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AUTOMORPHISM BASES FOR DEGREES OF UNSOLVABILITY†

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ABSTRACT

A set of degrees of unsolvability is said to *generate* the degrees if every degree is in the closure of the set under the lattice operations on degrees. (Of course the inf operation is only partially defined.) It is shown that every set of degrees which is comeager or of measure one generates the degrees and that the set of minimal degrees generates the degrees. (Thus any automorphism of degrees is determined by its action on, for example, the minimal degrees.) Also the degrees below $\mathbf{0}'$ which have the same jump as any given degree below $\mathbf{0}'$ and those which cup any given nonzero degree $\leqq \mathbf{0}'$ to $\mathbf{0}'$ are shown to generate the degrees below $\mathbf{0}'$.

§1. Introduction

Let $\mathcal{D}$ be the set of all degrees of unsolvability. A set $\mathcal{A} \subseteq \mathcal{D}$ is said to *generate* $\mathcal{D}$ if $\mathcal{D}$ is the closure of $\mathcal{A}$ under the l.u.b. operation $\cup$ and the partial g.l.b. operation $\cap$. It will be shown that every set $\mathcal{A} \subseteq \mathcal{D}$ which is comeager or of measure 1 generates $\mathcal{D}$. Also the set of minimal degrees generates $\mathcal{D}$. Let $\mathcal{D}(\leqq \mathbf{0}')$ denote the set of degrees $\leqq \mathbf{0}'$. It is also shown that if $\boldsymbol{b}$ is any fixed degree in $\mathcal{D}(\leqq \mathbf{0}')$, then $\{\boldsymbol{c} \leqq \mathbf{0}' : \boldsymbol{c}' = \boldsymbol{b}'\}$ generates $\mathcal{D}(\leqq \mathbf{0}')$, as does $\{\boldsymbol{c} \leqq \mathbf{0}' : \boldsymbol{c} \cup \boldsymbol{b} = \mathbf{0}'\}$ provided that $\boldsymbol{b} \neq \mathbf{0}$. In a future paper Posner will show that the minimal degrees below $\mathbf{0}'$ generate $\mathcal{D}(\leqq \mathbf{0}')$.

A set $\mathcal{A}$ of degrees is called an *automorphism base* for $\mathcal{D}$ if any two order automorphisms of $\mathcal{D}$ which agree on $\mathcal{A}$ are identical. (Automorphism bases for the inclusion lattice of r.e. sets were studied by R. Shore [18]. M. Lerman [8] investigated automorphism bases for r.e. degrees and suggested to the authors the study of such bases for degrees in general.) It is obvious that every set of

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degrees which generates $\mathscr{D}$ must be an automorphism base for $\mathscr{D}$, and thus our results provide many examples of automorphism bases for $\mathscr{D}$. However it is not known whether there exist nontrivial automorphisms of $\mathscr{D}$ and thus it may be that *every* set of degrees is an automorphism base for $\mathscr{D}$. Turning this around, it seems conceivable that results on automorphism bases could be useful in showing that there do not exist nontrivial automorphisms of $\mathscr{D}$, say by showing that every automorphism of $\mathscr{D}$ must agree with the identity on some automorphism base. For instance, it is shown in [1, 10, 11] that every automorphism of $\mathscr{D}$ must be the identity on some cone $\mathscr{D}\ (\geqq \boldsymbol{a})$ and so to prove the nonexistence of nontrivial automorphisms of $\mathscr{D}$ it would suffice to show that every cone is an automorphism base. In the current paper we combine our results on generating sets with the result that there is a fixed cone on which all automorphisms of $\mathscr{D}$ are the identity [1] to show that for almost every degree $\boldsymbol{a}$ (in the sense of measure or category) the cone $\mathscr{D}\ (\geqq \boldsymbol{a})$ is an automorphism base. Since all cones are closed under $\cup$, $\cap$ (when defined), and the jump operation, this yields examples of automorphism bases which fail to generate the degrees under $\cup$, $\cap$ and jump.

Our notation and terminology are standard. Unless otherwise specified, we use letters such as a, e, n for natural numbers, A, E, N for sets of natural numbers, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ for degrees, lower case Greek letters for strings, and script letters for higher type objects such as sets of degrees. Of course $\omega = \{0, 1, 2, \cdots\}$ and 2^ω is the power set of ω. The join and symmetric difference operation on subsets of ω are defined by

$$A \oplus B = \{2n : n \in A\} \cup \{2n+1 : n \in B\} \quad \text{and} \quad A \triangle B = (A - B) \cup (B - A).$$

The join operation is extended to strings in a natural way. The notations $\Phi_e(A)$ and $\{e\}^A$ are each used for the eth function partial recursive in A. We write $\deg(A)$ for the degree of A. If $\boldsymbol{a}$ is a degree, then

$$\mathscr{D}(\leqq \boldsymbol{a}) = \{\boldsymbol{b} : \boldsymbol{b} \leqq \boldsymbol{a}\} \quad \text{and} \quad \mathscr{D}(\geqq \boldsymbol{a}) = \{\boldsymbol{b} : \boldsymbol{b} \geqq \boldsymbol{a}\}.$$

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§2. Measure and category

Let 2^ω have its usual product measure μ and its usual product topology. These induce notions of measure and category for (certain) sets $\mathscr{A}$ of degrees in the usual way, i.e., by identifying $\mathscr{A} \subseteq \mathscr{D}$ with $\{A : \deg(A) \in \mathscr{A}\} \subseteq 2^\omega$. The first theorem of this section immediately implies that any set of degrees of measure 1 generates $\mathscr{D}$ and hence is an automorphism base for $\mathscr{D}$. Theorem 2.4 is the analogous result for category. Further such results for modified notions of category are obtained in §§3 and 4.

THEOREM 2.1. *If $\mathscr{A} \subseteq \mathscr{D}$ has measure 1, then every degree $\boldsymbol{b}$ is of the form $(\boldsymbol{a}_1 \cup \boldsymbol{a}_2) \cap (\boldsymbol{a}_3 \cup \boldsymbol{a}_4)$, where each $\boldsymbol{a}_i$ is in $\mathscr{A}$.*

PROOF. Let any degree $\boldsymbol{b}$ be fixed. We define a natural measure $\mu_{\boldsymbol{b}}$ on $\mathscr{D}(\geqq \boldsymbol{b})$ as follows. If $\mathscr{C} \subseteq \mathscr{D}(\geqq \boldsymbol{b})$, then $\mu_{\boldsymbol{b}}(\mathscr{C}) = \mu(\{\boldsymbol{a} : \boldsymbol{b} \cup \boldsymbol{a} \in \mathscr{C}\})$.

LEMMA 2.2. *If $\mathscr{A} \subseteq \mathscr{D}$ and $\mu(\mathscr{A}) = 1$, then $\mu_{\boldsymbol{b}}(\mathscr{A}^*) = 1$, where $\mathscr{A}^*$ is the set of degrees $\geqq \boldsymbol{b}$ of the form $\boldsymbol{a}_1 \cup \boldsymbol{a}_2$, with $\boldsymbol{a}_1, \boldsymbol{a}_2 \in \mathscr{A}$.*

PROOF. For any sets $A, B \subseteq \omega$

$$B \oplus A \equiv_T A \oplus (A \triangle B). \tag{1}$$

(This follows easily from the fact that $\oplus$ induces the l.u.b. operation on degrees and that $A \triangle (A \triangle B) = B$.) If $\mathscr{C} \subseteq 2^\omega$, let $\mathscr{C} \triangle B = \{C \triangle B : C \in \mathscr{C}\}$. If $\mathscr{C}$ is measurable, then $\mu(B \triangle \mathscr{C}) = \mu(\mathscr{C})$. (This may be easily verified directly but also follows from the fact that μ is Haar measure, and hence translation invariant, on the topological group $(2^\omega, \triangle)$.) Indeed we were led to Theorem 2.1 by J. Shoenfield's observation that Theorem 2.4 was analogous to some results for topological groups.) Applying this with $\mathscr{C}$ the set of reals whose degree is in $\mathscr{A}$ and B a fixed set of degree $\boldsymbol{b}$ we deduce, using the finite additivity of μ, that for almost every set $A \subseteq \omega$, both A and $A \triangle B$ are in $\mathscr{A}$. The lemma now follows from (1).

