NOTES ON SPLINE FUNCTIONS IV: A CARDINAL SPLINE ANALOGUE OF THE THEOREM OF THE BROTHERS MARKOV[†]

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ABSTRACT

A cardinal spline analog of the Markov theorem is given. It is applied to derive the necessary conditions for a function to be the limit of its cardinal spline interpolents as their degree trends to infinity. Sufficient conditions for this to happen are given in [8].

1. Introduction

DEFINITION 1. We will define three classes $\mathscr{S}_n$, $\mathscr{S}_n^*$, and $\tilde{\mathscr{S}}_n$ whose elements are cardinal spline functions or, simply, cardinal splines.

Let n be a non-negative integer and let $\mathscr{S}_n = \{S(x)\}$ denote the class of functions $f(x)$ from $\mathbb{R}$ to $\mathbb{C}$ satisfying the conditions:

(i) $S(x) \in C^{n-1}(\mathbb{R})$, and (ii) the restriction of $S(x)$ to every interval $[v, v+1)$ $(v = 0, \pm 1, \cdots)$ is an element of π_n. (Here π_n denotes the class of polynomials of degree not exceeding n.)

We also define the classes

$$\mathscr{S}_n^* = \{S(x);\, S(x+\tfrac{1}{2}) \in \mathscr{S}_n\},$$

and finally

$$\tilde{\mathscr{S}}_n = \begin{cases} \mathscr{S}_n & \text{if } n \text{ is odd} \\ \mathscr{S}_n^* & \text{if } n \text{ is even.} \end{cases}$$

For purposes of this paper we restate a fundamental theorem of Subbotin [9] as Theorem 2.

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THEOREM 2. ([9].) *If* (y_ν) *is a bounded bi-infinite sequence of numbers* $(\nu = 0, \pm 1, \cdots)$, *then there exists a unique* $S_n(x) \in \tilde{\mathscr{S}}_n$, *which is bounded on* $\mathbb{R}$ *such that*

$$S_n(\nu) = y_\nu \text{ for all } \nu. \tag{1}$$

For extensions of Theorem 2 to sequences and splines of power growth see [5] and [7, Lec. 4].

If $f(x)$ is a bounded function defined on $\mathbb{R}$, then Theorem 2 implies that there is a unique bounded spline $S_n(x)$ that interpolates $f(x)$ at all the integers. The question arises: which conditions insure that the interpolant $S_n(x)$ will converge to $f(x)$ as n approaches infinity? In this direction the following sufficient conditions are known.

THEOREM 3. ([8].) *If*

$$f(x) = \int_{-\pi}^{\pi} e^{iux} d\alpha(u), \tag{2}$$

where

(3) $\alpha(u)$ *is of bounded variation, with* $\alpha(-\pi + 0) - \alpha(-\pi) = \alpha(\pi) - \alpha(\pi - 0)$, *then for the interpolants of odd degree we have*

$$\lim_{m \to \infty} S_{2m-1}(x) = f(x) \text{ uniformly on } \mathbb{R}. \tag{4}$$

One of the main contributions of the present note is to derive the following necessary conditions for the validity of (4).

THEOREM 4. *If*

$$f(x) \text{ is bounded on } \mathbb{R} \tag{5}$$

and if (4) *holds, then*

(6) $f(x)$ *is the restriction to* $\mathbb{R}$ *of an entire function* $f(z)$ *of exponential type* $\leqq \pi$.

(A proof of Theorem 4 is given in Section 4.) From (5) and (6) it follows (by a theorem of Boas [1, p. 107]) that $f(x)$ admits a representation of the form

$$f(x) = \lim_{n \to \infty} \int_{-\pi}^{\pi} e^{iux} d\alpha_n(u) \text{ locally uniformly,} \tag{7}$$

where each $\alpha_n(u)$ is of bounded variation on $[-\pi, \pi]$.

