ISRAEL JOURNAL OF MATHEMATICS 91 (1995), 239–247

ON THE CODIMENSIONS OF THE VERBALLY PRIME P.I. ALGEBRAS

BY

ALLAN BERELE*

Department of Mathematics
DePaul University, Chicago, IL 60614, USA

AND

AMITAI REGEV**

Department of Mathematics
The Pennsylvania State University, University Park, PA 16802, USA
and
Department of Theoretical Mathematics
The Weizmann Institute of Science, Rehovot 76100, Israel

ABSTRACT

Razmyslov's theory of trace identities for the prime P.I. algebras $M_{k,\ell}$ is applied to give bounds for the cocharacters and the codimensions of these algebras $M_{k,\ell}$, as well as for the matrix algebras $M_k(E)$ over the Grassmann algebra E. These bounds are easier to obtain and are better (tighter) than earlier obtained bounds.

Introduction

The codimension series and the cocharacter series of $M_k(F)$, the $k \times k$ matrices over the characteristic zero field F, have been the object of much study. In [8], the second author obtained the asymptotic behavior of the codimensions $c_n(M_k(F))$:

$$c_n(M_k(F)) \simeq C \left(\frac{1}{n}\right)^{(k^2+1)/2} k^{2n}$$

* Work supported by NSF grant DMS 9100258.
** Work supported by NSF grant DMS 9101488.
Received December 26, 1993

where C is a certain constant which is computed explicitly. It follows that the codimensions and the trace codimensions are asymptotically equal. This work was partly based on some earlier work of Formanek in [3] and [4]. In [3] Formanek points out that the conductor can be used to compare the ordinary cocharacter of $M_k(F)$ to the trace cocharacter. In [4] he verifies that a certain polynomial constructed by Regev is indeed a non-trivial element of the conductor ideal. This particular polynomial generates an S_n-module with a particularly convenient structure. This yields not only the above information on codimensions, but also that "most" irreducible characters have the same multiplicities in the trace cocharacter and in ordinary cocharacter.

In this paper we generalize part of this theory to the other verbally prime p.i. algebras, $M_{k,\ell}$ and $M_n(E)$. First, we show that $M_{k,\ell}$ has a non-trivial conductor. This allows us to compare the trace and ordinary cocharacter and to calculate the polynomial and exponential behavior of $c_n(M_{k,\ell})$:

THEOREM 7: *There exist constants C_1 and C_2 such that, for all n,*

$$C_1\left(\frac{1}{n}\right)^{(k^2+\ell^2-1)/2}(k+\ell)^{2n}\le c_n(M_{k,\ell})\le C_2\left(\frac{1}{n}\right)^{(k^2+\ell^2-1)/2}(k+\ell)^{2n}.$$

It is worth remarking that in the case of $k=\ell=1$ the trace codimensions and the ordinary codimensions are *not* asymptotically equal. The trace codimensions are asymptotically twice as large. See [9], corollary 3.19.

Using Theorem 7 we obtain somewhat less sharp bounds on the cocharacter of $M_n(E)$. We obtain the exponential behavior exactly, but the polynomial behavior only to within $\frac{1}{2}k^2$.

THEOREM 8: *There exist constants C_3 and C_4 such that, for all n,*

$$C_3\left(\frac{1}{n}\right)^{(2k^2-1)/2}(2k^2)^n\le c_n(M_k(E))\le C_4\left(\frac{1}{n}\right)^{(k^2-1)/2}(2k^2)^n.$$

Finally, in Theorems 10 and 11 we compare the ordinary cocharacter of $M_{k,\ell}$ to the trace cocharacter. This comparison is of less direct benefit than Theorem 7, since the trace cocharacter is not all that well-understood. It is known that the nth trace cocharacter of $M_{k,\ell}$ equals the S_n-character

$$\left(\sum_{\lambda\in H(k,\ell;n+1)}\chi_\lambda\otimes\chi_\lambda\right)'$$

where the tensor is the inner tensor product and where the prime denotes inducing down from S_{n+1} to S_n. This sum seems quite interesting from the point of view of combinatorics and so Theorems 10 and 11 relate an unsolved p.i. problem to an unsolved combinatorics problem.