LEMMA 2.3 (Stillwell [20]). *If $\boldsymbol{c} \geqq \boldsymbol{b}$, then $\mu_{\boldsymbol{b}}(\{\boldsymbol{d} : \boldsymbol{c} \cap \boldsymbol{d} = \boldsymbol{b}\}) = 1$.*

Theorem 2.1 is now immediate from the two lemmas. Applying Lemma 2.2 choose any degree $\boldsymbol{c} \in \mathscr{A}^*$. By Lemmas 2.2 and 2.3 and the finite additivity of μ, there exists $\boldsymbol{d} \in \mathscr{A}^*$ with $\boldsymbol{c} \cap \boldsymbol{d} = \boldsymbol{b}$.

THEOREM 2.4. *If $\mathscr{A} \subseteq \mathscr{D}$ and $\mathscr{A}$ is comeager, then every degree $\boldsymbol{b}$ may be expressed as $(\boldsymbol{a}_1 \cup \boldsymbol{a}_2) \cap (\boldsymbol{a}_3 \cup \boldsymbol{a}_4)$, with each $\boldsymbol{a}_i \in \mathscr{A}$.*

PROOF. The proof of Theorem 2.1 yields this result *mutatis mutandis.* An effective version of this result will be given in Theorem 4.4.

COROLLARY 2.5. *If $\mathscr{A}$ is the class of degrees $\boldsymbol{a}$ such that $\boldsymbol{a}^{(n)} = \boldsymbol{a} \cup \boldsymbol{0}^{(n)}$ for all $n < \omega$ and no degree $\boldsymbol{b} \leqq \boldsymbol{a}$ is a minimal degree, then every degree $\boldsymbol{b}$ is of the form $(\boldsymbol{a}_1 \cup \boldsymbol{a}_2) \cap (\boldsymbol{a}_3 \cup \boldsymbol{a}_4)$, with each $\boldsymbol{a}_i \in \mathscr{A}$.*

PROOF. This may be deduced from either Theorem 2.1 or Theorem 2.4 since $\mathcal{A}$ is comeager and has measure 1 [5, 13, 20, 21].

COROLLARY 2.6. *There is a degree* $\boldsymbol{a} > \mathbf{0}$ *such that* $\mathcal{D}(\geqq \boldsymbol{a})$ *is an automorphism base. In fact the set of all such degrees is comeager and of measure* 1.

PROOF. We first show that for any fixed degree $\boldsymbol{c}$, the set of degrees $\boldsymbol{a}$ such that $\mathcal{D}(\geqq \boldsymbol{c}) \cup \mathcal{D}(\geqq \boldsymbol{a})$ generates $\mathcal{D}$ is of measure 1. For any degree $\boldsymbol{b}$, $\{\boldsymbol{a} : \boldsymbol{b} = (\boldsymbol{b} \cup \boldsymbol{c}) \cap (\boldsymbol{b} \cup \boldsymbol{a})\}$ is of measure 1, by Lemma 2.3. It follows by Funini's theorem that, for almost every degree $\boldsymbol{a}$, the equation $\boldsymbol{b} = (\boldsymbol{b} \cup \boldsymbol{c}) \cap (\boldsymbol{b} \cup \boldsymbol{a})$ holds for almost every degree $\boldsymbol{b}$. Fix any degree $\boldsymbol{a}$ such that the equation just mentioned holds for our fixed $\boldsymbol{c}$ and for almost every $\boldsymbol{b}$. Then $\mathcal{D}(\geqq \boldsymbol{c}) \cup \mathcal{D}(\geqq \boldsymbol{a})$ generates a set of degrees of measure 1 and hence, by Theorem 2.1, generates $\mathcal{D}$. Now choose $\boldsymbol{c}$ to be a degree so that every automorphism of $\langle \mathcal{D}, \leqq \rangle$ is pointwise fixed on $\mathcal{D}(\geqq \boldsymbol{c})$. (By [1] one may take $\boldsymbol{c}$ to be any degree, such as the degree $\boldsymbol{\mathcal{O}}$ of Kleene's $\mathcal{O}$, which is above all degrees of hyperarithmetic sets.) Thus for almost every degree $\boldsymbol{a}$, $\mathcal{D}(\geqq \boldsymbol{a})$ is an automorphism base.

The proof for category is similar. Indeed the argument shows that $\mathcal{D}(\leqq \boldsymbol{c}) \cup \mathcal{D}(\geqq \boldsymbol{a})$ generates $\mathcal{D}$ whenever $\boldsymbol{a}$ contains a set which is 1-generic relative to $\boldsymbol{c}$. (The notion of 1-genericity, here relativized in the obvious way, is defined just before the statement of Corollary 4.16, and properties of 1-generic sets are discussed in [5].) Since for any degree $\boldsymbol{c}$ there exists a set of degree $< \boldsymbol{c}'$ which is 1-generic relative to $\boldsymbol{c}$, it follows that there is a nonzero degree $\boldsymbol{a} < \boldsymbol{\mathcal{O}}'$ such that $\mathcal{D}(\geqq \boldsymbol{a})$ is an automorphism base. A more refined bound on such an $\boldsymbol{a}$ may be obtained by choosing $\boldsymbol{c}$, and then $\boldsymbol{a}$, to have hyperjump recursive in $\boldsymbol{\mathcal{O}}$. There is a degree $\boldsymbol{c}$ above all degrees of hyperarithmetic sets and having hyperjump recursive in $\boldsymbol{\mathcal{O}}$ because the sets with hyperjump recursive in $\boldsymbol{\mathcal{O}}$ form a basis for Σ^1_1. This basis theorem together with a proof using Σ^1_1 forcing, i.e. Gandy forcing, were pointed out to Jockusch by S. Simpson.

In [11] it is shown that every automorphism of $\mathcal{D}$ which fixes $\mathbf{0}'$ also fixed every degree $\geqq \mathbf{0}^{(3)}$. Thus repeating the previous argument with $\boldsymbol{c} = \mathbf{0}^{(3)}$ and $\boldsymbol{a} < \mathbf{0}^{(4)}$ 1-generic relative to $\boldsymbol{c}$, we obtain a degree $\boldsymbol{a} > \mathbf{0}$ such that $\boldsymbol{a}^{(4)} = \mathbf{0}^{(4)}$ and $\mathcal{D}(\geqq \boldsymbol{a}) \cup \{\mathbf{0}'\}$ is an automorphism base. In particular, for this $\boldsymbol{a}$, $\mathcal{D}(\geqq \boldsymbol{a})$ is a basis for jump preserving automorphisms, for which Richter's result [15] suffices. (To see that $\boldsymbol{a}^{(4)} = \mathbf{0}^{(4)}$, choose $\boldsymbol{a} \leqq \mathbf{0}^{(4)}$ 1-generic relative to $\mathbf{0}^{(3)}$ and use the 4-genericity of $\boldsymbol{a}$ to conclude that $\boldsymbol{a}^{(4)} = \boldsymbol{a} \cup \mathbf{0}^{(4)} = \mathbf{0}^{(4)}$.)

If $\boldsymbol{a}, \boldsymbol{b}$ are nonzero degrees such that $\mathcal{D}(\geqq \boldsymbol{a}) \cup \mathcal{D}(\geqq \boldsymbol{b})$ generates $\mathcal{D}$, it is obviously necessary that $\boldsymbol{a}$, $\boldsymbol{b}$ form a minimal pair. We do not know whether it is also sufficient.

§3. Minimal degrees

Our next goal is to show that the set of minimal degrees generates $\mathcal{D}$. This answers a question raised in correspondence by M. Lerman.