The closing of the gap between the sufficient conditions (2), (3), and the necessary conditions (5), (6) is an open problem.

At this point we restate the theorem of the brothers Markov as Theorem 5 where we use the sup-norm for the interval $[-1,1]$.

THEOREM 5. *If* $P(x)\in\pi_n$ *and* $\|P\|\leqq 1$, *then*

$$\|P^{(\nu)}\|\leqq T_n^{(\nu)}(1) \text{ for } \nu=1,2,\cdots,n.$$

For the best proof of Theorem 5 and references see [3].

The main tool in our proof of Theorem 4 is an analog of Theorem 5 for cardinal splines. A few preliminary remarks are in order. If we apply Theorem 2 to the special sequence $y_\nu=(-1)^\nu$, then the solution $S_n(x)$ is the so-called Euler spline $\mathscr{E}_n(x)$. This is a special cardinal spline that is uniquely characterized by the conditions

$$\mathscr{E}_n(\nu)=(-1)^\nu \text{ for all } \nu,\ \|\mathscr{E}_n\|_\infty=1,\ \mathscr{E}_n(x)\in\tilde{\mathscr{S}}_n. \tag{8}$$

It is called the Euler spline because its polynomial components are of the form $aE_n(x-b)$, where $E_n(x)$ is the classical Euler polynomial. Among its many properties (see [6, sects. 1, 2, 3]) we mention the relations

$$\|\mathscr{E}_n^{(\nu)}\|=\begin{cases}|\mathscr{E}^{(\nu)}(0)| & \text{if } \nu \text{ is even}\\ |\mathscr{E}^{(\nu)}(\tfrac{1}{2})| & \text{if } \nu \text{ is odd, } (\nu\leqq n).\end{cases} \tag{9}$$

Here and below we use the sup-norm on $\mathbb{R}$.

The analog of the Markov theorem we give as Theorem 6.

THEOREM 6. *If*

$$S(x)\in\tilde{\mathscr{S}}_n, \text{ and } \|S\|\leqq 1, \tag{10}$$

then

$$\|S^{(\nu)}\|\leqq\|\mathscr{E}_n^{(\nu)}\| \text{ for } \nu=1,2,\cdots,n. \tag{11}$$

As $\mathscr{E}_n(x)$ satisfies the conditions (10), by (8), we see that the bounds in (11) can not be improved. A comparison with Theorem 5 shows that the role of the Chebyshev polynomial $T_n(x)$ is here taken over by the Euler spline $\mathscr{E}_n(x)$. (A proof of Theorem 6 is given in Section 3.)

2. A few auxilliary tools

We need the so-called B-splines which seem indispensable in a study of cardinal splines.

DEFINITION 7. If $Q_1(x)$ denotes the characteristic function of the interval $[0, 1]$, we define $Q_n(x)$ as the n-fold convolution of $Q_1(x)$ with itself: $Q_n(x) = Q_1 * Q_1 * \cdots * Q_1(x)$. Explicitly we find that

$$Q_n(x) = \frac{1}{(n-1)!} \sum_{r=0}^{n} (-1)^r \binom{n}{r} (x-r)_+^{n-1}, \tag{12}$$

where $x_+ = \max(0, x)$. $Q_n(x) > 0$ in $(0, n)$, and $Q_n(x) = 0$ elsewhere.

The function $Q_n(x)$ is the forward B-spline which is well known because of the identity

$$\Delta^n f(0) = \int_{-\infty}^{\infty} Q_n(x) f^{(n)}(x)\, dx. \tag{13}$$

From (12) it is seen that $Q_n(x) \in \mathscr{S}_{n-1}$. It is often convenient to shift the origin to the point $x = \frac{1}{2}n$ and to define

$$M_n(x) = Q_n(x + \tfrac{1}{2}n), \tag{14}$$

this being an even function of x such that

$$M_n(x) = 0 \text{ if } x \leqq -\tfrac{1}{2}n, \text{ or if } x \geqq \tfrac{1}{2}n. \tag{15}$$

Notice also that $M_n(x) \in \tilde{\mathscr{S}}_{n-1}$ for $n = 1, 2, \cdots$.