1. Notation and background

Let X be the set $\{x_1, x_2, ...\}$. We will denote by $\bar{F}\{X\}$ the free associative F algebra with trace on X. Elements of $\bar{F}\{X\}$ are (mixed) trace polynomials. $\bar{F}\{X\}$ contains a number of subalgebras and subspaces that will be of interest. First, $F\{X\}$ will be the free algebra on X with no trace. Likewise, $TR\{X\}$ will be the subalgebra of $\bar{F}\{X\}$ generated by the traces. It is easy to see that $\bar{F}\{X\} = F\{X\} \otimes TR\{X\}$. We next identify the multilinear, homogeneous degree n polynomials in $x_1, \ldots, x_n$ in each of these algebras. The space of multilinear $f(x_1, \ldots, x_n)$ in $F\{X\}$ is generally denoted V_n. We denote the corresponding subspaces of $\bar{F}\{X\}$ and $TR\{X\}$ by MT_n and T_n (for mixed trace and pure trace), respectively. Note that each of these three spaces is an S_n-module under the permutation action, $\sigma f(x_1, \ldots, x_n) = f(x_{\sigma 1}, \ldots, x_{\sigma n})$.

It is worthwhile to recall a few facts about the structure of these S_n-modules. First, V_n is isomorphic to FS_n with action left multiplication. A permutation σ is identified with the monomial $x_{\sigma 1} \cdots x_{\sigma n}$. Next, T_n is isomorphic to FS_n, but with conjugation action. Let π be a permutation with cycle decomposition $\pi = (i_1, i_2, \ldots, i_a)(j_1, \ldots, j_b) \cdots (k_1, \ldots, k_c)$. Then π is identified with the pure trace monomial

$$\mathrm{tr}_\pi(x_1, \ldots, x_n) = \mathrm{tr}(x_{i_1} x_{i_2} \cdots x_{i_a}) \mathrm{tr}(x_{j_1} \cdots x_{j_b}) \cdots \mathrm{tr}(x_{k_1} \cdots x_{k_c}).$$

Then, for any permutation $\sigma, \mathrm{tr}_{\sigma\pi\sigma^{-1}}(x_1, \ldots, x_n) = \mathrm{tr}_\pi(x_{\sigma 1}, \ldots, x_{\sigma n})$. Finally, we point out that V_n and T_n are each submodules of MT_n in an obvious manner. But, a bit less obvious is that there is isomorphism between MT_n and T_{n+1} given by $f(x_1, \ldots, x_n) \to \mathrm{tr}(x_{n+1} f(x_1, \ldots, x_n))$.

Let I be any trace T-ideal. If I is the ideal of trace identities for the algebra A we will write $I = I(A)$. Then $I \cap V_n$, $I \cap T_n$ and $I \cap MT_n$ are S_n-submodules of V_n, T_n and MT_n, respectively. We will be interested in the quotients whose characters give the various cocharacters of A. The nth ordinary cocharacter $\chi_n(A)$ is the S_n character of $V_n/(V_n \cap I)$; the nth mixed trace cocharacter $\chi_n^{mtr}(A)$ is the S_n character of $MT_n/(MT_n \cap I)$; and the pure trace cocharacter $\chi_n^{ptr}(A)$ is the

S_n character of $T_n/(T_n \cap I)$. The corresponding dimensions will be denoted $c_n(A), c_n^{mtr}(A)$ and $c_n^{ptr}(A)$. For any A each of $\chi_n(A)$ and $\chi_n^{ptr}(A)$ is less than or equal to χ_n^{mtr} and $c_n(A)$ and $c_n^{ptr}(A)$ are each less than or equal to $c_n^{mtr}(A)$. It is also known that if the trace on A is non-degenerate then the above isomorphism $MT_n \to T_{n+1}$ induces an S_n-isomorphism

$$MT_n/(MT_n \cap I) \to T_{n+1}/(T_{n+1} \cap I).$$

Hence, $\chi_n^{mtr}(A) = \chi_{n+1}^{ptr}(A)'$, where by the prime symbol we mean induce down from S_{n+1} to S_n.

2. Construction of the conductor

We recall two of Razmyslov's results from [6]. (See also [1] for the former.)

Trace Identities for $M_{k,\ell}$: In the identification $T_n \equiv FS_n$ the trace identities for $M_{k,\ell}$, $I(M_{k,\ell}) \cap T_n$, corresponds to a two-sided ideal of FS_n. This ideal is the sum of all simple two-sided ideals corresponding to partitions outside of $H(k,\ell;n)$.

Razmyslov's Central Polynomial: $M_{k,\ell}$ satisfies a trace identity of the form

$$p(x_1,\ldots,x_m,y) = c(x_1,\ldots,x_m)\mathrm{tr}(y),$$

where $p(x_1,\ldots,x_m,y) \in F\{X\}$ is a central polynomial and where c is of the form $c(x_1,\ldots,x_m) = \mathrm{tr}(c'(x_1,\ldots,x_m))$ and $c'(x_1,\ldots,x_m) \in F\{X\}$.