THEOREM 3.1. *For every degree $\boldsymbol{b}$ there are minimal degrees $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, $\boldsymbol{a}_4$ such that $\boldsymbol{b} = (\boldsymbol{a}_1 \cup \boldsymbol{a}_2) \cap (\boldsymbol{a}_3 \cup \boldsymbol{a}_4)$.*

PROOF. The proof combines Lachlan's construction [7] of a minimal degree using recursive coinfinite conditions, a simple coding method for making a set of the given degree $\boldsymbol{b}$ recursive in the join of two constructed sets of minimal degree, and a (non-r.e.) minimal pair construction. (Lachlan's construction is used rather than some other minimal degree construction because it is compatible with the coding method.) Lachlan's construction may be viewed as a category argument (with respect to the system of recursive trees corresponding to recursive coinfinite conditions) in the sense of [21]. This generalized category point of view is useful in the proof at hand because it simplifies technical details, yields added generality, and brings out the analogy between Theorems 2.1 and 3.1. Since two different notions of comeager arise in the proof we define a generalized version of this notion essentially as in [21].

Let X be a compact Hausdorff space, and let $\mathcal{S}$ be a nonempty family of nonempty closed subsets of X. A set $Y \subseteq X$ is called *$\mathcal{S}$-dense* if for each $F_0 \in \mathcal{S}$ there exists $F_1 \in \mathcal{S}$ such that $F_1 \subseteq F_0 \cap Y$. A set $Y \subseteq X$ is called *$\mathcal{S}$-comeager* if Y contains the intersection of some countable family of $\mathcal{S}$-dense subsets of X. As in the Baire category theorem, one easily shows that every $\mathcal{S}$-comeager set is nonempty. (If Y_e is $\mathcal{S}$-dense for each e, choose closed sets $F_0, F_1, \cdots$ inductively with $F_0 \in \mathcal{S}$ and $F_{e+1} \subseteq F_e \cap Y_e$, $F_{e+1} \in \mathcal{S}$ for all e. Then $\bigcap_e F_e \subseteq \bigcap_e Y_e$ and $\bigcap_e F_e$ is nonempty since X is compact.)

We shall apply this notion first to the family of closed subsets of 2^ω defined by recursive coinfinite conditions. Let $\mathcal{R}$ be the family of all such conditions, i.e., of all pairs (P, N) where P, N are disjoint recursive subsets of ω with $P \cup N$ coinfinite. For each condition $(P, N) \in \mathcal{R}$, let $\mathcal{N}(P, N)$ be the family of all sets $A \subseteq \omega$ such that $P \subseteq A$ and $N \cap A = \emptyset$. (Thus P represents positive information and N represents negative information.) Clearly $\mathcal{N}(P, N)$ is a nonempty closed subset of 2^ω (in its usual topology). We abuse notation by writing $\mathcal{R}$ for the family of all such closed sets $\mathcal{N}(P, N)$, for $(P, N) \in \mathcal{R}$.

We remark that conditions in $\mathcal{R}$ define the same neighborhoods in 2^ω as uniform recursive trees T such that for all strings σ, the strings $T(\sigma * 0)$, $T(\sigma * 1)$ differ on exactly one argument. Such trees are called 1-trees by Lachlan [7].

Given $(P, N) \in \mathcal{R}$, let $a_0, a_1, \cdots$ be the elements of the complement of $P \cup N$ in increasing order. The 1-tree T corresponding to (P, N) is such that, for every string σ, $|T(\sigma)| = a_{|\sigma|}$ and, for $i < a_{|\sigma|}$, $T(\sigma)(i)$ is 1 if $i \in P$, 0 if $i \in N$, and $\sigma(j)$ if $i = a_j$ $(j < |\sigma|)$. Using this equivalent formulation of $\mathcal{R}$ in terms of 1-trees, Lachlan [7] showed that the family of all sets of minimal degree is $\mathcal{R}$-comeager (by showing that, for each e, the family of sets A such that $\Phi_e(A)$ is nontotal, recursive, or Turing equivalent to A is $\mathcal{R}$-dense). Thus it suffices to prove the following lemma.

LEMMA 3.2. *If $\boldsymbol{b}$, $\boldsymbol{c}$ are any degrees with $\boldsymbol{b} \lneq \boldsymbol{c}$ and $\mathcal{A}$ is any $\mathcal{R}$-comeager subset of 2^ω, then there are degrees $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ such that $\boldsymbol{b} = (\boldsymbol{a}_1 \cup \boldsymbol{a}_2) \cap \boldsymbol{c}$ and each $\boldsymbol{a}_i$ contains a set in $\mathcal{A}$.*

To prove the theorem from the lemma, let $\mathcal{A}$ be the family of all sets of minimal degree. Apply the lemma with $\boldsymbol{c}$ any degree above the given degree $\boldsymbol{b}$ to obtain minimal degrees $\boldsymbol{a}_3$, $\boldsymbol{a}_4$ with $\boldsymbol{b} \lneq \boldsymbol{a}_3 \cup \boldsymbol{a}_4$. Apply the lemma again with $\boldsymbol{c} = \boldsymbol{a}_3 \cup \boldsymbol{a}_4$ to obtain minimal degrees $\boldsymbol{a}_1, \boldsymbol{a}_2$ with $\boldsymbol{b} = (\boldsymbol{a}_1 \cup \boldsymbol{a}_2) \cap (\boldsymbol{a}_3 \cup \boldsymbol{a}_4)$.

PROOF OF LEMMA 3.2. It will be shown that the family $\mathcal{B}$ of pairs $(A_1, A_2) \in \mathcal{A} \times \mathcal{A}$ which satisfy the lemma with $\deg(A_i) = \boldsymbol{a}_i$ is $\mathcal{F}$-comeager for a certain family $\mathcal{F}$ of nonempty closed subsets of $2^\omega \times 2^\omega$, and thus is nonempty. The family $\mathcal{F}$ will depend on B, where B is a fixed set of degree $\boldsymbol{b}$. The key step is picking a coding procedure for ensuring $B \leqq_T A_1 \oplus A_2$ which is compatible with the requirements $A_1, A_2 \in \mathcal{A}$.

The coding method we use is due to M. Lerman [9] and is simpler than our original method. We require that the symmetric difference of A_1 and A_2 (denoted $A_1 \triangle A_2$) be infinite and that, for all n,

$$n \in B \Leftrightarrow a_n \in A_1 \tag{1}$$

where $a_0, a_1, \cdots$ are the elements of $A_1 \triangle A_2$ in strictly increasing order. We now choose a family $\mathcal{F}$ of closed subsets of $2^\omega \times 2^\omega$ so that the family $\mathcal{G}$ of pairs (A_1, A_2) with $A_1 \triangle A_2$ is infinite and (1) true for all n is $\mathcal{F}$-comeager. Specifically let $\mathcal{F}$ be the family of all sets $\mathcal{N}(T_1) \times \mathcal{N}(T_2)$ where each T_i is itself a pair $(P_i, N_i) \in \mathcal{R}$ and each condition (2)–(4) holds:

$$P_1 \cup N_1 = P_2 \cup N_2, \tag{2}$$

$$P_1 \triangle P_2 \text{ is finite,} \tag{3}$$

(4) If $a_0, \cdots, a_{k-1}$ are the elements of $P_1 \triangle P_2$ in strictly increasing order, then condition (1) holds for all $n < k$.

As before we abuse notation by writing $\mathcal{F}$ for both the family of closed subsets of $2^\omega \times 2^\omega$ defined above and the corresponding set of pairs $(T_1, T_2) \in \mathcal{R}^2$ satisfying (2)–(4). If $T_i = (P_i, N_i) \in \mathcal{R}$, we write $T_2 \supseteq T_1$ if $P_2 \supseteq P_1$ and $N_2 \supseteq N_1$ (i.e., if $\mathcal{N}(T_2) \subseteq \mathcal{N}(T_1)$). Similarly if $(T_1', T_2') \supseteq (T_1, T_2)$ if $T_1' \supseteq T_1$ and $T_2' \supseteq T_2$, i.e., if $\mathcal{N}(T_1') \times \mathcal{N}(T_2') \subseteq \mathcal{N}(T_1) \times \mathcal{N}(T_2)$.

