{ks}\|Q(\|v_1+v\|,\|v_1\|)\|v\| < (\varepsilon/2)K_i$ $(i=1,2)$ and $\|v_1+v\|^s < 2\|v_1\|^s$. Finally choose $\delta>0$ as in the first part of the proof, for this choice of ε_0, and so that $\delta<\delta'$. Then $u\in\tilde{D}(v_1,\delta)$ implies $u\in\tilde{D}(v_1,\delta')$ and $u = w(\alpha)+\alpha\beta(T(\alpha)(v_1+v)+\bar{x}+\bar{z})$ with $(v,\bar{x},\bar{z})\in U\times X_0\times Z$, $|\alpha|, |\beta|, \|v\|$ each less than ε,

$$\|\bar{x}-\beta^{s-1}x_0\| < \beta^{s-1}\varepsilon \quad\text{and}\quad \|\bar{z}-\alpha^{k-1}\beta^{s-1}z_0\| < |\alpha|^{k-1}|\beta|^{s-1}\varepsilon.$$

Set $x = \beta^{-s+1}\bar{x}-x_0$ and $z=\alpha^{-k+1}\beta^{-s+1}\bar{z}-z_0$. Then $\|x\|<\varepsilon$, $\|z\|<\varepsilon$ and $u = w(\alpha)+\alpha\beta T(\alpha)(v_1+v)+\alpha\beta^s(x_0+x)+\alpha^k\beta^s(z_0+z)$.

LEMMA 2.9. *Given $\varepsilon>0$ there exists $\delta>0$ such that $S(v_0,\delta)\subset\tilde{S}(v_0,\varepsilon)$.*

PROOF. First observe from Lemma 2.4 that $T(\alpha)$ maps X to X, is continuous in $\alpha\in I_r$, and $T(0)x=x$ for each $x\in X$. Therefore, there exists $\varepsilon'>0$ such that if $|\alpha|<\varepsilon'$ then $\|T(\alpha)x\|\geqq\frac{1}{2}\|x\|$ for all $x\in X$. Thus, it is sufficient to show that given $\varepsilon'>0$, there exists $\delta>0$ such that each $u\in S(v_0,\delta)$ may be written in the form $u=w(\alpha)+\alpha x$ with $|\alpha|<\varepsilon'$ and $\|x\|<\varepsilon'$. Then $u = w(\alpha)+\alpha T(\alpha)[T(\alpha)]^{-1}x$ and $\|[T(\alpha)]^{-1}x\|\leqq 2\|x\|<2\varepsilon'$. Choosing $\varepsilon'<\varepsilon/2$, we have $u\in\tilde{S}(v_0,\varepsilon)$.

Let $f\colon E\to\mathbf{R}$ be a continuous linear functional satisfying $f(v_0)=1$, $f(x)=0$ for each $x\in X$. Consider the map $\phi\colon I_r\to\mathbf{R}$ given by $\phi(\alpha)=f(w(\alpha))$. ϕ is continuously differentiable, $\phi(0)=0$ and $\phi'(0)=1$. Thus ϕ has a continuous inverse ϕ^{-1} on an interval I_η $(\eta>0)$ and $\phi^{-1}(0)=0$. Choose $\delta>0$ so that $3\delta/2<\eta$, $|\phi^{-1}(a)|<\varepsilon'$ for $|a|<3\delta/2$ and $|f(v)|<\frac{1}{2}$ for $v\in E$, $\|v\|<\delta$. Suppose

$u \in S(v_0, \delta)$, so that $u = \alpha(v_0 + v)$ with $|\alpha| < \delta$ and $\|v\| < \delta$. Let $\alpha' = \alpha(1 + f(v))$. Then $|\alpha'| < 3\delta/2$ so set $\xi = \phi^{-1}(\alpha')$. Then $|\xi| < \varepsilon'$. Since

$$\alpha(v_0 + v) = w(\xi) + \xi\{\alpha' v_0 - w(\phi^{-1}(\alpha')) + \alpha(v - f(v)v_0)\}/\phi^{-1}(\alpha')$$

we need only show that $\delta > 0$ may be chosen small enough that

$$\|av_0 - w(\phi^{-1}(a))\| < (\varepsilon'/2)|\phi^{-1}(a)| \quad \text{if } |a| < 3\delta/2 \tag{2.11}$$

and

$$|a|\,\|v - f(v)v_0\| < (\varepsilon'/2)|\phi^{-1}(a(1 + f(v)))| \quad \text{if } |a| < \delta, \|v\| < \delta. \tag{2.12}$$