A number of integral relations hold, among which we need

$$\int_{-\infty}^{\infty} Q_p(x-r)\, Q_q(x-t)\, dx = Q_{p+q}(r - t + p),$$

where p and q are natural numbers, while r and t are real (see [4, p. 177]). In particular, using (14), we find that

$$\int_{-\infty}^{\infty} Q_n(x-i)\, Q_1\left(x - j - \frac{n-1}{2}\right) dx = M_{n+1}(i-j). \tag{16}$$

Observe that $M_{n+1}(v)$ is an even sequence whose terms are different from zero (in fact positive) as long as $-\frac{1}{2}(n+1) < v < \frac{1}{2}(n+1)$, by (15). Of importance for us is the rational function

$$F(z) = \sum_{v} M_{n+1}(v) z^v.$$

Setting $z = e^{iu}$, we find that

$$\phi_{n+1}(u) = F(e^{iu})$$

is a cosine polynomial such that

$$\phi_{n+1}(u) > 0 \text{ for all real } u, \ \phi_{n+1}(0) = 1, \tag{17}$$

and $\min_4 \phi_{n+1}(u) = \phi_{n+1}(\pi) = 2\left(\frac{2}{\pi}\right)^{n+1} \sum_{r=1}^{\infty} \frac{(-1)^{(r-1)(n+1)}}{(2r-1)^{n+1}}.$

It follows from (17) that the reciprocal of $F(z)$ admits a Laurent expansion

$$(18) \qquad 1/F(z) = \sum_{-\infty}^{\infty} \omega_\nu z^\nu \text{ on } |z| = 1,$$

which is identical with the Fourier series

$$(19) \qquad 1/\phi_{n+1}(u) = \sum_{-\infty}^{\infty} \omega_\nu e^{i\nu u}, \text{ where } \omega_\nu = \omega_{-\nu}.$$

Of particular importance for us is the fact that

$$(20) \qquad (-1)^\nu \omega_\nu > 0 \text{ for all } \nu.$$

Finally, observe that (19) and (20) show that

$$(21) \qquad \sum_{-\infty}^{\infty} |\omega_\nu| = \Sigma(-1)^\nu \omega_\nu = 1/\Phi_{n+1}(\pi).$$

For these results we refer to [4, Sects. 1, 2]. They appear again in [7, Lec. 3, 4]. In fact the above coefficients ω_ν allow us to express the fundamental function $L_n(x)$ of cardinal spline interpolation in $\tilde{\mathscr{S}}_n$ in the form

$$L_n(x) = \sum_{-\infty}^{\infty} \omega_\nu M_{n+1}(x-\nu).$$

The unique solution $S_n(x)$ satisfying the condition (1) is then expressible in the form

$$S_n(x) = \sum_{-\infty}^{\infty} y_\nu L_n(x-\nu).$$

3 Proof of Theorem 6

Let us assume that (10) holds and let us first establish (11) for $\nu = n$, hence that

$$(22) \qquad \| S^{(n)} \| \leq \| \mathscr{E}_n^{(n)} \|.$$

If we apply Theorem 5 to each polynomial component of $S(x)$ in its respective unit interval, we find that

$$(23) \qquad \| S^{(n)} \| \leq T_n^{(n)}(1) = 2^{n-1} n!.$$

Therefore $S^{(n)}(x)$ is a bounded step-function with discontinuities at the points $j + \frac{1}{2}(n-1)$, for integer j. We may therefore write

