ISRAEL JOURNAL OF MATHEMATICS, Vol. 43, No. 2, 1982

FINITENESS CONDITIONS IN KRULL SUBRINGS OF A RING OF POLYNOMIALS

BY
BRONISLAW WAJNRYB

ABSTRACT

Let R be a Krull subring of a ring of polynomials $k[x_1,\cdots,x_n]$ over a field k. We prove that if R is generated by monomials over k then R is affine. We also construct an example of a non-affine Krull ring R, such that $k[x,xy]\subset R\subset k[x,y]$, and a non-Noetherian Krull ring S, such that $k[x,xy,z]\subset S\subset k[x,y,z]$.

1. Introduction

Let k be a fixed field. By a ring we shall mean an integral domain containing k. An affine ring is a ring finitely generated over k. By a minimal prime ideal of a ring R we shall mean a prime ideal of height 1.

Let R be a Krull subring of a ring of polynomials $k[x_1,\cdots,x_n]$, and let K be the quotient field of R. Every element of R belongs to at most a finite number of minimal prime ideals of R. This finiteness condition has many implications but in general R is neither affine nor Noetherian. Hilbert asked in his 14-th problem whether R is affine, assuming that $R=K\cap k[x_1,\dots,x_n]$? Zariski proved that R is affine if $\mathrm{trdim}_k K<3$ and Nagata found a counterexample with $\mathrm{trdim}_k K=4$. If we allow $x_1,\cdots,x_n$ to be algebraically dependent then counterexamples exist for $\mathrm{trdim}_k K=3$. Many affirmative results are known under special conditions ([9]). A slightly different question was asked by Heinzer in [7]: Let R_0 be an affine Krull ring with the quotient field K and let R be a Krull ring, such that $R_0\subset R\subset K$. Is R Noetherian? The answer is again yes if $\mathrm{trdim}_k K<3$. An example of such a 3-dimensional non-Noetherian ring R is given in [4], but R is not contained in a ring of polynomials.

In section 3 we give two new examples of non-finiteness in Krull subrings of a ring of polynomials. We construct a non-affine Krull ring R, such that $k[x,xy]\subset R\subset k[x,y]$, and a non-Noetherian Krull ring S, such that $k[x,xy,z]\subset S\subset$

Received May 22, 1981 and in revised form February 25, 1982

$k[x, y, z]$. The divisor class group of R and of S is equal to $\mathbf{Z}$, the group of integers. In fact it could not be smaller because of the following result of Zaks ([12]): If R is a Krull ring with the divisor class group torsion and if $A \subset R \subset A[x]$ then R is finitely generated over A.

In section 2 we prove the following theorem.

THEOREM. *Let R be a Krull ring generated by some monomials in a ring of polynomials $k[x_1, \cdots, x_n]$. Then R is affine.*

The ring R in the theorem is a semigroup ring. Such rings have been extensively investigated (e.g. [1], [2], [3], [5], [6], [8], [11]). Anderson and Hochster were particularly interested in Krull rings generated by monomials. Anderson obtained many interesting results assuming that R is affine. Hochster proved that if R is Noetherian then it is affine and Cohen–Macauley. Our result shows that these assumptions are satisfied by every Krull ring generated by monomials.

The author is grateful to Professor Zaks for many helpful discussions which contributed to this paper, and to the referee for suggesting a simplification of the proof of Lemma 1.

2. Krull rings generated by monomials

In this section R is a Krull ring generated by some monomials in a ring of polynomials $k[x_1, \cdots, x_n]$ over a field k. We want to prove the following

THEOREM. *The ring R is affine.*

Let us fix some notation. Since R is a Krull ring, the localization R_p of R at a minimal prime ideal p of R is a discrete valuation ring. We shall denote by v_p the corresponding discrete valuation of the quotient field K of R. We shall denote by Γ the semigroup of monomials in R and by G the group of monomials generated by Γ. Then G is freely generated by algebraically independent monomials $y_1, \cdots, y_m$ and $K = k(y_1, \cdots, y_m)$. The ring R is a semigroup ring, $R = k[\Gamma]$.