Now $w(\xi) = \xi v_0 + \xi t(\xi)$ where $t\colon I_r \to E$ is continuous and $t(0) = 0$. Thus, if $\xi = \phi^{-1}(a)$, then $a = \xi + \xi f(t(\xi))$ and so

$$av_0 - w(\phi^{-1}(a)) = \phi^{-1}(a)\{(f \circ t \circ \phi^{-1})(a)v_0 - (t \circ \phi^{-1})(a)\}.$$

Choose $\delta > 0$ so that $|(f \circ t \circ \phi^{-1})(a)|\,\|v_0\| < \varepsilon'/4$ and $\|t \circ \phi^{-1}(a)\| < \varepsilon'/4$ for $|a| < 3\delta/2$. Then we have proved (2.11).

To prove (2.12), note that

$$(1 + f(v))a = \phi^{-1}(a(1 + f(v)))\{1 + (f \circ t \circ \phi^{-1})(a(1 + f(v)))\}.$$

Thus, if $|a| < \delta$, $\|v\| < \delta$, then $|a| < 2(1 + \varepsilon'/4\|v_0\|)|\phi^{-1}(a(1 + f(v)))|$. Now choose $\delta > 0$ as above and so that $\|v - f(v)v_0\| < \varepsilon'\|v_0\|/(4\|v_0\| + \varepsilon')$ if $\|v\| < \delta$. With this choice of $\delta > 0$, (2.12) holds, and the proof of the lemma is complete.

3. The main results

THEOREM 3.1 (Magnus [11]). *Let $w_0 \in V$ be a non-degenerate solution of* (2.6) *and let W be a subspace of V such that* $\mathrm{span}\{w_0\} \oplus W = N(\phi(w_0))$. *Then there exists $r > 0$ and a function g from $A_r = \{(\alpha, w) \in \mathbf{R} \times W : |\alpha| < r, \|w\| < r\}$ into E such that $g(0,0) = 0$ and $G(g(A_r)) = \{0\}$. Furthermore, g is of class C^{n-k} on A_r, and of class C^n on $\tilde{A}_r = \{(\alpha, w) \in A_r : \alpha \neq 0\}$, and there exists $\rho > 0$ such that $S(w_0, \rho) \subset g(A_r)$.*

If $0 < \varepsilon \leqq r$, let $\sigma(w_0, \varepsilon)$ denote the set $g(\tilde{A}_\varepsilon)$.

As shown earlier, if $v_0 \in E$ corresponds to an arc C of singularities, as defined by (H1), (H2), then $v_0 \in V$ and v_0 is a degenerate solution of (2.6). In this case, Theorem 3.1 does not apply, but the following theorem establishes that each non-degenerate solution of (2.7) yields a set of solutions of (1.1) near C.

THEOREM 3.2. *Suppose $v_0 \in E$ corresponds to an arc of singularities, and that $v_1 \in V_1$ is a non-degenerate solution of* (2.7). *Let U_0, W be subspaces of V_1 such* *that* $W \oplus \operatorname{span}\{v_1\} = N(\psi(v_1))$ *and* $N(\psi(v_1)) \oplus U_0 = V_1$. *Then there exists* $r > 0$ *and a function h from the ball* $B_r = \{(\alpha, \beta, w) \in \mathbf{R}^2 \times W : |\alpha| < r, |\beta| < r, \|w\| < r\}$ *into E such that*

(a) $h(0,0,0) = 0$,

(b) $G(h(\alpha, \beta, w) = 0$ *for each* $(\alpha, \beta, w) \in B_r$,

(c) *h is of class* C^{n-k} *on* B_r, *of class* C^{n-s} *on* $\tilde{B}_r = \{(\alpha, \beta, w) \in B_r : \beta \neq 0\}$ *and of class* C^n *on* $B'_r = \{(\alpha, \beta, w) \in B_r : \alpha\beta \neq 0\}$,

(d) *there exists* $\rho > 0$ *such that* $\tilde{D}(v_1, \rho) \subset h(B_r)$.