LEMMA 1: *$M_{k,\ell}$ satisfies a trace identity of the form* $\mathrm{tr}(x_1)\mathrm{tr}(x_2)\cdots\mathrm{tr}(x_n)$ *= a linear combination of trace monomials with fewer than n traces in each (n may be taken to be $(k+1)(\ell+1)$).*

Proof: If $n \geq (k+1)(\ell+1)$ then there is a partition of n outside of $H(k,\ell;n)$. So, there is an element of FS_n, say $\Sigma a_\sigma \sigma$, which corresponds to a trace identity of $M_{k,\ell}$. Since the trace identities form a two-sided ideal in FS_n, we may assume that a_{id}, the coefficient of the identity permutation, is 1. Translating back to pure trace polynomials yields an identity of the desired type.

COROLLARY 2: *Modulo $I(M_{k,\ell})$ every element of $\bar{F}\{X\}$ can be written as a linear combination of terms with at most $(k+1)(\ell+1)-1$ traces.*

We will now consider the cases of $k=\ell$ and $k \neq \ell$ separately. If $k \neq \ell$, then we may set $y = I$, the identity matrix, in Razmyslov's central polynomial to get

$p(x_1,\ldots,x_m,I)=(k-\ell)c(x_1,...,x_m)$. Hence, modulo $I(M_{k,\ell}), c(x_1,...,x_m)$ is in V_m. Let $r=m[(k+1)(\ell+1)-1]$ and let

$$d(x_1,\ldots,x_r)=c(x_1,\ldots,x_m)c(x_{m+1},\ldots,x_{2m})\cdots c(x_{r-m+1},...,x_r).$$

THEOREM 3(a): *Let $f(x_1,\ldots,x_n)$ be any element of $\bar{F}\{X\}$. Then, modulo $I(M_{k,\ell}), d(x_1,\ldots,x_r)f(x_1,\ldots,x_n)\in F\{X\}$, i.e., it can be written with no traces.*

Proof: By Corollary 2, $f(x_1,\ldots,x_n)$ can be written as a sum of an identity of $M_{k,\ell}$ and terms with $(k+1)(\ell+1)-1$ or fewer traces. Consider such a term, say $v=\mathrm{tr}(u_1)\cdots\mathrm{tr}(u_t)$ with $t\leq(k+1)(\ell+1)-1$. Then

$$vd=(c(x_1,\ldots,x_m)\mathrm{tr}(u_1))\cdots(c(x_{(t-1)m+1},...,x_{tm})\mathrm{tr}(u_t))\cdots c(x_{r-m+1},\ldots,x_r).$$

It now follows from the definition of c that this is in $I(M_{k,\ell})+F\{X\}$.

This is what we need for the $k\neq\ell$ case. If $k=\ell$, then it is not the case that $c(x_1,\ldots,x_m)\in F\{X\}(\mathrm{mod}\, I(M_{k,\ell}))$. However, given $2m$ variables, $x_1,\ldots,x_m$ and $y_1,\ldots,y_m$, then

$$c(x_1,\ldots,x_m)c(y_1,\ldots,y_m)=c(x_1,\ldots,x_m)\mathrm{tr}(c'(y_1,\ldots,y_m)).$$

And $M_{k,\ell}$ satisfies the identity

$$c(x_1,\ldots,x_m)\mathrm{tr}(c'(y_1,\ldots,y_m))=p(x_1,\ldots,x_m,c'(y_1,\ldots,y_m)),$$

the right hand side of which involves no traces. As above, we let $r=m[(k+1)^2-1]$ and

$$d(x_1,\ldots,x_r)=c(x_1,\ldots,x_m)c(x_{m+1},\ldots,x_{2m})\cdots c(x_{r-m+1},\ldots,x_r).$$