We now show that $\mathcal{G}$, and hence the family of pairs (A_1, A_2) with $B \leqq_T A_1 \oplus A_2$, is $\mathcal{F}$-comeager. To prove this it obviously suffices to show that for each n, the family of pairs (A_1, A_2) with $|A_1 \triangle A_2| > n$ and (1) holding with a_n the $(n+1)$st element of $A_1 \triangle A_2$ is $\mathcal{F}$-dense. Let $n \in \omega$ and $(T_1, T_2) \in \mathcal{F}$ be given. Suppose $T_i = (P_i, N_i)$ for $i = 1, 2$. Let $a_0, \cdots, a_{k-1}$ be the elements of $P_1 \triangle P_2$ in strictly increasing order. Choose numbers $a_k, \cdots, a_{k+m}$ in strictly increasing order, all exceeding a_{k-1} and none in the coinfinite set $P_1 \cup N_1$. Let $P_1' = P_1 \cup \{a_{k+i} : 0 \leqq i \leqq n$ and $k + i \in B\}$. Let $P_2' = P_2 \cup \{a_{k+i} : 0 \leqq i \leqq n$ and $k + i \in B\}$. For $i = 1, 2$, let $N_i' = N_i \cup \{x \leqq a_{k+n} : x \notin P_i'\}$. For $i = 1, 2$, let $T_i' = (P_i', N_i')$. Observe that $P_1' \triangle P_2' = \{a_0, \cdots, a_{k+n}\}$ and every number less than or equal to a_{k+n} is in $P_1' \cup N_1'$. Thus if $(A_1, A_2) \in \mathcal{N}(T_1') \times \mathcal{N}(T_2')$, then $P_1' \triangle P_2'$ is an initial segment of $A_1 \triangle A_2$. Using these observations the reader may easily show that $(T_1', T_2') \in \mathcal{F}$ and that (1) holds for the given n of every pair $(A_1, A_2) \in \mathcal{N}(T_1') \times \mathcal{N}(T_2')$. Since $(T_1', T_2') \supseteq (T_1, T_2)$, the proof that the set of pairs (A_1, A_2) with $B \leqq_T A_1 \oplus A_2$ is $\mathcal{F}$-comeager is complete.

The following lemma will be used to show that $\mathcal{A} \times \mathcal{A}$ is $\mathcal{F}$-comeager from the assumption that $\mathcal{A}$ is $\mathcal{R}$-comeager.

Lemma 3.3. *If $(T_1, T_2) \in \mathcal{F}$, $T_1' \in \mathcal{R}$, and $T_1' \supseteq T_1$, then there exists $T_2' \in \mathcal{R}$ with $T_2' \supseteq T_2$ and $(T_1', T_2') \in \mathcal{F}$.*

Proof. Suppose $T_i = (P_i, N_i)$ and $T_1' = (P_1', N_1')$. Let $P_2' = P_2 \cup (P_1' - N_2)$ and $N_2' = N_2 \cup (N_1' - P_2)$. P_2', N_2' are disjoint because P_2, N_2 and P_1', N_1' are each disjoint. Furthermore $P_2' \cup N_2' = P_1' \cup N_1'$ since $P_2 \cup N_2 = P_1 \cup N_1 \subseteq P_1' \cup N_1'$. In particular $P_2' \cup N_2'$ is coinfinite since $P_1' \cup N_1'$ is. (It is at this point that (2) in the definition of $\mathcal{F}$ is particularly crucial since without it we would have no way to show that $P_2' \cup N_2'$ is coinfinite.) Thus $T_2' \in \mathcal{R}$, where $T_2' = (P_2', N_2')$. We claim now that $P_1' \triangle P_2' = P_1 \triangle P_2$. Given a number n, if $n \notin P_1 \cup N_1 = P_2 \cup N_2$ then P_1', P_2' agree on n (i.e., $n \in P_1' \Leftrightarrow n \in P_2'$). Therefore n belongs to neither $P_1 \triangle P_2$ nor $P_1' \triangle P_2'$. Now assume $n \in P_1 \cup N_1 = P_2 \cup N_2$. Since $T_i' \supseteq T_i$ and P_i', N_i' are disjoint for $i = 1, 2$, P_i' and P_i agree on n. Therefore $P_1' \triangle P_2'$ and $P_1 \triangle P_2$ agree on n in this case also. Since $P_1' \triangle P_2' = P_1 \triangle P_2$, clauses (3) and (4) in the definitions of $\mathcal{F}$ carry over immediately from (T_1, T_2) to (T_1', T_2') and so $(T_1', T_2') \in \mathcal{F}$. Clearly $(T_1', T_2') \supseteq (T_1, T_2)$ so the lemma is proved.

It follows at once from Lemma 3.3 that if $\mathscr{D}$ is an $\mathscr{R}$-dense subset of 2^ω, then $\mathscr{D}\times 2^\omega$ is an $\mathscr{F}$-dense subset of $2^\omega \times 2^\omega$. Let $\mathscr{A}$ be $\mathscr{R}$-comeager, so $\mathscr{A} \supseteq \bigcap_i \mathscr{D}_i$ where each $\mathscr{D}_i$ is $\mathscr{R}$-dense. Then

$$\mathscr{A}\times 2^\omega \supseteq (\bigcap_i \mathscr{D}_i)\times 2^\omega = \bigcap_i (\mathscr{D}_i \times 2^\omega),$$

so $\mathscr{A}\times 2^\omega$ is $\mathscr{F}$-comeager since each $\mathscr{D}_i\times 2^\omega$ is $\mathscr{F}$-dense. By a similar argument (or symmetry), $2^\omega\times\mathscr{A}$ is $\mathscr{F}$-comeager. Therefore $(\mathscr{A}\times 2^\omega)\cap(2^\omega\times\mathscr{A}) = \mathscr{A}\times\mathscr{A}$ is $\mathscr{F}$-comeager.

Recall that $\mathscr{G}$ is the class of pairs (A_1, A_2) such that $A_1 \triangle A_2$ is infinite and (1) holds for all n. The following lemma shows that the "minimal pair" conditions for satisfying $\boldsymbol{b} = (\boldsymbol{a}_1 \cup \boldsymbol{a}_2)\cap \boldsymbol{c}$ are $\mathscr{F}$-dense in $\mathscr{G}$.

LEMMA 3.4. *Let f be any function not recursive in B, and let e be a number. Then the complement (in $2^\omega\times 2^\omega$) of $\mathscr{G}\cap\{(A_1,A_2):\Phi_e(A_1\oplus A_2)=f\}$ is $\mathscr{F}$-dense, where $\mathscr{G}$ is the subset of $2^\omega\times 2^\omega$ defined above.*