Finally, h has an explicit form as follows:

$$h(\alpha, \beta, w) = w(\alpha) + \alpha\beta T(\alpha)(v_1 + w + \hat{u}(\alpha, \beta, w)) + \alpha\beta^s(x_0 + \hat{x}(\alpha, \beta, w))$$
$$+ \alpha^k \beta^s (z_0 + \hat{z}(\alpha, \beta, w)),$$

where $x_0 \in X_0$, $z_0 \in Z$ *are given by* (2.9), *and* $\hat{u}, \hat{x}, \hat{z}$ *are functions from* B_r *into* U_0, X_0, Z *respectively such that* $\hat{u}(0,0,0) = \hat{x}(0,0,0) = \hat{z}(0,0,0) = 0$. $\hat{u}, \hat{x}, \hat{z}$ *are of class* C^{n-k} *on* B_r, C^{n-s} *on* $\tilde{B}_r$ *and of class* C^n *on* B'_r.

Proof. Let $F: \mathbf{R}^2 \times V_1 \times X_0 \times Z \to Y$ be the function defined in Lemma 2.6. Then $F(0,0,v_1,x_0,z_0) = 0$ and the Frechet derivative of F with respect to (v, x, z) at the point $(0,0,v_1,x_0,z_0)$ is the linear mapping $H: V_1 \times X_0 \times Z \to Y$ given by $H(v,x,z) = sF_{ks}v_1^{s-1}v + B_k x + G'(0)z$. By definition of U_0, X_0, Z, H is a linear homeomorphism between $U_0 \times X_0 \times Z$ and Y. An application to F of the implicit function theorem, as stated in [11], proves all the conclusions of the theorem except (d), noting that if $(\alpha, \beta, w) \in B_r$ and $\alpha\beta = 0$, then $h(\alpha, \beta, w) \in C$ so that $G(h(\alpha, \beta, w)) = 0$.

To prove (d) note that, by the implicit function theorem (and continuity), there exists $\lambda > 0$ such that $\|\hat{u}(\alpha, \beta, w)\|$, $\|\hat{x}(\alpha, \beta, w)\|$, $\|\hat{z}(\alpha, \beta, w)\|$ are each less than λ for all $(\alpha, \beta, w) \in B_r$, and so that if $F(\alpha, \beta, v_1 + w + u, x_0 + x, z_0 + z) = 0$ with $(\alpha, \beta, w) \in B_r$, $u \in U_0$, $x \in X_0$, $z \in Z$ each with norm less than λ, then $u = \hat{u}(\alpha, \beta, w)$, $x = \hat{x}(\alpha, \beta, w)$ and $z = \hat{z}(\alpha, \beta, w)$. Set $\varepsilon = \min\{\lambda, r\}$ and choose $\rho > 0$ less than δ as in Lemma 2.8 so that each $y \in \tilde{D}(v_1, \rho)$ may be written as $y = w(\alpha) + \alpha\beta T(\alpha)(v_1 + w + u) + \alpha\beta^s(x_0 + x) + \alpha^k\beta^s(z_0 + z)$ with $(\alpha, \beta, w, u, x, z) \in \mathbf{R}^2 \times W \times U_0 \times X_0 \times Z$, each with norm less than ε. Then either $\alpha\beta = 0$ or $F(\alpha, \beta, v_1 + w + u, x_0 + x, z_0 + z) = 0$, so that $y \in h(B_r)$. Thus $\tilde{D}(v_1, \rho) \subset h(B_r)$ and the proof is complete.

For $0 < \varepsilon \leqq r$ let B'_ε denote the set

$$\{(\alpha, \beta, w) \in \mathbf{R}^2 \times W : |\alpha| < \varepsilon, |\beta| < \varepsilon, \|w\| < \varepsilon, \alpha\beta \neq 0\}$$

and define $\tau(v_1, \varepsilon) = h(B'_\varepsilon)$. Clearly, for each $\rho > 0$ there exists $\varepsilon > 0$ $(\varepsilon \leqq r)$

such that $\tau(v_1, \varepsilon) \subset S(v_0, \rho)$, and, for each $\varepsilon > 0$ ($\varepsilon \leqq r$), there exists $\xi > 0$ such that $\tilde{D}(v_1, \xi) \subset \tau(v_1, \varepsilon) \cup C$.