THEOREM 3(b): *Let $f(x_1,\ldots,x_n)$ be any element of $\bar{F}\{X\}$. Then, modulo $I(M_{k,k})$,*

$$d(x_1,\ldots,x_r)f(x_1,\ldots,x_n)\in F\{X\}+c(x_{r-m+1},\ldots,x_r)F\{X\}.$$

Proof: The proof is similar to the proof of Theorem 3(a). The crucial difference is this: Let $v=\mathrm{tr}(u_1)\cdots\mathrm{tr}(u_t)$ be a product of traces with $t\leq r$. In the product dv we pair the $\mathrm{tr}(u)$ with the $c(x)$. If $t<r$, then we pair the remaining $c(x)$'s with each other. And, if $r-t$ is odd, we let $c(x_{r-m+1},\ldots,x_r)$ be the leftover term. That is, we write

$$dv=(c(x_1,\ldots,x_m)\mathrm{tr}(u_1))\cdots(c(x_{(t-1)m+1},\ldots,x_{tm})\mathrm{tr}(u_t))\times$$
$$(c(x_{tm+1},\ldots,x_{t(m+1)})c(x_{t(m+1)+1},\ldots,x_{t(m+2)}))\cdots.$$

Each product of two polynomials is in $F\{X\}$ and the last term might only contain one polynomial.

3. Applications to codimensions and cocharacters

Definition: Let Δ be the map from MT_n to MT_{n+r} given by

$$f(x_1,\ldots,x_n) \longrightarrow f(x_1,\ldots,x_n)d(x_{n+1},...,x_{n+r}).$$

Here are some basic facts about Δ:

LEMMA 4:

(a) *Δ is an FS_n-module homomorphism.*

(b) *If I is an ideal of $\bar{F}\{X\}$, then $\Delta(MT_n \cap I) \subseteq (MT_{n+r} \cap I)$.*

(c) *If $f \in MT_n$ and $f \notin I(M_{k,\ell})$, then $\Delta f \notin I(M_{k,\ell})$.*

(d) *If $k \neq \ell$ and $f \in MT_n$, then $\Delta f \in V_{n+r} + I(M_{k,\ell})$, and if $k = \ell$ then $\Delta f \in V_{n+r} + V_{n+r-m}c(x_{n+r-m+1},\ldots,x_{n+r}) + I(M_{k,\ell})$.*

Proofs: (a) and (b) are obvious. (c) follows from the fact that $M_{k,\ell}$ is verbally prime. (d) follows from Theorem 3.

We wish to use Lemma 4 to get information about $c_n(M_{k,\ell})$ and $\chi_n(M_{k,\ell})$, the ordinary codimension and cocharacter. Lemma 4 will let us compare the ordinary cocharacter with the trace cocharacter. And the trace cocharacter is more or less known.

THEOREM 5: *$\chi_n^{ptr}(M_{k,\ell}) = \Sigma_{\lambda\in H(k,\ell;n)} \chi_\lambda \otimes \chi_\lambda$, where χ_λ is the irreducible S_n-character on λ.*

Proof: It follows from Razmyslov's description of the trace identities for $M_{k,\ell}$, that $\chi_n^{ptr}(M_{k,\ell})$ is the character of

$$\sum_{\lambda\in H(k,\ell;n)} I_\lambda$$

where I_λ is the two-sided ideal of FS_n and the S_n-action is by conjugation. But, the character of I_λ under conjugation is $\chi_\lambda \otimes \chi_\lambda$.

Although the S_n-character $\sum_{\lambda\in H(k,\ell;n)} \chi_\lambda \otimes \chi_\lambda$ is not yet fully understood, we do know the asymptotics of its degree. Let χ_λ have degree d_λ. Then the degree of $\sum_{\lambda\in H(k,\ell;n)} \chi_\lambda \otimes \chi_\lambda$ equals $\sum_{\lambda\in H(k,\ell;n)} d_\lambda^2$. This sum is investigated in [2]. Here is the main result (Theorem 7.21 taking $z = 1$).

THEOREM: *$\sum_{\lambda\in H(k,\ell;n)} d_\lambda^2 \simeq C(\frac{1}{n})^{(k^2+\ell^2-1)/2}(k+\ell)^{2n}$, where C is a constant which can be calculated explicitly.*

COROLLARY 6: *$c_n^{ptr}(M_{k,\ell})$ and $c_n^{mtr}(M_{k,\ell})$ are asymptotic to*

$$C\left(\frac{1}{n}\right)^{(k^2+\ell^2-1)/2}(k+\ell)^{2n} \quad \text{and} \quad C(k+\ell)^2\left(\frac{1}{n}\right)^{(k^2+\ell^2-1)/2}(k+\ell)^{2n},$$

respectively.

Proof: $c_n^{ptr}(M_{k,\ell}) = \sum_{\lambda\in H(k,\ell;n)} d_\lambda^2$ and $c_n^{mtr}(M_{k,\ell}) = c_{n+1}^{ptr}(M_{k,\ell})$.