LEMMA 1. *There exists a finite number of minimal prime ideals $p_1, \cdots, p_r$ of R, such that $\Gamma = \{g \in G;\ v_{p_i}(g) \geqq 0$ for $i = 1, \cdots, r\}$.*

PROOF. Clearly $\Gamma = \{g \in G;\ v_p(g) \geqq 0$ for all minimal prime ideals p of $R\}$. For each y_i, $v_p(y_i) = 0$ for all but finitely many p. Let $\{p_1, \cdots, p_r\}$ be the set of minimal prime ideals (necessarily finite) for which $v_{p_j}(y_i) \neq 0$ for some i. Then for

$p \notin \{p_1, \cdots, p_r\}$, $v_p(g) = 0$ for all $g \in G$. Thus $\Gamma = \{g \in G;\ v_{p_i}(g) \geqq 0$ for $i = 1, \cdots, r\}$. □

LEMMA 2. *Let F be a free abelian group of rank m. Let $v_1, \cdots, v_r$ be homomorphisms of F into the integers and let $F_0 = \{g \in F;\ v_i(g) \geqq 0$ for $i = 1, \cdots, r\}$. Then F_0 is a finitely generated semigroup.*

PROOF. We can regard F as a set of points with integer coordinates in a real linear space $V = R^m$. Let $\mathbf{o}$ denote the origin of V. We can extend v_i to a linear functional on V represented in the standard basis by a vector with integer coefficients. Let $H_i = \{x \in V;\ v_i(x) \geqq 0\}$ for $i = 1, \cdots, r$. Then the cone $C = \bigcap_{i=1}^{r} H_i$ is a convex hull of a finite number of halflines $L_1, \cdots, L_s$ originating at $\mathbf{o}$, and each L_i contains a point $g_i \in F$ (see the remark below). Every point of C is a positive combination of g_i's. Let

$$N = \left\{x \in F \cap C;\ x = \sum_{i=1}^{s} a_i g_i,\ 0 \leqq a_i \leqq 1 \text{ for } i = 1, \cdots, s\right\}.$$

Then N is a finite set and it generates the semigroup $F_0 = F \cap C$. Indeed every point in F_0 can be written as $(\Sigma_{i=1}^{s} n_i g_i + (\text{something in } N))$, where n_i is a non-negative integer for $i = 1, \cdots, s$. Therefore F_0 is finitely generated. □

REMARK. The set of halflines $L_1, \cdots, L_s$ considered in the above proof can be obtained in the following way. If C contains a whole line we shall first divide it into smaller cones by using the coordinate hyperplanes. We can intersect each small cone with a suitable hyperplane in order to get a bounded intersection. The intersection is a convex bounded polyhedron which is equal to the convex hull of its vertices. Each vertex lies on a halfline through $\mathbf{o}$, which is defined by equations with integer coefficients, hence contains a point of F. Clearly C is the convex hull of all these halflines.

PROOF OF THE THEOREM. The ring R is generated by the monomials in Γ. By Lemmas 1 and 2, Γ is finitely generated. Therefore R is finitely generated. □

3. Non-affine Krull subrings of $k[x, y]$ and $k[x, y, z]$

Let k be a field and let $k[x, y]$ denote a ring of polynomials over k. Let $a_1, a_2, \cdots$ be an infinite sequence of elements of k. Let $g_0, g_1, \cdots$ be a sequence of polynomials defined by induction as follows: $g_0 = x$, $g_i = yg_{i-1} + a_i$ for $i \geqq 1$. Let $R = k[g_0, g_1, \cdots]$.

We shall prove that R is a Krull ring for a suitable choice of the field k and of the sequence (a_i), but first we shall prove some general facts.

LEMMA 3. *R is freely generated by $1, g_1, g_2, \cdots$ as a $k[x]$-module.*

PROOF. Clearly $1, g_1, g_2, \cdots$ are independent over $k[x]$. It is enough to show that $g_n \cdot g_m$ belongs to the module for every n, m. We shall prove by induction on m that for every $n \geqq m \geqq 1$ we have

$$g_n g_m = x(g_{n+m} - a_{n+m}) + \sum_{i=0}^{m-1} a_{m-i} g_{n+i} - \sum_{i=1}^{m-1} a_{n+i} g_{m-i}.$$

For $m = 1$

$$g_n g_1 = g_n(yx + a_1) = xyg_n + a_1 g_n = x(g_{n+1} - a_{n+1}) + a_1 g_n.$$

Assume that the formula is true for m and let $n \geqq m + 1$.