Now theorem 7 of [11] states that if every solution of (2.6) is non-degenerate then there exists $\gamma > 0$ such that the set of solutions of (1.1) with norm less than γ lies in the set $\overline{\bigcup \sigma(w_0, r)}$, where the union is over a finite number of solutions $w_0 \in V$ of (2.6) and $r = r(w_0)$ is given in Theorem 3.1.

Since $v_0 \in V$ in Theorem 3.2 is a degenerate solution of (2.6) this result no longer applies, but we shall show that a corresponding result holds. That is, if every solution of (2.7) is non-degenerate, then there exists $\rho = \rho(v_0) > 0$ such that $S(v_0, \rho) \subset \overline{\bigcup \tau(v_1, r)}$, where the union is over a finite number of solutions $v_1 \in V_1$ of (2.7) and $r = r(v_1)$ is given in Theorem 3.2.

The following theorem, for non-degenerate solutions of (2.7), is the analogue of theorems 2–5 of [11], for non-degenerate solutions of (2.6). Let $J(v_0) = \{v \in V_1 \colon P_1 G^{(k)}(0) v_0^{k-s} v^s = 0, \| v \| = 1\}$.

THEOREM 3.3. (i) *The mapping* f *given by*

$$f(\alpha, \beta, w, u, x, z) = h(\alpha, \beta, w) + u + x + z$$

is a homeomorphism of $B_r' \times U_0 \times X_0 \times Z$ *onto an open subset of* E.

(ii) *There exists* $\varepsilon > 0$ *such that, for each* $x \in \tau(v_1, \varepsilon)$, $G'(x)$ *has range* Y; $N(G'(x))$ *is complementary to* $U_0 \oplus X_0 \oplus Z$ *in* E *and linearly homeomorphic to* $N(\psi(v_1)) \oplus \operatorname{span}\{v_0\}$.

(iii) *There exists* $\varepsilon > 0$ *such that* $\tau(v_1, \varepsilon)$ *is an open subset of the metric space* $G^{-1}(0)$.

(iv) *Let* I *be a subset of* $J(v_0)$ *consisting of non-degenerate solutions of* (2.7). *For each* $v \in I$, *let* $\tau(v, r)$ *be the set of solutions of* (1.1) *given by Theorem* 3.2, *and choose* $\varepsilon = \varepsilon(v) > 0$ *as in* (ii), (iii) *above. Then the set* $M = \bigcup_{v \in I} \tau(v, \varepsilon(v))$ *is a submanifold of* E *of class* C^n.

PROOF. The proof is a straightforward generalisation of the proof of corresponding results in [11].

(i) Let S denote the set of elements of the form $w(\alpha) + \alpha\beta T(\alpha)(v_1 + w) + u + x + z$ with $0 < |\alpha| < r$, $0 < |\beta| < r$ and $\| w \| < r$. Then S is open and $R(f) \subset S$. Suppose $y = w(\alpha') + \alpha'\beta' T(\alpha')(v_1 + w') + u' + x' + z' \in S$. Then $f(\alpha, \beta, w, u, x, z) = y$ is solved uniquely by

$$\alpha = \alpha', \quad \beta = \beta', \quad w = w',$$

$$u = u' - \alpha'\beta'\hat{u}(\alpha', \beta', w'),$$

$$x = x' - \alpha'(\beta')^s(x_0 + \hat{x}(\alpha', \beta', w')) - \alpha'\beta' L(\alpha')\hat{u}(\alpha', \beta', w'),$$

$$z = z' - (\alpha')^k(\beta')^s(z_0 + \hat{z}(\alpha', \beta', w')) - \alpha'(\beta')^k M(\alpha')\hat{u}(\alpha', \beta', w').$$

These equations define the inverse of f which is clearly of class C^n.