PROOF. Suppose $(T_1, T_2)\in\mathscr{F}$ is given. We must construct $(T_1', T_2')\in\mathscr{F}$ with $(T_1', T_2')\supseteq(T_1,T_2)$ such that no pair (A_1,A_2) in $\mathscr{G}\cap(\mathscr{N}(T_1')\times\mathscr{N}(T_2'))$ satisfies $\Phi_e(A_1\oplus A_2)=f$. For any pair $(T_1',T_2')\in\mathscr{R}\times\mathscr{R}$ with $T_i'=(P_i',N_i')$ we write $\Phi_e((T_1',T_2');x)=y$ if $\Phi_e(P_1'\oplus P_2';x)=y$ and furthermore all numbers whose nonmembership in $P_1'\oplus P_2'$ is used in the computation are in $N_1'\oplus N_2'$. Thus if $\Phi_e((T_1',T_2');x)=y$, then $\Phi_e((A_1,A_2);x)=y$ for every pair (A_1,A_2) in $\mathscr{N}(T_1')\times\mathscr{N}(T_2')$. Hence if there is a pair (T_1',T_2') in $\mathscr{F}$ and a number x such that $(T_1',T_2')\supseteq(T_1,T_2)$ and $\Phi_e((T_1',T_2');x)$ is defined and different from $f(x)$, then any such pair has the desired properties mentioned above. Thus assume there is no such pair. We now claim that for any pair (A_1,A_2) in $\mathscr{G}\cap(\mathscr{N}(T_1)\times\mathscr{N}(T_2))$ with $\Phi_e(A_1\oplus A_2)$ total, $\Phi_e(A_1\oplus A_2)$ is recursive in B. (Thus (T_1,T_2) itself may serve as the desired $(T_1',T_2')\supseteq(T_1,T_2)$.) The proof of the claim is the usual "nonsplitting" case from the minimal pair construction except that strings are replaced by pairs from $\mathscr{F}$ and 2^ω is replaced by $\mathscr{G}$. To prove the claim, assume that (A_1,A_2) is such a pair and an argument x is given. Search recursively in B for four finite sets D_1,D_2,F_1,F_2 such that $(T_1',T_2')\in\mathscr{F}$ and $\Phi_e((T_1',T_2');x)$ is defined, where $T_i'=(P_i\cup D_i, N_i\cup F_i)$ for $i=1,2$. Such a search may be carried out recursively in B because S is recursive in B, where S is the set of quintuples (D_1,D_2,F_1,F_2,x) such that the above conditions hold. Furthermore, such a quintuple must exist for each x by our hypothesis on (A_1,A_2). (Specifically, let m be larger than $\max(P_1\triangle P_2)$ and also larger than any number whose membership or nonmembership in A_1 or A_2 is used to compute $\Phi_e(A_1\oplus A_2;x)$. For $i=1,2$, let $D_i=\{u<m: u\in A_i\}$ and $F_i=\{u<m: u\notin A_i\}$. Let $T_i'=(P_i\cup D_i, N_i\cup F_i)$.

It must be shown that $(T_1', T_2') \in \mathscr{F}$. Here the main point is that $P_1' \triangle P_2'$ is an initial segment of $A_1 \triangle A_2$ so (4) holds of (T_1', T_2') because (1) holds of $A_1 \oplus A_2$. Clearly $\Phi_e((T_1', T_2'); x)$ is defined so the desired quintuple (D_1, D_2, F_1, F_2, x) exists.) Finally the value of $\Phi_e((T_1', T_2'); x)$ must be $f(x)$ for any such pair (T_1', T_2') since otherwise we would be in the trivial case which was previously ruled out. Thus f is recursive in B, contrary to hypothesis.

Since $\mathscr{G}$ is $\mathscr{F}$-comeager and only countably many functions are recursive in C, it follows from Lemma 3.4 that the set of pairs (A_1, A_2) whose degrees $\boldsymbol{a}_1, \boldsymbol{a}_2$ satisfy $\boldsymbol{c} \cap (\boldsymbol{a}_1 \cup \boldsymbol{a}_2) = \boldsymbol{b}$ is $\mathscr{F}$-comeager for each $\boldsymbol{c} \geqq \boldsymbol{b}$. Thus this set of pairs must intersect the $\mathscr{F}$-comeager set $\mathscr{A} \times \mathscr{A}$ so the proof of Lemma 3.2 (and hence of Theorem 3.1) is complete.

We now remark on how the machinery used to prove Theorem 3.1 may be used to extend it to some subclasses of the minimal degrees. A degree $\boldsymbol{a}$ is called *hyperimmune-free* if every function of degree $\leqq \boldsymbol{a}$ is (everywhere) majorized by some recursive function. The construction of a nonzero hyperimmune-free degree may be easily adapted to recursive coinfinite conditions, so the family of hyperimmune-free minimal degrees is $\mathscr{R}$-comeager. It then follows from Lemma 3.2 and the discussion after it that this family generates $\mathscr{D}$. Now let $\mathscr{A}_0$ be the family of hyperimmune-free minimal degrees which contain a bi-immune set, i.e. a set A with no infinite r.e. set contained in A or $\bar{A}$. In [2] the question was raised whether $\mathscr{A}_0$ is nonempty. In correspondence, S. G. Simpson observed that $\mathscr{A}_0$ is in fact $\mathscr{R}$-comeager. (Given a set A, let $\Delta(A) = \{n : |\{i \in A : i < n\}|$ is even$\}$. Then for any infinite set W, $\{A : W \not\subseteq \Delta(A)$ and $W \not\subseteq \omega - \Delta(A)\}$ is $\mathscr{R}$-dense. Since $\Delta(A) \equiv_T A$ for all A, it follows that $\{A : (\exists B)[B \equiv_T A$ and B bi-immune$]\}$ is $\mathscr{R}$-comeager.) Since $\mathscr{A}_0$ is $\mathscr{R}$-comeager, it then follows as before that $\mathscr{A}_0$ generates $\mathscr{D}$.

It is not known whether Theorem 3.1 can be proved using the original minimal degree construction of Spector [19] or its modern variants using arbitrary recursive perfect trees (e.g., [17]). More precisely, it is not known whether the degrees of every $\mathscr{F}$-comeager family of sets generate the degrees when $\mathscr{F}$ is the family of closed sets defined by admissible triples [19, 16] or by recursive perfect trees [17].

§4. Degrees below $\mathbf{0}'$

THEOREM 4.1. *Let $\boldsymbol{b}$ be any fixed degree in $\mathscr{D}(\leqq \mathbf{0}')$. Then $\{\boldsymbol{c} \leqq \mathbf{0}' : \boldsymbol{c}' = \boldsymbol{b}'\}$ generates $\mathscr{D}(\leqq \mathbf{0}')$.*

PROOF. The proof makes use of some definitions and results from [6]. For each $n \geq 1$ let GH_n be the class of degrees $\boldsymbol{a}$ such that $\boldsymbol{a}^{(n)} = (\boldsymbol{a} \cup \boldsymbol{0}')^{(n)}$ and let GL_n be the class of degrees $\boldsymbol{a}$ such that $\boldsymbol{a}^{(n)} = (\boldsymbol{a} \cup \boldsymbol{0}')^{(n-1)}$. The restrictions of these classes to $\mathcal{D}(\leqq \boldsymbol{0}')$ are the "high" and "low" classes H_n consisting of all degrees $\boldsymbol{a} \leqq \boldsymbol{0}'$ satisfying $\boldsymbol{a}^{(n)} = \boldsymbol{0}^{(n+1)}$ and L_n consisting of all degrees $\boldsymbol{a} \leqq \boldsymbol{0}'$ satisfying $\boldsymbol{a}^{(n)} = \boldsymbol{0}^{(n)}$.

By corollary 8 of [6], if $\boldsymbol{a} \leqq \boldsymbol{0}'$ and $\boldsymbol{a} \notin L_2$ then there exist degrees $\boldsymbol{b}, \boldsymbol{c}$ in L_1 such that $\boldsymbol{a} = \boldsymbol{b} \cup \boldsymbol{c}$. Thus, since $L_1 \subseteq L_2$, L_2 generates $\mathcal{D}(\leqq \boldsymbol{0}')$.

By theorem 2 of [6], if $\boldsymbol{a} \notin GL_2$ then for any $\boldsymbol{c} \geqq \boldsymbol{a} \cup \boldsymbol{0}'$ such that $\boldsymbol{c}$ is r.e. in $\boldsymbol{a}$, there exists a degree $\boldsymbol{b} \leqq \boldsymbol{a}$ such that $\boldsymbol{b}' = \boldsymbol{c}$. The proof of this result is easily modified to show that under the same hypotheses there exists a minimal pair of degrees $\boldsymbol{b}_1, \boldsymbol{b}_2$ such that $\boldsymbol{b}_1' = \boldsymbol{b}_2' = \boldsymbol{c}$. (This is accomplished by simply "throwing in" the standard requirements for the construction of a minimal pair via e-splittings and modifying the proof of theorem 2 of [6] accordingly.) Note that if $\boldsymbol{d} \in L_2$ then $\boldsymbol{d}'' = \boldsymbol{0}''$, so $\boldsymbol{0}'$ is not in $L_2^{\boldsymbol{d}}$ (L_2 relativized to $\boldsymbol{d}$). Thus we can relativize the result above to $\boldsymbol{d}$ to obtain:

(*) Let $\boldsymbol{d}$ be an element of L_2 and let $\boldsymbol{c}$ be $\geqq \boldsymbol{d}'$ and r.e. in $\boldsymbol{0}'$. Then there exist degrees $\boldsymbol{b}_1$ and $\boldsymbol{b}_2 \leqq \boldsymbol{0}'$ such that $\boldsymbol{d} = \boldsymbol{b}_1 \cap \boldsymbol{b}_2$ and $\boldsymbol{c} = \boldsymbol{b}_1' = \boldsymbol{b}_2'$.