(24) $$S^{(n)}(x) = \sum_{j=-\infty}^{\infty} c_j Q_1\left(x - j - \frac{n-1}{2}\right),$$

where c_j is the value of $S^{(n)}(x)$ in the interval $(j + \frac{1}{2}(n-1), j + \frac{1}{2}(n+1))$. Using (13), (24), and (16), we may write

$$\begin{aligned}\Delta^n S(i) &= \int_{-\infty}^{\infty} Q_n(x-i) S^{(n)}(x)\,dx \\ &= \sum_j c_j \int_{-\infty}^{\infty} Q_n(x-i) Q_1\left(x - j - \frac{n-1}{2}\right) dx,\end{aligned}$$

and finally

(25) $$\Delta^n S(i) = \sum_{j=-\infty}^{\infty} M_{n+1}(i-j) c_j \text{ for all } i.$$

By (23), (c_j) is a bounded sequence. If we regard the sequence convolution (25) as a bounded linear transformation of the space B of bounded sequences into itself then it follows from (17) and (18), that (25) admits an inverse given by

(26) $$c_i = \sum_{j=-\infty}^{\infty} \omega_{i-j} \Delta^n S(j) \text{ for all } i,$$

(see [4, 5]), which gives the only bounded solution of the system (25). To estimate c_i from (26), observe that

$$\Delta^n S(j) = (-1)^n \sum_{r=0}^{n} (-1)^r \binom{n}{r} S(j+r).$$

Since $\| S \| \leqq 1$, this gives

$$|\Delta^n S(j)| \leqq \sum \binom{n}{r} = 2^n,$$

and, using (26), we obtain that

$$|c_i| \leqq 2^n \sum_j |\omega_j| \text{ for all } i.$$

Finally, (21) allows us to write

(27) $$\| S^{(n)} \| = \sup_i |c_i| \leqq 2^n / \phi_{n+1}(\pi).$$

Notice that the Euler spline $\mathscr{E}_n(x)$ also satisfies the assumptions (10) of Theorem 6. Applying the above to $\mathscr{E}_n(x)$, rather than $S(x)$, we may write

(28) $$\mathscr{E}_n^{(n)}(x) = \sum_j \tilde{c}_j Q_1\left(x - j - \frac{n-1}{2}\right),$$

and, as above, we obtain that

(29) $$\tilde{c}_i = \sum_j \omega_{i-j}\Delta^n\mathscr{E}_n(j).$$

However, by (8), $\Delta^n\mathscr{E}_n(j) = (-1)^n \sum_{j=0}^{n} (-1)^r \binom{n}{r}(-1)^{j+r} = (-1)^{n+j}2^n,$

and substituting into (29) we obtain

$$\tilde{c}_i = \sum_j \omega_{i-j}(-1)^{n+j}2^n = (-1)^{n+i}2^n \sum_j (-1)^{i-j}\omega_{i-j} = (-1)^{n+i}2^n \sum_i |\omega_i|,$$

and finally, by (21), that

(30) $$\tilde{c}_i = (-1)^{n+i}\, 2^n/\phi_{n+1}(\pi).$$

Hence, by (28) and (30), we get

(31) $$\| \mathscr{E}_n^{(n)} \| = \sup_i |\tilde{c}_i| = 2^n/\phi_{n+1}(\pi).$$

Now (27) and (31) show that (22) indeed holds.

In order to complete a proof of Theorem 6 we appeal to a theorem of Kolomogorov [2] (for further references see [6]) which remains valid for functions from $\mathbb{R}$ to $\mathbb{C}$. We restate it as Theorem 7.

THEOREM 7. ([2].) *If $f(x)$ is a function having a bounded* nth *derivative, and is such that*

$$\|f\| \leqq 1, \ \|f^{(n)}\| \leqq \|\mathscr{E}_n^{(n)}\|,$$

then

$$\|f^{(\nu)}\| \leqq \|\mathscr{E}_n^{(\nu)}\| \ \textit{for}\ \nu = 1,2,\cdots,n-1.$$

Applying this to our spline $S(x)$, we see that (22) implies the validity of the remaining inequalities (11).