(ii) For $(\alpha, \beta, w) \in B_r$ set $\hat{v}(\alpha, \beta, w) = v_1 + w + \hat{u}(\alpha, \beta, w)$ and define mappings $J(\alpha, \beta, w)$: $U_0 \to Y$ and $K(\alpha, \beta, w)$: $X_0 \to Y$ by

$$J(\alpha, \beta, w)u = \alpha^{-k+1}\beta^{-s+1}G'(h(\alpha, \beta, w))u \quad \text{if} \quad \alpha\beta \neq 0,$$

$$J(0, \beta, w)u = \frac{\beta^{-s+1}}{(k-1)!} G^{(k)}(0)(v_0 + \hat{v}(0, \beta, w) + \beta^s(x_0 + \hat{x}(0, \beta, w)))^{k-1}u$$

$$\text{if} \quad \beta \neq 0,$$

$$J(\alpha, 0, w)u = \frac{\alpha^{-k+s}}{(s-1)!} G^{(s)}(w(\alpha))(T(\alpha)\hat{v}(\alpha, 0, w))^{s-1}u \quad \text{if} \quad \alpha \neq 0,$$

$$J(0, 0, w)u = \frac{1}{(k-s)!(s-1)!} G^{(k)}(0)v_0^{k-s}(\hat{v}(0, 0, w))^{s-1}u,$$

$$K(\alpha, \beta, w)x = \alpha^{-k+1}G'(h(\alpha, \beta, w))x \quad \text{if} \quad \alpha \neq 0,$$

$$K(0, \beta, w)x = \frac{1}{(k-1)!} G^{(k)}(0)(v_0 + \hat{v}(0, \beta, w) + \beta^s(x_0 + \hat{x}(0, \beta, w)))^{k-1}x.$$

Then K and J depend continuously on $(\alpha, \beta, w) \in B_r$, and for $(\alpha, \beta, w) \in B'_r$,

$$G'(h(\alpha, \beta, w))(u + x + z) = G'(h(\alpha, \beta, w))z + \alpha^{k-1}\beta^{s-1}J(\alpha, \beta, w)u + \alpha^{k-1}K(\alpha, \beta, w)x. \tag{3.1}$$

Now the map $u + x + z \to G'(0)z + J(0,0,0)u + K(0,0,0)x$ is a linear homeomorphism between $U_0 \oplus X_0 \oplus Z$ and Y. Thus, there exists $\varepsilon > 0$ such that the map $u + x + z \to G'(h(\alpha, \beta, w))z + J(\alpha, \beta, w)u + K(\alpha, \beta, w)x$ is also a linear homeomorphism between $U_0 \oplus X_0 \oplus Z$ and Y for each $(\alpha, \beta, w) \in B_\varepsilon$. By composing this mapping with the invertible map $(u, x, z) \to (\alpha^{k-1}\beta^{s-1}u, \alpha^{k-1}x, z)$ from $U_0 \times X_0 \times Z$ into itself for each (α, β), $\alpha\beta \neq 0$ we obtain (3.1), and this proves (ii).

(iii) Let $\varepsilon > 0$ be as in (ii) and suppose there exists a sequence $\{y_n\} \subset G^{-1}(0) - \tau(v_1, \varepsilon)$ such that $y_n \to y \in \tau(v_1, \varepsilon)$ as $n \to \infty$. Since $y \in R(f)$, we may suppose that $y_n \in R(f)$ for each n. Let $\xi_n = (\alpha_n, \beta_n, w_n, u_n, x_n, z_n) \in B'_r \times U_0 \times X_0 \times Z$ be such that $y_n = f(\xi_n)$, $\alpha_n\beta_n \neq 0$, and at least one of $\|u_n\|$, $\|x_n\|$, $\|z_n\|$ is non-zero (since $y_n \notin \tau(v_1, \varepsilon)$). Then $\alpha_n \to \alpha$, $\beta_n \to \beta$, $w_n \to w$, $u_n \to 0$, $x_n \to 0$ and $z_n \to 0$, where $y = f(\alpha, \beta, w, 0, 0, 0) = h(\alpha, \beta, w)$. Let $H(\xi) = G(f(\xi))$ for