In particular, taking $\boldsymbol{c}$ to be $\boldsymbol{0}''$ it follows that every degree in L_2 is the g.l.b. of a pair of degrees in H_1. Thus since L_2 generates $\mathcal{D}(\leqq \boldsymbol{0}')$, H_1 generates $\mathcal{D}(\leqq \boldsymbol{0}')$. Further, since no H_1 degree is in L_2 and (as previously remarked) every non-L_2 degree below $\boldsymbol{0}'$ is the l.u.b. of a pair of L_1 degrees, we conclude that L_1 generates $\mathcal{D}(\leqq \boldsymbol{0}')$.

Finally, let $\boldsymbol{b}$ be any fixed degree in $\mathcal{D}(\leqq \boldsymbol{0}')$. Then since $\boldsymbol{b}' \geqq \boldsymbol{0}'$ and $\boldsymbol{b}'$ is r.e. in $\boldsymbol{0}'$ we may apply (*) with $\boldsymbol{c} = \boldsymbol{b}'$ to conclude that every degree $\boldsymbol{d}$ in L_1 is the g.l.b. of a pair of degrees in $\mathcal{D}(\leqq \boldsymbol{0}')$ having jump $\boldsymbol{b}'$. Thus $\{\boldsymbol{c} \leqq \boldsymbol{0}' : \boldsymbol{c}' = \boldsymbol{b}'\}$ generates L_1 and so generates $\mathcal{D}(\leqq \boldsymbol{0}')$.

We next wish to obtain an analogue of Theorem 2.4 for $\mathcal{D}(\leqq \boldsymbol{0}')$. For any string σ, let $\mathcal{N}(\sigma)$ denote the set of all B in 2^ω such that $B \supseteq \sigma$, and for any set of strings D let $\mathcal{N}(D)$ be the union over all σ in D of $\mathcal{N}(\sigma)$. (Obviously $\mathcal{N}(D)$ is open in the usual topology on 2^ω.) We say that D is *dense* if $\mathcal{N}(D)$ is dense in the usual topology. Following Yates [21], we say that a set $\mathcal{A} \subseteq 2^\omega$ is $\boldsymbol{a}$-comeager, where $\boldsymbol{a}$ is a fixed degree, if there is a uniformly $\boldsymbol{a}$-recursive sequence of dense sets of strings $\{D_i\}$ such that $\mathcal{A} \supseteq \bigcap_i \mathcal{N}(D_i)$. (Equivalently, $\mathcal{A}$ is $\boldsymbol{a}$-comeager if player II has an $\boldsymbol{a}$-recursive winning strategy in the Banach–Mazur game for $\mathcal{A}$.) This definition is extended to sets $\mathcal{A} \subseteq 2^\omega \times 2^\omega, 2^\omega \times 2^\omega \times 2^\omega$, etc. in the obvious way.

It is clear that the intersection of finitely many $\boldsymbol{a}$-comeager sets is $\boldsymbol{a}$-comeager.

In addition, an analysis of the proofs of the Baire Category Theorem and the Kuratowski–Ulam Theorem (see [12]) yields the following two propositions. (The first is a special case of the "Generalized Baire Category Theorem" of [21].)

PROPOSITION 4.2. *Suppose $\mathscr{A} \subseteq 2^{\omega}$ is $\boldsymbol{a}$-comeager. Then there is a $B \in \mathscr{A}$ such that the degree of B is $\leqq \boldsymbol{a}$.*

PROPOSITION 4.3. *Suppose $\mathscr{A} \subseteq 2^{\omega} \times 2^{\omega}$ is $\boldsymbol{a}$-comeager. Then $\boldsymbol{a}$-comeager many cross sections of $\mathscr{A}$ are $\boldsymbol{a}$-comeager.*

The following is an effectivization of Theorem 2.4.

THEOREM 4.4. *Suppose $\mathscr{A}$ is $\boldsymbol{a}$-comeager. Then for any degree $\boldsymbol{b}$ there exist $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4$ in $\mathscr{A} \cap \mathscr{D}(\leqq(\boldsymbol{a} \cup \boldsymbol{b}'))$ such that $\boldsymbol{b} = (\boldsymbol{a}_1 \cup \boldsymbol{a}_2) \cap (\boldsymbol{a}_3 \cup \boldsymbol{a}_4)$.*

PROOF. Let $\boldsymbol{b}$ be fixed.

LEMMA. *$\{\boldsymbol{c} : \exists \boldsymbol{a}_1, \boldsymbol{a}_2 \in \mathscr{A}$ such that $\boldsymbol{c} \cup \boldsymbol{b} = \boldsymbol{a}_1 \cup \boldsymbol{a}_2\}$ is $(\boldsymbol{a} \cup \boldsymbol{b})$-comeager.*

PROOF. Observe that if $B \in \boldsymbol{b}$ then $B \triangle \mathscr{A}$ is $(\boldsymbol{b} \cup \boldsymbol{a})$-comeager. The remainder of the proof then follows as in the proof of the corresponding lemma in the proof of Theorem 2.4.

LEMMA. *$\{(\boldsymbol{c}, \boldsymbol{d}) : \boldsymbol{b} = (\boldsymbol{c} \cup \boldsymbol{b}) \cap (\boldsymbol{d} \cup \boldsymbol{b})\}$ is $\boldsymbol{b}'$-comeager.*

PROOF. Let $B \in \boldsymbol{b}$ and for each $e = \langle e_0, e_1 \rangle$ let D_e be the set of all pairs of strings (σ, τ) such that

(1) $\{e_0\}^{B \oplus \sigma}$ and $\{e_1\}^{B \oplus \tau}$ are incompatible, or

(2) $\forall \gamma, \delta \supseteq \sigma$, $\{e_0\}^{B \oplus \gamma}$ and $\{e_0\}^{B \oplus \delta}$ are compatible, or

(3) $\exists x \leq lh(\tau)$, $\forall \gamma \supseteq \tau$, $\{e_1\}^{B \oplus \gamma}$ is undefined at x. The D_e are easily seen to be dense and uniformly recursive in $\boldsymbol{b}'$. Further, if $(C, D) \in \mathscr{N}(D_e)$ then

(1) $\{e_0\}^{B \oplus C}$ is incompatible with $\{e_1\}^{B \oplus D}$, or

(2) $\{e_0\}^{B \oplus C}$ is either nontotal or recursive in B, or

(3) $\{e_1\}^{B \oplus D}$ is undefined at some x.

Thus if $(C, D) \in \bigcap_e \mathscr{N}(D_e)$ then $\boldsymbol{b} = (\deg(C) \cup \boldsymbol{b}) \cap (\deg(D) \cup \boldsymbol{b})$.