$$\begin{aligned} g_n g_{m+1} &= g_n(yg_m + a_{m+1}) \\ &= y\left(xg_{n+m} - xa_{n+m} + \sum_{i=0}^{m-1} a_{m-i} g_{n+i} - \sum_{i=1}^{m-1} a_{n+i} g_{m-i}\right) + a_{m+1} g_n \\ &= x(g_{n+m+1} - a_{n+m+1}) - a_{n+m}(g_1 - a_1) \\ &\quad + \sum_{i=0}^{m-1} a_{m-i}(g_{n+i+1} - a_{n+i+1}) - \sum_{i=1}^{m-1} a_{n+i}(g_{m+1-i} - a_{m+1-i}) + a_{m+1} g_n. \end{aligned}$$

Since $\Sigma_{i=0}^{m-1} a_{m-i}(g_{n+i+1} - a_{n+i+1}) = \sum_{i=1}^{m} a_{m+1-i}(g_{n+i} - a_{n+i})$ we get finally

$$g_n g_{m+1} = x(g_{n+m+1} - a_{n+m+1}) + \sum_{i=0}^{m} a_{m+1-i} g_{n+i} - \sum_{i=1}^{m} a_{n+i} g_{m+1-i}. \qquad \square$$

It follows from Lemma 3 that every element of R can be written in a unique way as $\Sigma F_i(x) g_i + c$, $c \in k$. The set $M = \{f \in R; f = \Sigma F_i(x) g_i\}$ is a maximal ideal of R. We want R to be a Krull ring. In particular R should satisfy the following condition

(*) For every $f \in R \setminus \{0\}$ there exists $n > 0$, such that $fy^n \not\in R$.

Since yg_i belongs to R for every i and y is not in R, the condition (*) is equivalent to the existence of n, such that $fy^n \in R \setminus M$. Suppose that R satisfies (*). We can define a function $v(f) = \max\{n; fy^n \in R\}$ for all non-zero elements of R. Then $fy^{v(f)} \in R \setminus M$. Clearly $v(fg) = v(f) + v(g)$ and $v(f+g) \geqq \min\{v(f), v(g)\}$, hence v can be extended to a discrete valuation of $k(x, y)$ with the valuation ring V. Every polynomial f in $k[x, y]$ can be written as $f = g + A(y)$, where g belongs to R and $A(y)$ is in $k[y]$. Since $v(A(y)) = v(y) \deg A(y)$ and $v(y) < 0$ we have $R = V \cap k[x, y]$. Therefore R is a Krull ring. Since $\dim R = 2$ and the minimal prime ideal M is maximal R is not affine.

It remains to choose a sequence (a_i) in such a way that the corresponding ring R satisfies (*). The choice is not arbitrary. If we choose for example $a_i = i$, and let $f = g_2 - 2g_1 + g_0$ then $fy^n \in M$ for all n.

EXAMPLE 1. Let $k = Q(z)$ be a purely transcendental extension of the field of rational numbers. Let $a_n = \Pi_{i=1}^{n}(z-i)$. Define polynomials $g_0(x,y) = x$, $g_n(x,y) = yg_{n-1} + a_n = xy^n + \Sigma_{i=1}^{n} a_i y^{n-i}$ for $n \geqq 1$. Let $R = k[g_0, g_1, \cdots]$. We shall prove that *R is a non-affine Krull ring.*

We shall keep the notation introduced in this section. We have to prove that R satisfies (*). Suppose that there exists $f = \Sigma_{i=0}^{n}\Sigma_{j=0}^{m} b(i,j)x^j g_i$, $b(i,j) \in k$, such that $fy^r \in M$ for every r. After multiplication by an element of k we may assume that $f \in Q[x,y,z]$ and that not all of $b(i,j)$ are divisible by $(z-n-2)$ in $Q[z]$. Let $S = Q[x,y,z] \cap R$ and let $P = M \cap S$. By our assumptions $fy^r \in P$ for every r. We shall prove that this cannot happen.

By the definition of R and S every element of $Q[x,y,z]$ can be written in the unique way in a form $f = g + \Sigma d_w y^w$ where $g \in P$ and $d_w \in Q[z]$, $w \geqq 0$. We shall call d_0 the constant term of f.