$\xi \in B_r' \times U_0 \times X_0 \times Z_0$. Then H is continuously differentiable, and given $\delta > 0$, the following holds for all sufficiently large n:

$$\|H(\xi_n) - H(\alpha_n, \beta_n, w_n, 0,0,0) - DH(\alpha, \beta, w, 0,0,0)\cdot(0,0,0,u_n, x_n, z_n)\| \leqq \delta(\|u_n\| + \|x_n\| + \|z_n\|).$$

Since $H(\xi_n) = H(\alpha_n, \beta_n, w_n, 0,0,0) = 0$ for each n, and $DH(\alpha, \beta, w, 0,0,0)\cdot(0,0,u_n, x_n, z_n) = G'(h(\alpha, \beta, w))(u_n + x_n + z_n)$, we have

$$\|G'(y)(u_n + x_n + z_n)\| \leqq \delta(\|u_n\| + \|x_n\| + \|z_n\|).$$

But this contradicts (ii) if we choose $\delta > 0$ small enough.

(iv) By (iii), M is an open subset of $G^{-1}(0)$ and, by (ii), for each $x \in M$, $G'(x)$ is onto Y and $N(G'(x))$ possess a complementary subspace in E. The result now follows from standard differential topology.

LEMMA 3.4. *Let v_0 correspond to an arc C of singularities.*

(a) *If $x_0 \neq 0$ and $\tilde{D}(x_0, \varepsilon) - C$ is non-empty for all $\varepsilon > 0$, then $x_0 \in V_1$ and $x_0/\|x_0\| \in J(v_0)$.*

(b) *If $J(v_0)$ is empty then there exists $\rho > 0$ such that $S(v_0, \rho) \subset C$.*

(c) *If $J(v_0)$ is non-empty, then given $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in S(v_0, \delta)$ then $x \in \tilde{D}(v_1, \varepsilon)$ for some $v_1 \in J(v_0)$.*

PROOF. (a) There exist sequences $\{(\alpha_n, \beta_n)\} \subset \mathbf{R}^2$, $\{x_n\} \subset X$ such that $\alpha_n\beta_n \neq 0$, $\alpha_n \to 0$, $\beta_n \to 0$, $x_n \to 0$ as $n \to \infty$, and

$$G(w(\alpha_n) + \alpha_n T(\alpha_n)(v_0 + \beta_n(x_0 + x_n))) = 0.$$

Set $x_0 = w_0 + y_0$, $x_n = w_n + y_n$ with $w_i \in V_1$, $y_i \in X_0 \oplus Z$ for $i \geqq 0$. By Lemma 2.7 (a), (b), there exist constants $c_1 > 0$, $c_2 > 0$ such that

$$\|\beta_n(y_0 + y_n)\| \leqq c_1|\alpha_n|^{k-1}\|\beta_n(w_0 + w_n)\|^s + c_2\|\beta_n(w_0 + w_n)\|^s$$

for all sufficiently large n. Thus

$$\|y_0 + y_n\| \leqq (c_1|\alpha_n|^{k-1} + c_2)|\beta_n|^{s-1}\|w_0 + w_n\|^s.$$

Let $n \to \infty$. Since $s \geqq 2$, we have $y_0 = 0$.

Now let $y_n = u_n + z_n$ with $u_n \in X_0$, $z_n \in Z$. By Lemma 2.7 (c), given $\varepsilon > 0$

$$\|\beta_n\alpha_n^k B_k u_n + \alpha_n\beta_n G'(0)z_n + \alpha_n^k\beta_n^s F_{ks}(w_0 + w_n)^s\| \leqq \varepsilon|\alpha_n|^k|\beta_n|^s\|w_0 + w_n\|^s$$

for all sufficiently large n. Since $\alpha_n\beta_n \neq 0$,

$$\|P_1 F_{ks}(w_0 + w_n)^s\| \leqq \varepsilon\|P_1\|\|w_0 + w_n\|^s.$$

Now let $n \to \infty$. Then $\|P_1 G^{(k)}(0) v_0^{k-s} w_0^s\| < (k-s)! s! \varepsilon \|P_1\| \|w_0\|^s$ and since ε is arbitrary, $w_0 / \|w_0\| \in J(v_0)$.