Now let $\mathscr{A}_1 = \{\boldsymbol{c} : \boldsymbol{c} \cup \boldsymbol{b} = \boldsymbol{a}_1 \cup \boldsymbol{a}_2$ for some $\boldsymbol{a}_1, \boldsymbol{a}_2 \in \mathscr{A}\}$ and let $\mathscr{A}_2 = \{\boldsymbol{c} : \{\boldsymbol{d} : \boldsymbol{b} = (\boldsymbol{c} \cup \boldsymbol{b}) \cap (\boldsymbol{d} \cup \boldsymbol{b})\}$ is $\boldsymbol{b}'$-comeager$\}$. By the preceding two lemmas and Proposition 4.3, $\mathscr{A}_1$ and $\mathscr{A}_2$ are $(\boldsymbol{b}' \cup \boldsymbol{a})$-comeager. Thus, by Proposition 4.2, there is a degree $\boldsymbol{c} \in \mathscr{A}_1 \cap \mathscr{A}_2$ such that $\boldsymbol{c} \leqq \boldsymbol{b}' \cup \boldsymbol{a}$. Since $\boldsymbol{c} \in \mathscr{A}_2$, $\{\boldsymbol{d} : \boldsymbol{b} = (\boldsymbol{c} \cup \boldsymbol{b}) \cap (\boldsymbol{d} \cup \boldsymbol{b})\}$ is $(\boldsymbol{b}' \cup \boldsymbol{a})$-comeager and so there is a $\boldsymbol{d} \in \mathscr{A}_1 \cap \mathscr{D}(\leqq(\boldsymbol{a} \cup \boldsymbol{b}'))$ such that $\boldsymbol{b} = (\boldsymbol{c} \cup \boldsymbol{b}) \cap (\boldsymbol{d} \cup \boldsymbol{b})$. Since $\boldsymbol{c}, \boldsymbol{d} \in \mathscr{A}_1$, there are $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4 \in \mathscr{A}$ such that $\boldsymbol{c} =$

$\boldsymbol{a}_1 \cup \boldsymbol{a}_2$ and $\boldsymbol{d} = \boldsymbol{a}_3 \cup \boldsymbol{a}_4$. We thus have $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4 \in \mathscr{A} \cap \mathscr{D}(\leqq (\boldsymbol{a} \cup \boldsymbol{b}'))$ such that $\boldsymbol{b} = (\boldsymbol{a} \cup \boldsymbol{a}_2) \cap (\boldsymbol{a}_3 \cup \boldsymbol{a}_4)$ as required.

A set of degrees $\mathscr{A} \subseteq \mathscr{D}(\leqq \mathbf{0}')$ is said to be comeager in $\mathscr{D}(\leqq \mathbf{0}')$ if $\mathscr{A} = \mathscr{A}^* \cap \mathscr{D}(\leqq \mathbf{0}')$ for some $\mathbf{0}'$-comeager set of degrees $\mathscr{A}^*$. By Proposition 4.2, if $\mathscr{A}$ is comeager in $\mathscr{D}(\leqq \mathbf{0}')$ then $\mathscr{A}$ is nonempty. We in fact have the following analogue to Theorem 2.4.

THEOREM 4.5. *Suppose $\mathscr{A}$ is comeager in $\mathscr{D}(\leqq \mathbf{0}')$. Then $\mathscr{A}$ generates $\mathscr{D}(\leqq \mathbf{0}')$.*

PROOF. By Theorem 4.1 it will suffice to show that $\mathscr{A}$ generates L_1. Let $\mathscr{A}^*$ be a $\mathbf{0}'$-comeager set of degrees such that $\mathscr{A} = \mathscr{A}^* \cap \mathscr{D}(\leqq \mathbf{0}')$ and let $\boldsymbol{b}$ be any degree in L_1. By Theorem 4.4, there exist $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4 \in \mathscr{A}^* \cap \mathscr{D}(\leqq (\mathbf{0}' \cup \boldsymbol{b}'))$ such that $\boldsymbol{b} = (\boldsymbol{a}_1 \cup \boldsymbol{a}_2) \cap (\boldsymbol{a}_3 \cup \boldsymbol{a}_4)$. Since $\boldsymbol{b}' = \mathbf{0}'$, $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4$ are $\leqq \mathbf{0}'$ and so in $\mathscr{A}$.

We give several corollaries to Theorem 4.5. If σ is a string we say that $\{e\}^\sigma(x)$ is *strongly undefined* if $\{e\}^\gamma(x)$ is undefined for all $\gamma \supseteq \sigma$. A set B is called 1-*generic* if for each e there is a string $\sigma \subseteq B$ such that $\{e\}^\sigma(e)$ is defined or strongly undefined.

COROLLARY 4.6. *The set of degrees $\boldsymbol{b} \leqq \mathbf{0}'$ such that $\boldsymbol{b}$ contains a* 1-*generic set generates $\mathscr{D}(\leqq \mathbf{0}')$.*

PROOF. For each e let D_e be the set of all strings σ such that $\{e\}^\sigma(e)$ is defined or strongly undefined. The D_e are easily seen to be dense and uniformly recursive in $\mathbf{0}'$. Thus the class of 1-generic sets is $\mathbf{0}'$-comeager and so the corollary is immediate by Theorem 4.5.

COROLLARY 4.7. *For any nonrecursive degree $\boldsymbol{a} \leqq \mathbf{0}'$, $\{\boldsymbol{b} : \boldsymbol{a} \nleqq \boldsymbol{b}\}$ generates $\mathscr{D}(\leqq \mathbf{0}')$.*

PROOF. Suppose $\mathbf{0} < \boldsymbol{a} < \mathbf{0}'$. Let $A \in \boldsymbol{a}$ and for each e let D_e be the set of all strings σ such that $\{e\}^\sigma$ is incompatible with A or $\forall \gamma, \delta \supseteq \sigma$, $\{e\}^\gamma$ and $\{e\}^\delta$ are compatible. The D_e are dense and, since $\boldsymbol{a} \leqq \mathbf{0}'$, uniformly recursive in $\mathbf{0}'$. Further, if $B \in \mathscr{N}(D_e)$ then either $\{e\}^B$ is incompatible with A or if $\{e\}^B$ is total it is recursive. Thus $\{e\}^B \neq A$. Hence, if $B \in \cap_e \mathscr{N}(D_e)$ then $A \nleq_T B$, and so the class of all B such that $\boldsymbol{a} \nleq \deg(B)$ is $\mathbf{0}'$-comeager. The corollary now follows by Theorem 4.5.

COROLLARY 4.8. *Suppose $\boldsymbol{a} \in L_2$. Then $\{\boldsymbol{b} : \boldsymbol{b} \cap \boldsymbol{a} = \mathbf{0}\}$ generates $\mathscr{D}(\leqq \mathbf{0}')$.*

PROOF. Since $\boldsymbol{a} \leqq \mathbf{0}'$ and $\boldsymbol{a}'' = \mathbf{0}''$, the class of sets recursive in $\boldsymbol{a}$ is uniformly recursive in $\mathbf{0}'$ [3, theorem 1 relativized to $\boldsymbol{a}$]. Let $\{A_i\}$ be a $\mathbf{0}'$-recursive enumeration of the sets recursive in $\boldsymbol{a}$. For each e and i let $D_{\langle e,i\rangle}$ be defined as in the proof of Corollary 4.7 with A_i in place of A. We again have that the $D_{\langle e,i\rangle}$ are dense and, since $\{A_i\}$ is recursive in $\mathbf{0}'$, the $D_{\langle e,i\rangle}$ are uniformly recursive in $\mathbf{0}'$. Also, if $B \in \mathcal{N}(D_{\langle e,i\rangle})$ then either $\{e\}^B \neq A_i$ or A_i is recursive. Thus, if $B \in \bigcap_{\langle e,i\rangle} \mathcal{N}(D_{\langle e,i\rangle})$ then $\deg(B) \cap \boldsymbol{a} = \mathbf{0}$. Hence, the set of degrees below $\mathbf{0}'$ which form a minimal pair with $\boldsymbol{a}$ is comeager in $\mathcal{D}(\leqq \mathbf{0}')$ and so, by Theorem 4.5, generates $\mathcal{D}(\leqq \mathbf{0}')$.

In a future paper Posner will show that for *any* degree $\boldsymbol{a} < \mathbf{0}'$ $\{\boldsymbol{b} \leqq \mathbf{0}' : \boldsymbol{b} \cap \boldsymbol{a} = \mathbf{0}\}$ generates $\mathcal{D}(\leqq \mathbf{0}')$. In fact it will be shown that the set of minimal degrees $\boldsymbol{b} \leqq \mathbf{0}'$ which form a minimal pair with $\boldsymbol{a}$ generates $\mathcal{D}(\leqq \mathbf{0}')$.