Let $f = g + \Sigma d_w y^w$ and $f' = g' + \Sigma d'_w y^w$ be two polynomials with $g, g' \in P$. We shall write $d_w \equiv d'_w$ if $d_w - d'_w$ is divisible by $(z-n-2)$ and we shall write $f \equiv f'$ if $d_w \equiv d'_w$ for every w. In particular $a_i \equiv 0$ if and only if $i > n+1$. We shall prove by induction that

(1) $$y^t g_i \equiv g_{i+t} - \sum a_{t+i-u} y^u \qquad \text{where } t > u \geqq t+i-n-1, \quad u \geqq 0.$$

In particular $g_{n+1}y^t \equiv g_{n+1+t} \equiv 0$.

(2) $$x^j y^u \equiv \sum d_w y^w \qquad \text{for some } d_w \in Q[z], \quad w \geqq u-(n+1)j, \quad w \geqq 0.$$

Statement (1) is true for $t = 0$. Suppose that it is true for $t-1$. Then $y^t g_i \equiv y(g_{i+t-1} - \Sigma a_{t+i-u-1}y^u) \equiv g_{i+t} - a_{i+t} - \Sigma a_{t+i-u-1}y^{u+1}$, $t-1 > u \geqq t+i-n-2$, $u \geqq 0$. If $t+i \leqq n+1$ then the lowest value of u is 0 and $-a_{i+t}$ appears in formula (1). If $t+i > n+1$ then $a_{i+t} \in P$ and we get again formula (1). Statement (2) is true for $j = 1$ because $xy^u = g_u - \Sigma a_i y^{u-i} \equiv g_u - \Sigma_{i=1}^{n+1} a_i y^{u-i}$. Suppose that it is true for $j-1$. Then $x^j y^u \equiv \Sigma d_w x y^w \equiv \Sigma d_w(\Sigma c_{w,v} y^v)$ for some d_w, $c_{w,v}$ and $v \geqq w-n-1$, $w \geqq u-(n+1)(j-1)$, $v \geqq 0$. □

Consider our function f. Let $e = \max\{(n+1)(j+1) - i;\, b(i,j) \neq 0\}$. Since $i \leqq n$ there is a unique $b(r,s) \neq 0$, such that $e = (n+1)(s+1) - r$. We shall prove that $y^e f = \Sigma b(i,j)x^j y^e g_i \notin P$. By (1) and (2)

$$x^j y^e g_i \equiv x^j \left(g_{i+e} - \sum a_{e+i-u} y^u \right)$$

$$\equiv - \sum a_{e+i-u} \left(\sum d_{w,u} y^w \right) \quad \text{for some } d_{w,u},$$

$$u \geqq e+i-n-1, \quad w \geqq u-(n+1)j \geqq e+i-(n+1)(j+1)$$
$$= (n+1)(s+1) - r - (n+1)(j+1) + i \geqq 0.$$

By our assumptions the polynomial $\Sigma\, d_{w,u} y^w$ may have a constant term only if $i = r$, $j = s$, $u = e + i - n - 1 = (n+1)s$. We have

$$x^s y^{(n+1)s} = \left(g_{n+1} - \sum_{i=1}^{n+1} a_i y^{n+1-i} \right)^s.$$

Since $g_{n+1} y^t \equiv 0$ we have

$$x^s y^{(n+1)s} \equiv (-a_{n+1})^s + \sum_{w>0} d_w y^w$$

for some d_w. Therefore

$$y^e f = \sum b(i,j) x^j y^e g_i \equiv -b(r,s)(-a_{n+1})^{s+1} + \sum_{v>0} d_v y^v$$

for some d_v. Hence $y^e f \notin P$. This contradiction shows that there is no "bad" element f of R and the ring R satisfies condition (*).

REMARK. It follows from a theorem of Heinzer in [7] that R is Noetherian. One can prove in particular that the ideal M is generated by g_0 and g_1.

EXAMPLE 2. Consider the ring $S = R \cap Q[x, y, z]$ defined in Example 1. Clearly S is a Krull ring. Every element of S can be written in the form $\Sigma\, F_i(x, z) g_i + C(z)$. The set $P = M \cap S = \{f; f = \Sigma\, F_i(x, z) g_i\}$ is a minimal prime ideal of S. We shall prove that *no P-primary ideal is finitely generated.*

(This strongly non-Noetherian property is also satisfied by a certain minimal prime ideal in an example in [4].)