(b) By Lemma 3.2 and Lemma 2.7 (c), given $\varepsilon > 0$ there exists $\delta > 0$ such that $u \in \tilde{S}(v_0, \delta) - C$ implies $u = w(\alpha) + \alpha T(\alpha)(v + y)$ with $\alpha \in \mathbf{R}$, $v \in V_1 - \{0\}$, $y \in X_0 \oplus Z$ and $\|P_1 F_{ks}(v / \|v\|)^s\| < \varepsilon$. Let $\Lambda = \{v \in V_1 : \|v\| = 1\}$. Then Λ is a compact metric space. By lemma 4 (a) of [11], there exists $\gamma^1 > 0$ such that $\|P_1 E_{ks} v^s\| > \gamma^1$ for all $v \in \Lambda$. Hence, there exists $\gamma > 0$ such that $\tilde{S}(v_0, \gamma) \subset C$. By Lemma 3.2, there exists $\rho > 0$ such that $S(v_0, \rho) \subset \tilde{S}(v_0, \gamma)$.

(c) By lemma 4(b) of [11], given $\varepsilon > 0$, there exists $\gamma > 0$ such that if $u \in \tilde{S}(v_0, \gamma) - C$, $u = w(\alpha) + \alpha T(\alpha)(v + y)$, $\alpha \in \mathbf{R}$, $v \in V_1$, $y \in X_0 \oplus Z$, then $\|v\| \neq 0$ and $\|v / \|v\| - v_1\| < \frac{1}{2}\varepsilon$ for some $v_1 \in J(v_0)$. Now use Lemma 2.7 (a), (b) to choose $\gamma > 0$ as above, less than ε and so that $\|v\| < \varepsilon$, $\|y\| < \varepsilon \|v\| / 2$.

Then

$$u = w(\alpha) + T(\alpha)\alpha \|v\| (v_1 + v / \|v\| - v_1 + y / \|v\|)), \qquad \|v\| < \varepsilon$$

and $\|v / \|v\| - v_1 + y / \|v\|\| < \varepsilon$. Thus $u \in \tilde{D}(v_1, \varepsilon)$. Clearly $\tilde{S}(v_0, \gamma) \cap C \subset \tilde{D}(v_1, \varepsilon)$. By Lemma 3.2, there exists $\delta > 0$ such that $S(v_0, \delta) \subset \tilde{S}(v_0, \gamma)$, and the proof is complete.

THEOREM 3.5. *Suppose $v_0 \in V$ is a quasi-degenerate solution of (2.6) corres ponding to an arc C of singularities. Then there exist $\xi > 0$, a finite subset $\{v_1, \cdots, v_N\}$ of $J(v_0)$ and a corresponding set $\{\tau(v_1, \varepsilon_1), \cdots, \tau(v_N, \varepsilon_N)\}$ such that*

$$S(v_0, \xi) \subset \bigcup_{i=1}^{N} \tau(v_i, \varepsilon_i) \cup C.$$

PROOF. For each $v \in J(v_0)$ choose $\varepsilon(v) > 0$ so that $\tau(v, \varepsilon(v))$ satisfies the conclusion of Theorem 3.3 (iii) and let $\rho(v) > 0$ be such that $\tilde{D}(v, \rho(v)) \subset \tau(v, \varepsilon(v)) \cup C$. For each $v \in J(v_0)$, let $K(v)$ be the set $\{w \in J(v_0)$: $\|v - w\| < \frac{1}{2}\rho(v)\}$. Since $J(v_0)$ is compact, there exist $v_1, \cdots, v_N$ in $J(v_0)$ such that $J(v_0) = K(v_1) \cup \cdots \cup K(v_N)$.

Set $\varepsilon_i = \varepsilon(v_i)$ and $\varepsilon = \frac{1}{2}\min\{\rho(v_i) : 1 \leqq i \leqq N\}$. By Lemma 3.4 (c), there exists $\xi > 0$ such that if $x \in S(v_0, \xi)$ then $x \in \tilde{D}(v, \varepsilon)$ for some $v \in J(v_0)$. Thus $x = w(\alpha) + \alpha\beta T(\alpha)(v + x')$ with $|\alpha|$, $|\beta|$, $\|x'\|$ less than ε. But $v \in K(v_i)$ for some $i = 1, \cdots, N$, so that $x \in \tilde{D}(v_i, \varepsilon + \frac{1}{2}\rho(v_i))$. Thus $x \in \tilde{D}(v_i, \rho(v_i)) \subset \tau(v_i, \varepsilon(v_i)) \cup C$, which completes the proof.