4. Proof of Theorem 4

If in (2) we let $\alpha(x) = 0$ in $(-\pi,\pi)$, while $\alpha(-\pi) = -\frac{1}{2}$, $\alpha(\pi) = \frac{1}{2}$, then the conditions (3) are satisfied, and we find that (2) shows that $f(x) = \cos\pi x$. The cardinal spline $S_{2m-1}(x)$ is, of course, identical with $\mathscr{E}_{2m-1}(x)$. By Theorem 3 we conclude that

(32) $$\lim_{m\to\infty} \mathscr{E}_{2m-1}(x) = \cos\pi x.$$

However, this would be circular reasoning, since (32) is used in the proof of

Theorem 3 as given in [8]. A direct proof of (32) which will give us much more is provided by the Fourier series expansion

(33) $$\mathscr{E}_n(x) = \sum_{r=1}^{\infty} (2r-1)^{-n-1}\cos(2r-1)\pi x \Big/ \sum_{r=1}^{\infty} (2r-1)^{-n-1}.$$

This immediately shows that $\mathscr{E}_n(x) = \cos \pi x + O(3^{-n})$ as $n \to \infty$.

Moreover, differentiation of (33) ν times shows, by crude estimates, that for all real x

$$\left|\mathscr{E}_n^{(\nu)}(x)\right| < \pi^\nu\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) \text{ if only } \nu < n.$$

As the sum of the series $= \frac{1}{8}\pi^2 < 2$, we conclude that

$$\left\| \mathscr{E}_n^{(\nu)} \right\| < 2\pi^\nu \text{ if } \nu < n.$$

Let us now assume that $f(x)$ satisfies the assumptions of Theorem 4. This means that $f(x)$ is bounded and that for odd values of n

(34) $$\lim_{n\to\infty} S_n(x) = f(x) \text{ uniformly on } \mathbb{R}.$$

This implies that for an appropriate M

(35) $$\| S_n \| < M \text{ for all odd } n.$$

Moreover, in view of Theorem 6 and (35) we conclude that

(36) $$\| S_n^{(\nu)} \| < 2M\pi^\nu \text{ if } \nu < n.$$

The relations (34) and (36) allow us to draw some strong conclusions. We claim that $f(x) \in C^\infty$, that

(37) $$\lim_{n\to\infty} S_n^{(\nu)}(x) = f^{(\nu)}(x) \text{ locally uniformly on } \mathbb{R}, \text{ for all } \nu,$$

and finally, that

(38) $$\|f^{(\nu)}\| \leqq 2M\pi^\nu \text{ for all } \nu.$$

All these statements follow by familiar elementary reasoning. By (36), for $\nu = 1$, $S_n'(x)$ are equi-bounded, and by (36), for $\nu = 2$, they are also equi-uniformly continuous since they satisfy the same Lipschitz condition. By the Arzelà-Ascoli theorem, we can extract from (n) a subsequence (n') such that $\lim S_{n'}'(x) = g(x)$ as n' approaches infinity, locally uniformly on $\mathbb{R}$. But then

$$\int_0^x g(t)\,dt = \lim_{n'\to\infty} \int_0^x S_{n'}'(t)\,dt = \lim (S_{n'}(x) - S_{n'}(0)) = f(x) - f(0),$$

by (34). It follows that $f(x) \in C^1$ and that $g(x) = f'(x)$ for all x. This determines $g(x)$ uniquely, and shows that the sequence $S_n'(x)$ was convergent in the first place. We can repeat this reasoning on the higher derivatives, and by induction, (37) and (38) are established.

The inequalities (38) now show that $f(x)$ is the restriction to $\mathbb{R}$ of the entire function

$$f(z) = \sum_0^\infty f^{(\nu)}(0) z^\nu / \nu!.$$

Moreover, by (38), we obtain that $|f(z)| \leqq 2Me^{\pi|z|}$. Therefore $f(z)$ is of exponential type $\leqq \pi$, and this concludes our proof.

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