COROLLARY 4.9. *There exists a pair of nonrecursive degrees $\boldsymbol{c}, \boldsymbol{d} \leqq \mathbf{0}'$ such that $\{\boldsymbol{a} \leqq \mathbf{0}' : \boldsymbol{a} \geqq \boldsymbol{c}$ or $\boldsymbol{a} \geqq \boldsymbol{d}\}$ generates $\mathcal{D}(\leqq \mathbf{0}')$. In fact, the set of such pairs is comeager in $\mathcal{D}(\leqq \mathbf{0}')$.*

PROOF. We first show that the set of triples $(\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d})$ such that $\boldsymbol{b} = (\boldsymbol{b} \cup \boldsymbol{c}) \cap (\boldsymbol{b} \cup \boldsymbol{d})$ is $\mathbf{0}'$-comeager. For each i, j let $D_{\langle i,j\rangle}$ be the set of triples (β, γ, δ) such that

(1) $\{i\}^{\beta \oplus \gamma}$ is incompatible with $\{j\}^{\beta \oplus \delta}$, or

(2) $\forall \sigma, \tau \supseteq \gamma$, $\forall \alpha \supseteq \beta$ $\{i\}^{\alpha \oplus \sigma}$ is compatible with $\{i\}^{\alpha \oplus \tau}$, or

(3) $\exists x \leq \mathrm{lh}(\delta)$, $\forall \sigma \geq \delta$, $\forall \alpha \supseteq \beta$, $\{j\}^{\alpha \oplus \sigma}$ is undefined at x.

The reader may easily verify that the $D_{\langle i,j\rangle}$ are dense and uniformly recursive in $\mathbf{0}'$. Further, if $(B, C, D) \in \mathcal{N}(D_{\langle i,j\rangle})$ then

(1) $\{i\}^{B \oplus C}$ is incompatible with $\{j\}^{B \oplus D}$, or

(2) $\{i\}^{B \oplus C}$ is partial recursive in B, or

(3) $\{j\}^{B \oplus D}$ is undefined at some argument.

Thus, if $(B, C, D) \in \bigcap_{\langle i,j\rangle} \mathcal{N}(D_{\langle i,j\rangle})$ then

$$\deg(B) = (\deg(B) \cup \deg(C)) \cap (\deg(B) \cup \deg(D))$$

and so

$$\{(\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}) : \boldsymbol{b} = (\boldsymbol{b} \cup \boldsymbol{c}) \cap (\boldsymbol{b} \cup \boldsymbol{d})\}$$

is $\mathbf{0}'$-comeager, as claimed.

Now, applying Proposition 4.3 to the set of triples defined above we see that $\{(\boldsymbol{c}, \boldsymbol{d}) : \{\boldsymbol{b} : \boldsymbol{b} = (\boldsymbol{b} \cup \boldsymbol{c}) \cap (\boldsymbol{b} \cup \boldsymbol{d})\}$ is $\mathbf{0}'$-comeager$\}$ is $\mathbf{0}'$-comeager. Thus, the set of such pairs $(\boldsymbol{c}, \boldsymbol{d})$ where $\boldsymbol{c}, \boldsymbol{d} \leqq \mathbf{0}'$ is comeager in $\mathcal{D}(\leqq \mathbf{0}')$. For any such pair, $\{\boldsymbol{a} \leqq \mathbf{0}' : \boldsymbol{a} \geqq \boldsymbol{c}$ or $\boldsymbol{a} \geqq \boldsymbol{d}\}$ generates a set which is comeager in $\mathcal{D}(\leqq \mathbf{0}')$ and so, by Theorem 4.5, generates all of $\mathcal{D}(\leqq \mathbf{0}')$.

THEOREM 4.10. *Suppose $\mathbf{0} < \boldsymbol{b} \leqq \mathbf{0}'$. Then the set of $\boldsymbol{c} < \mathbf{0}'$ such that $\boldsymbol{c} \cup \boldsymbol{b} = \mathbf{0}'$ generates $\mathcal{D}(\leqq \mathbf{0}')$.*

PROOF. The proofs of Corollaries 4.6 and 4.7 show that L_1 and $\{\boldsymbol{d}<\boldsymbol{0}':\boldsymbol{d}\not\geq\boldsymbol{b}\}$ are comeager in $\mathcal{D}(\leq\boldsymbol{0}')$. (Here we have used the well-known result that 1-generic degrees are in GL_1, see [4].) The intersection of these sets is again comeager in $\mathcal{D}(\leq\boldsymbol{0}')$ so, by Theorem 4.5, it will suffice to show that $\{\boldsymbol{c}<\boldsymbol{0}': c\cup\boldsymbol{b}=\boldsymbol{0}'\}$ generates $\{\boldsymbol{d}<\boldsymbol{0}':\boldsymbol{d}'=\boldsymbol{0}' \text{ and } \boldsymbol{b}\not\leq\boldsymbol{d}\}$.

Suppose $\boldsymbol{d}'=\boldsymbol{0}'$ and $\boldsymbol{b}\not\leq\boldsymbol{d}$. We will show that there exist $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ such that $\boldsymbol{d}=\boldsymbol{c}_1\cap\boldsymbol{c}_2$ and $\boldsymbol{c}_1\cup\boldsymbol{b}=\boldsymbol{c}_2\cup\boldsymbol{b}=\boldsymbol{0}'$. Since $\boldsymbol{b}\not\leq\boldsymbol{d}$, we have $\boldsymbol{d}<\boldsymbol{d}\cup\boldsymbol{b}\leq\boldsymbol{0}'$. Thus, since $\boldsymbol{d}'=\boldsymbol{0}'$, the results of [14] may be relativized to $\boldsymbol{d}$ to obtain $\boldsymbol{c}_1,\boldsymbol{c}_2\geq\boldsymbol{d}$ such that $\boldsymbol{c}_1\cap\boldsymbol{c}_2=\boldsymbol{d}$ and $\boldsymbol{c}_1\cup(\boldsymbol{b}\cup\boldsymbol{d})=\boldsymbol{c}_2\cup(\boldsymbol{b}\cup\boldsymbol{d})=\boldsymbol{0}'$. Since $\boldsymbol{c}_1,\boldsymbol{c}_2$ are $\geq\boldsymbol{d}$, $\boldsymbol{c}_1\cup(\boldsymbol{b}\cup\boldsymbol{d})=\boldsymbol{c}_1\cup\boldsymbol{b}$ and $\boldsymbol{c}_2\cup(\boldsymbol{b}\cup\boldsymbol{d})=\boldsymbol{c}_2\cup\boldsymbol{b}$. Thus, $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ have the desired properties and so the proof of Theorem 4.10 is complete.

In the above proof, $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ can be taken to be degrees having jump $\boldsymbol{0}'$. Thus if $\boldsymbol{0}<\boldsymbol{b}<\boldsymbol{0}'$, $\{\boldsymbol{c}\leq\boldsymbol{0}':\boldsymbol{c}'=\boldsymbol{0}' \text{ and } \boldsymbol{c}\cup\boldsymbol{b}=\boldsymbol{0}'\}$ generates $\mathcal{D}(\leq\boldsymbol{0}')$. Further, if $\boldsymbol{0}<\boldsymbol{b}<\boldsymbol{0}'$ and $\boldsymbol{b}''=\boldsymbol{0}''$ then the set of degrees which are complementary to $\boldsymbol{b}$ in $\mathcal{D}(\leq\boldsymbol{0}')$, i.e., $\boldsymbol{c}$ such that $\boldsymbol{c}\cup\boldsymbol{b}=\boldsymbol{0}'$ and $\boldsymbol{c}\cap\boldsymbol{b}=\boldsymbol{0}'$, generates $\mathcal{D}(\leq\boldsymbol{0}')$.

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