We are now in a position to capture the polynomial and exponential behavior of the codimensions. Consider Lemma 4. By 4(b), Δ induces a map from $MT_n/(MT_n \cap I(M_{k,\ell}))$ to $MT_{n+r}/(MT_{n+r} \cap I(M_{k,\ell}))$. By 4(c) this map will be one-to-one. And, by 4(d), the image will be in $V_{n+r}/(V_{n+r} \cap I(M_{k,\ell}))$ if $k \neq \ell$, and in $(V_{n+r} + V_{n+r-m}c(x_{n+r-m+1}, \ldots, x_{n+r}))/(V_{n+r} \cap I(M_{k,\ell}))$ if $k = \ell$.

THEOREM 7: *There exist constants C_1 and C_2 such that, for all n,*

$$C_1\left(\frac{1}{n}\right)^{(k^2+\ell^2-1)/2}(k+\ell)^{2n} \leq c_n(M_{k,\ell}) \leq C_2\left(\frac{1}{n}\right)^{(k^2+\ell^2-1)/2}(k+\ell)^{2n}.$$

Proof: The upper bound follows from $c_n(M_{k,\ell}) \leq c_n^{mtr}(M_{k,\ell}) = c_{n+1}^{ptr}(M_{k,\ell}) = \sum_{\lambda\in H(k,\ell;n+1)} d_\lambda^2$. For the lower bound, first consider the case $k \neq \ell$. The above discussion implies that $c_n(M_{k,\ell}) \geq c_{n-r}^{mtr}(M_{k,\ell}) = c_{n-r+1}^{ptr}(M_{k,\ell}) = \sum_{\lambda\in H(k,\ell;\,n-r+1)} d_\lambda^2$. And, if $k = \ell$, then

$$c_n(M_{k,k}) + c_{n-m}(M_{k,k}) \geq c_{n-r}^{mtr}(M_{k,k}) = \sum_{\lambda\in H(k,k;\,n-r+1)} d_\lambda^2.$$

But, since $1 \in M_{k,k}$, the codimension sequence is increasing and $c_n(M_{k,k}) + c_{n-m}(M_{k,k}) \leq 2c_n(M_{k,k})$ and so $c_n(M_{k,k}) \geq \frac{1}{2}\sum_{\lambda\in H(k,\ell;n-r+1)} d_\lambda^2$. We leave it to the reader to show that each of these bounds is of the required form.

We may use Theorem 7 to get upper and lower bounds for the codimensions of $M_k(E)$. Although this won't capture the polynomial behavior precisely, it is better than the bounds calculated up to now.

THEOREM 8: *There exist constants C_3 and C_4 such that, for all n,*

$$C_3\left(\frac{1}{n}\right)^{(2k^2-1)/2}(2k^2)^n \leq c_n(M_k(E)) \leq C_4\left(\frac{1}{n}\right)^{(k^2-1)/2}(2k^2)^n.$$

Proof: In addition to Theorem 6, the proof uses three ingredients. The first two were proven by the second author in [7] and with Krackowski in [5], respectively. The third is due to Kemer. See [10] for a proof.

(a) For any *p.i.* algebras A and B, $\chi_n(A\otimes B) \le \chi_n(A)\otimes\chi_n(B)$, and so $c_n(A\otimes B)\le c_n(A)c_n(B)$.
(b) $c_n(E) = 2^{n-1}$.
(c) $M_{k,k}$ is p.i. equivalent to $M_k(E)\otimes E$.

Now we first use the simple equation $M_k(F)\otimes E = M_k(E)$. Hence, $c_n(M_k(E)) \le c_n(E)c_n(M_k(F))$, so

$$c_n(M_k(E)) \le 2^{n-1}c_n(M_k(F)) \le 2^{n-1}C\left(\frac{1}{n}\right)^{(k^2-1)/2} k^{2n}.$$

This implies the upper bound.

To get the lower bound, we use (c). $c_n(M_{k,k}) \le c_n(E)c_n(M_k(E))$, so

$$c_n(M_k(E)) \ge c_n(E)^{-1}c_n(M_{k,k}) \ge 2^{-(n-1)}C\left(\frac{1}{n}\right)^{(2k^2-1)/2} (2k)^{2n}.$$

Again, this implies the desired bound.