Since P is a minimal prime ideal of a Krull ring, the primary ideals belonging to P are exactly the symbolic powers $P^{(m)}$ of P. The discrete valuation belonging to P is such that $P^{(m)} = \{f \in S; f y^{m-1} \in P\}$. We shall prove first that $g_r \notin (g_0, g_1, \cdots, g_m)$ for $r > m$. Consider an element

$$\sum_{i=0}^{m} \left(\sum_{j=0}^{n} F_{i,j}(x, z) g_j + C_i(z) \right) g_i = \sum_{t=0}^{s} G_t(x, z) g_t$$

which belongs to the ideal $(g_0, g_1, \cdots, g_m)$. By the product formula of Lemma 3, $g_j g_i$ is a combination of g_t's with coefficients belonging to the ideal $(z-1, x)$ of $Q[x, z]$. In particular the coefficient $G_t(x, z)$ belongs to $(z-1, x)$ for $t > m$. Therefore $g_r \not\in (g_0, g_1, \cdots, g_m)$ for $r > m$.

We shall prove now that there exist $d_0, d_1, \cdots, d_m \in Q[z]$, such that $y^m(g_r - \Sigma_{i=0}^m d_i g_i) \in P$. Then $g_r - \Sigma_{i=0}^m d_i g_i \in P^{(m+1)}$ hence $P^{(m+1)} \not\subseteq (g_0, \cdots, g_m)$ for every m. If $P^{(s)} \in (g_0, \cdots, g_m)$ for some m and s, then either $P^{(s)} \in (g_0, \cdots, g_{s-1})$ or $P^{(m+1)} \in (g_0, \cdots, g_m)$. Therefore the existence of $d_0, d_1, \cdots, d_m$ as above will imply that no P-primary ideal is contained in a finitely generated ideal contained in P.

It is easy to prove by induction that $y^m g_i = g_{i+m} - \Sigma_{j=0}^{m-1} a_{m+i-j} y^j$. Therefore $y^m(g_r - \Sigma_{i=0}^m d_i g_i) \in P$ if and only if the d_i's satisfy the following system of equations:

$$\sum_{i=0}^{m} a_{m+i-j} d_i = a_{r+m-j} \qquad \text{for } j = 0, 1, \cdots, m-1.$$

The following lemma implies the existence of such $d_0, d_1, \cdots, d_m$.

Lemma 4. *For $r > s \geqq m$ there exist polynomials $d_0, d_1, \cdots, d_m \in Q[z]$, which satisfy*

$$\text{(3)} \qquad L_j: \ \sum_{i=0}^{m} a_{s+i-j} d_i = a_{r+m-j} \qquad \text{for } j = 0, 1, \cdots, m-1.$$

Proof. We shall prove the lemma by induction on m. For $m = 1$ we have $a_s d_0 + a_{s+1} d_1 = a_{r+m}$. Since all the coefficients are divisible by a_s, we can find a polynomial solution. Suppose that the lemma is true for $m-1$. We have $a_n = \Pi_{i=1}^n (z-i)$. Therefore $a_n - (z-i)a_{n-1} = (i-n)a_{n-1}$. Consider a new system of equations $R_j = L_j - (z - s + j)L_{j+1}$, $j = 0, 1, \cdots, m-2$. We have

$$R_j: \ \sum_{i=0}^{m} -i a_{s+i-j-1} d_i = (s - r - m) a_{r+m-j-1}$$

or

$$R_j: \ \sum_{i=1}^{m} a_{s+i-j-1}(-i d_i / s - r - m) = a_{r+m-j-1} \qquad \text{for } j = 0, 1, \cdots, m-2.$$

By the induction hypothesis there exists a polynomial solution $d_1, d_2, \cdots, d_m$ of this system. In the equation L_{m-1} all coefficients are divisible by a_{s+1-m}. Put $d_0 = (a_{r+1} - \Sigma_{i=1}^m a_{s+i+1-m} d_i)/a_{s+1-m}$. Then $d_0, d_1, \cdots, d_m$ satisfy the system (3). □

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