Under the conditions of Theorem 3.5, let $\Pi(v_0)$ denote the set $\bigcup_{i=1}^{N} \tau(v_i, \varepsilon_i) \cup C$. Let J denote the set of solutions of (2.6) with unit norm. Note that if $v_0 \in V$ corresponds to an arc of singularities C, then $v_0 / \|v_0\| \in J$ and $v_0 / \|v_0\|$ also corresponds to C.

THEOREM 3.6. *Suppose every solution of (2.6) is either non-degenerate or quasi-degenerate, and corresponds to an arc of singularities. Then there exist finite subsets* $\{w_1, \cdots, w_p\}$, $\{u_1, \cdots, u_q\}$ *of J consisting of non-degenerate and quasi-degenerate elements respectively, together with corresponding sets* $\sigma(w_i, \varepsilon_i)$ $(\varepsilon_i > 0)$, $i = 1, \cdots, p$ *and* $\Pi(u_i)$, $i = 1, \cdots, q$ *such that the set* $\Omega = \bigcup_{i=1}^{p} \sigma(w_i, \varepsilon_i) \cup \bigcup_{i=1}^{q} \Pi(u_i)$ *has the following property*:

There exists $\gamma > 0$ *such that if* $x \in E$, $G(x) = 0$ *and* $\|x\| < \gamma$, *then* $x \in \Omega$.

PROOF. If v is a non-degenerate element of J, choose $\varepsilon(v)$, $\xi(v)$ greater than zero as in [11], so that $S(v, \xi(v)) \subset \sigma(v, \varepsilon(v))$. If v is a quasi-degenerate element of J, choose $\xi(v) > 0$ as in Theorem 3.5 so that $S(v, \xi(v)) \subset \Pi(v)$. For $v \in J$, let $L(v)$ be the set $\{w \in J : \|v - w\| < \frac{1}{2}\xi(v)\}$. Then there exist $v_1, \cdots, v_M$ in J such that $J = L(v_1) \cup \cdots \cup L(v_M)$.

Set $\varepsilon = \frac{1}{2}\min\{\xi(v_i): 1 \leqq i \leqq M\}$. By Theorem 6 (c) of [11], there exists $\gamma > 0$ such that if $x \in E$, $G(x) = 0$ and $\|x\| < \gamma$, then $x \in S(v, \varepsilon)$ for some $v \in J$. Thus $x = \alpha(v + x')$ with $|\alpha| < \varepsilon$, $\|x'\| < \varepsilon$. But $v \in L(v_i)$ for some $i = 1, \cdots, M$, and the remainder of the proof is identical to that of the previous theorem.

Concluding remarks

In order to set bifurcation problems in the form (1.1), with G satisfying (2.1) for some k, and so that $PG^{(k)}(0)$ possesses non-degenerate zeroes, it is often necessary to rescale the parameter space (see for instance [9], [14]). Furthermore, in order to obtain the condition (2.4) together with non-degenerate solutions of (2.7), it may be necessary to further rescale separately for each curve of singularities. This does not affect Theorems 3.2 or 3.5, but Theorem 3.6 needs modification to cope with the general situation. However, in applications, the number of curves of singularities is invariably finite and both the rescaling and the modification of Theorem 3.6 required is obvious. An abstract formulation of rescaling for the case of an isolated singular point is described in [10].

Non-isolated singular points and degenerate solutions of (2.6) can occur as a result of a variety of types of degeneracy in G. The recent paper of Buchner, Marsden and Schecter [2] considers a class of such degeneracies with $k = 2$, and presents a generalised non-degeneracy condition (called pretty general position) defined with respect to known degenerate directions for G, under which the existence of curves of singular points of (1.1) through zero is established.

ACKNOWLEDGEMENT

The author would like to thank Professor J. K. Hale for the helpful discussions about bifurcation theory which prompted this paper. The author would also like to acknowledge the support of the Science Research Council.

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