Since the map Δ in Lemma 4 is an S_n-map, we may also compare the ordinary cocharacter of $M_{k,\ell}$ to the trace cocharacters. It is always the case that the ordinary cocharacter and the pure trace cocharacter are less than or equal to the mixed trace cocharacter. And, since $M_{k,\ell}$ has a unit, the mixed trace cocharacter can be gotten from the pure trace cocharacter by inducing down. But, now that we have a conductor we also get lower bounds for the cocharacters.

THEOREM 9: *Assume $k\neq\ell$. Then*
(a) $\chi_n(M_{k,\ell}) \le \sum_{\lambda\in H(k,\ell;n+1)}(\chi_\lambda\otimes\chi_\lambda)'$,
(b) $\sum_{\lambda\in H(k,\ell;n)}\chi_\lambda\otimes\chi_\lambda \le \chi_{n+r}(M_{k,\ell})\downarrow_{S_n}$,
(c) $\sum_{\lambda\in H(k,\ell;n+1)}(\chi_\lambda\otimes\chi_\lambda)' \le \chi_{n+r}(M_{k,\ell})\downarrow_{S_n}$.

Proof: (a) Follows from the remark in Section 1 that the ordinary cocharacter is always less than or equal to the mixed trace cocharacter.

(b) and (c): By Lemma 4(a,b,d), the map Δ induces S_n-maps from $MT_n/MT_n\cap I(M_{k,\ell})$ and $T_n/T_n\cap I(M_{k,\ell})$ to $V_{n+r}/V_{n+r}\cap I(M_{k,\ell})$ and by 4(c) the induced maps are one-to-one. The theorem follows.

In the case of $k=\ell$ the situation is similar, with an extra factor of 2.

THEOREM 10:
(a) $\chi_n(M_{k,k}) \le \sum_{\lambda\in H(k,k;n+1)}(\chi_\lambda\otimes\chi_\lambda)'$,

(b) $\sum_{\lambda \in H(k,k;n)} \chi_\lambda \otimes \chi_\lambda \leq 2\chi_{n+r}(M_{k,k}) \downarrow_{S_n}$,
(c) $\sum_{\lambda \in H(k,k;n+1)} (\chi_\lambda \otimes \chi_\lambda)' \leq 2\chi_{n+r}(M_{k,k}) \downarrow_{S_n}$.

Proof: The proof is essentially the same as for Theorem 7, with this adjustment: The injection induced from Δ allows us to prove that

$$\chi_n^{mtr}(M_{k,k}) \leq \chi_{n+r}(M_{k,k}) \downarrow_{S_n} + \chi_{n+r-m}(M_{k,k}) \downarrow_{S_n} .$$

But, since $1 \in M_{k,k}, \chi_n(M_{k,k}) \leq \chi_{n+1}(M_{k,k}) \downarrow_{S_n}$ for all n, and so $\chi_{n+r-m}(M_{k,k}) \downarrow_{S_n} \leq \chi_{n+r}(M_{k,k}) \downarrow_{S_n}$. Hence $\chi_n^{mtr}(M_{k,k}) \leq 2\chi_{n+r}(M_{k,k}) \downarrow_{S_n}$. And likewise for the pure traces.

References

[1] A. Berele, *Trace identities and $Z/2Z$-graded invariants*, Transactions of the American Mathematical Society **309** (1988), 581–589.

[2] A. Berele and A. Regev, *Hook Young diagrams with applications to combinatorics and to representations of Lie superalgebras*, Advances in Mathematics **64** (1987), 118–175.

[3] E. Formanek, *Invariants and the ring of generic matrices*, Journal of Algebra **89** (1984), 178–223.

[4] E. Formanek, *A conjecture of Regev about the Capelli polynomial*, Journal of Algebra **109** (1987), 93–114.

[5] D. Krackowski and A. Regev, *The polynomial identities of the Grassmann algebra*, Transactions of the American Mathematical Society **181** (1973), 429–438.

[6] Yu. Razmyslov, *Trace identities and central polynomials in the matrix superalgebra $M_{k,\ell}$*, Math USSR-Sbornik **56** (1987), 187–206.

[7] A. Regev, *The Kronecker product of S_n-characters and an $A \otimes B$ theorem for Capelli identities*, Journal of Algebra **66** (1980), 505–510.

[8] A. Regev, *Codimensions and trace codimensions of matrices are asymptotically equal*, Israel Journal of Mathematics **47** (1984), 246–248.

[9] A. Regev, *Sign trace identities*, Linear and Multilinear Algebra **21** (1987), 1–28.

[10] A. Regev, *Tensor products of matrix algebras over the Grassmann algebra*, Journal of Algebra **133** (1990), 512–523.